

Sampling–type sets and composition operators on Bloch–type spaces¹

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The purpose of this article is to survey the recent results on sampling–type sets and composition operators on Bloch–type spaces. Most of them were presented by the authors in two conferences held in Bogotá 2011, on the occasion of celebrating the 60th anniversary of the Departamento de Matemáticas, Universidad Nacional de Colombia.

Keywords: Composition operators, Bloch space, sampling sets.

El propósito de este artículo es el de divulgar resultados recientes sobre conjuntos tipo muestreo y del operador de composición en espacios de Bloch. La mayoría de estos resultados fueron presentados por los autores en dos conferencias, las cuales tuvieron lugar en Bogotá 2011, en ocasión de la celebración del 60 aniversario del Departamento de Matemáticas de la Universidad Nacional de Colombia.

Palabras claves: operador de composición, espacio de Bloch, conjuntos de muestreos.

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1 Introduction

Let \mathbb{D} denote the unit disk in the complex plane \mathbb{C} and let $H(\mathbb{D})$ be the space of all holomorphic functions on \mathbb{D} with the topology of uniform convergence on compact subsets of \mathbb{D} . A function $f \in H(\mathbb{D})$ is said to be a Bloch function if

$$|f'(z)| \leq \frac{K_f}{1 - |z|^2}, \quad (1)$$

for all $z \in \mathbb{D}$ and for some positive constant K_f depending on f . The set of all Bloch functions is an F -space which is denoted as \mathcal{B} . The idea of consider this kind of functions is derived from the proof of the Bloch's theorem (see [7] or [15]). If we denote by $\|f\|_{\mathcal{B}}$ the smallest constant K_f satisfying the relation (1), then we have that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|. \quad (2)$$

Thus, a function f belongs to \mathcal{B} if and only if $\|f\|_{\mathcal{B}} < \infty$. Furthermore, the relation $\|\cdot\|_{\mathcal{B}}$ is a seminorm for \mathcal{B} and this space becomes a Banach space when it is equipped with the norm $\|f\| := |f(0)| + \|f\|_{\mathcal{B}}$ (see [51]).

The Bloch space appears in a natural way in different contexts; for instance, a holomorphic function f belongs to the Bloch space if and only if there is an univalent function g on \mathbb{D} such that $f = \log(g')$ (see [1] and [30, Ch. 4]). Also, an analytic function f belongs to the Bloch space \mathcal{B} if it is Lipschitz when we consider the domain image with the Euclidean metric and the disk \mathbb{D} with hyperbolic metric, where for $z, w \in \mathbb{D}$ the hyperbolic distance from z to w , denoted as $\beta(z, w)$, is given by

$$\beta(z, w) = \frac{1}{2} \log \left(\frac{1 + \rho(z, w)}{1 - \rho(z, w)} \right).$$

Here $\rho(z, w) = |\varphi_w(z)|$ denotes the pseudo-hyperbolic distance from z to w , being $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$ the automorphism of the unit disk that interchanges the points 0 and a . In fact, it is known that

$$\|f\|_{\mathcal{B}} = \sup \left\{ \frac{|f(z) - f(w)|}{\beta(z, w)} : z, w \in \mathbb{D}, z \neq w \right\}. \quad (3)$$

For the details of the above mentioned result we refer to the book by Zhu [51]. Other expressions equivalent to $\|\cdot\|_{\mathcal{B}}$ can be found in Axler [4, Theorem 5.5], in particular the following relation was demonstrated by Yamashita [46]

$$\|f\|_{\mathcal{B}} \sim \sup_{z \in \mathbb{D}} \left[\int_{\Delta(z,r)} |f'(w)|^2 dA(w) \right]^{\frac{1}{2}}, \quad (4)$$

for all $f \in \mathcal{B}$, where $r \in (0, 1)$ is fixed and for $a \in \mathbb{D}$ and $\eta \in (0, 1)$, $\Delta(a, \eta)$ denotes the pseudo-hyperbolic disk with pseudo-hyperbolic center a and pseudo-hyperbolic radius η , that is,

$$\Delta(a, \eta) := \{z \in \mathbb{D} : \rho(z, a) < \eta\}.$$

$\Delta(a, \eta)$ is the image of the Euclidean disk $|\zeta| < \eta$ for the Möbius transformation $w = \varphi_a(z)$. Note that $w \in \Delta(a, \eta)$ if and only if $\beta(a, w) < M$, where $\tanh \frac{M}{2} = \eta$.

Spaces which generalize the Bloch space were considered by Zhu in [50], he considered a positive parameter $\alpha > 0$, and defined the α -Bloch space, denoted as \mathcal{B}^α , as the set of all holomorphic functions f on \mathbb{D} such that

$$\|f\|_\alpha := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

It is known that \mathcal{B}^α is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$ and it is clear that when $\alpha = 1$, we get the Bloch space. Other kind of Bloch-type space appears when one studies properties of some operators in certain spaces of holomorphic functions; for instance, in [3] Attele proved that the Hankel operator induced by a function f in the Bergman space (see [17, 51, 22] and the reference therein for a general theory of Bergman spaces) is bounded if and only if

$$\|f\|_{\log} := \sup_{z \in \mathbb{D}} (1 - |z|^2) \log \left(\frac{2}{1 - |z|^2} \right) |f'(z)| < \infty. \quad (5)$$

The set of all holomorphic functions f on \mathbb{D} satisfying (5) is a F -space, which is denoted as \mathcal{B}^{\log} , and it is known as the *Log-Bloch space* or the *weighted Bloch space*. Also, this space is a Banach space with the norm $\|f\|_{\mathcal{B}^{\log}} = |f(0)| + \|f\|_{\log}$.

Recently many authors have studied other classes of Bloch-like spaces. The idea is to try to extend the known results for the Bloch spaces to a more general space which generalize the Bloch space. Some authors have considered the class \mathcal{B}^μ of all analytic functions f on \mathbb{D} such that

$$\|f\|_{\mu} := \sup_{z \in \mathbb{D}} \mu(1 - |z|^2) |f'(z)| < \infty,$$

where μ is a continuous positive function defined on the interval $0 < t \leq 1$. We can see that when $\mu(t) = t^{\alpha}$ with $\alpha > 0$, we obtain the α -Bloch spaces and when $\mu(t) = t \log \frac{2}{t}$ the space \mathcal{B}^{μ} coincides with \mathcal{B}^{\log} . Quite recently Stević in [23, 40] introduced the so called logarithmic Bloch type space with $\mu(t) = t^{\alpha} \ln^{\beta} \frac{e}{t}$, $\alpha > 0$ and $\beta \geq 0$, where some properties of this space are studied. Another Bloch type space, using Young's functions, have been recently introduced by Ramos Fernández in [33], here $\mu(t) = 1/\varphi^{-1}(\frac{1}{t})$ being $\varphi : [0, \infty) \rightarrow [0, \infty)$ an \mathcal{N} -function (see [35]). In [21] the authors consider Bloch-type spaces where μ is a normal function; that is, $\mu : (0, 1] \mapsto [0, \infty)$ is a continuous function such that $h_1(x) := \mu(x)/x^{\alpha}$ ($0 < x < 1$) is an almost increasing function for some $\alpha > 0$, and $h_2(x) := \mu(x)/x^{\beta}$ ($0 < x < 1$) is an almost decreasing function for some $\beta > 0$ (a function $h \in \mathbb{R}^{(0,1)}$ is said to be almost increasing if there exists a constant $C > 0$ such that $h(x) \leq Ch(y)$ whenever $x < y$). Giménez, Malavé and Ramos Fernández [20] have considered μ -Bloch spaces, where the weight μ can be extended to non vanishing complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1, 1)$.

In a more general setting, a function $f \in H(\mathbb{D})$ is called a μ -Bloch function, denoted as $f \in \mathcal{B}^{\mu}$, if it satisfies

$$\|f\|_{\mu} := \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty,$$

where μ is a *weight function*, that is, μ is a bounded continuous and positive function μ defined on \mathbb{D} . It is readily seen that \mathcal{B}^{μ} is a Banach space with the norm $\|f\|_{\mathcal{B}^{\mu}} := |f(0)| + \|f\|_{\mu}$. Recently, this type of space has been considered by Stević and Sharma [42], Wolf [43] and Ramos Fernández [34], among others. H_{μ}^{∞} denotes the spaces of growth analytic functions on \mathbb{D} such that

$$\|f\|_{H_{\mu}^{\infty}} := \sup_{z \in \mathbb{D}} \mu(z) |f(z)| < \infty.$$

The relationship between μ -Bloch spaces and H_{μ}^{∞} spaces had been studied exhaustively in [5, 6]. It is clear that $f \in \mathcal{B}^{\mu}$ if and only if $f' \in H_{\mu}^{\infty}$. Indeed, in many cases the \mathcal{B}^{μ} space coincides with some $H_{\mu_1}^{\infty}$ for some weight function μ_1 on \mathbb{D} ; for example, in [50] it is shown that the α -Bloch space with $\alpha > 1$ coincides with $H_{\mu_1}^{\infty}$ where $\mu_1(z) = (1 - |z|^2)^{\alpha-1}$.

The spaces H_μ^∞ with $\mu(z) = (1 - |z|^2)^\alpha$ and $\alpha > 0$ are also known as *Korenblum spaces* (see [17, 22]) and because of that, H_μ^∞ spaces are also called *Korenblum-type spaces*.

2 Sampling sets for μ -Bloch spaces

A sequence $\Gamma = \{z_n\}$ of distinct points in \mathbb{D} is said to be a *sampling sequence* for \mathcal{B}^α if there is a positive constant L such that

$$\sup_{k \in \mathbb{N}} (1 - |z_k|^2)^\alpha |f'(z_k)| \geq L \|f\|_\alpha,$$

for all $f \in \mathcal{B}^\alpha$. Sampling sequences were studied by Seip for Korenblum spaces and his methods were exploited by Schuster, Varolin and many others to characterize sampling (and interpolation) sequences in Bergman spaces (see [39], [37] and [36]).

Before to announce the result due to Seip, we gather some preliminaries properties of pseudo-hyperbolic disk. An important pseudo-hyperbolic geometric property of \mathbb{D} is the following (see [17, Lemma 12] or [51]). *For any $r \in (0, 1]$, there exists a sequence $\{a_n\}$ in \mathbb{D} , and a positive integer N , such that*

1. *The unit disk \mathbb{D} is covered by $\{\Delta(a_n, r)\}$,*
2. *Every point in \mathbb{D} belongs to at most N of the disks $\Delta(a_n, \frac{1}{2}(1+r))$,*
3. *If $n \neq m$, then $\rho(a_n, a_m) = |\varphi_{a_n}(a_m)| \geq \frac{r}{2}$.*

A sequence $\Gamma = \{a_n\}$ in \mathbb{D} satisfying a property such as **3** is called *separated*. The number

$$\delta(\Gamma) := \inf_{n \neq m} \rho(a_n, a_m),$$

is called the separation constant of Γ . An *r-net* is a sequence $\{a_n\} \subset \mathbb{D}$ that satisfy property **1**.

In [38], Seip introduced the following notions of densities to characterize sampling (and interpolation) sequences in the Bergman space. Given a separated sequence $\Gamma = \{z_k\} \subset \mathbb{D}$ and a real number $r \in (\frac{1}{2}, 1)$, a quantity that somehow measures the average distribution pattern of the points of Γ with respect to the disks $\Delta(\zeta, s)$, for $0 < s < r$ (see [17]), is

$$D(\Gamma, \zeta, r) := \left(\log \frac{1}{1-r} \right)^{-1} \sum_{z_k \in \Omega(\zeta, \frac{1}{2}, r)} \log \frac{1}{|\varphi_\zeta(z_k)|},$$

where $\Omega(\zeta, r_1, r_2)$ denotes the pseudohyperbolic annulus: $r_1 < \rho(\zeta, z) < r_2$.

The *lower uniform density* of Γ is defined to be

$$D^-(\Gamma) := \liminf_{r \rightarrow 1} \inf_{\zeta \in \mathbb{D}} D(\Gamma, \zeta, r),$$

and the *upper uniform density* is

$$D^+(\Gamma) := \limsup_{r \rightarrow 1} \sup_{\zeta \in \mathbb{D}} D(\Gamma, \zeta, r).$$

It is clear from the definitions that $0 \leq D^-(\Gamma) \leq D^+(\Gamma)$. The following result holds (see [18, Lemma 4]).

Theorem 1 ([18]). *A sequence Γ has lower density $D^-(\Gamma) > 0$ if and only if Γ is an ε -net for some $\varepsilon \in (0, 1)$.*

Now, Seip's result can be enunciated as follows.

Theorem 2 ([38]). *A sequence $\Gamma \subset \mathbb{D}$ is a sampling sequence for \mathcal{B}^α if and only if it has a separated subsequence Γ' with lower density $D^-(\Gamma') > \alpha$.*

In particular.

Corollary 1. *A sequence $\Gamma \subset \mathbb{D}$ is a sampling sequence for \mathcal{B} if and only if it has a separated subsequence Γ' with lower density $D^-(\Gamma') > 1$.*

Remark 1. *As a consequence of Theorems 1 and 2, we observe that any sampling sequence for \mathcal{B}^α is an ε -net for some $\varepsilon \in (0, 1)$ and, conversely, if $\Gamma = \{z_k\}$ is a separated ε -net for $\varepsilon \in (0, 1)$ small enough, then Γ is a sampling sequence for \mathcal{B}^α .*

In [8] Boe and Nicolau, consider for the definition of sampling set the relation given in (3). More precisely, a sequence of points $\{z_n\}$ in the unit disk is called *sampling for the Bloch space* if there exists a constant $C > 0$ such that

$$\sup_{n \neq m} \frac{|f(z_n) - f(z_m)|}{\beta(z_n, z_m)} \geq C \|f\|_{\mathcal{B}} \quad (6)$$

for any function f in the Bloch space. With this definition, the result obtained by them is as follows.

Theorem 3 ([8]). *Suppose $\Gamma = \{\zeta_n\}$ is a separated sequence of points in the unit disk. The following statements are equivalents:*

- (i) Γ is a sampling sequence for the Bloch space.
- (ii) Γ is an η -net for some $\eta \in (0, 1)$.
- (iv) $D^-(\Gamma) > 0$.

Observe that the characterization obtained by Boe and Nicolau using the expression (3) differ from the one obtained by Seip using the expression (2); in this way the sampling sets of the Bloch type spaces will depend on the norm (or seminorm) considered on this spaces. Next, we propose the following

Problem 1. State and characterize the sampling sets of the Bloch space using the expression (4).

The authors believe that if $\Gamma = \{\zeta_n\}$ is a separated sequence of points in the unit disk, then Γ is a sampling sequence for the Bloch space (using expression (4)) if and only if $D^-(\Gamma) > \frac{1}{2}$.

The results due to Seip had been extended by Domański and Lindström, in [16] to Korenblum-type growth spaces for a wide class of essential weights v , depending on the distance to the boundary of the domain \mathbb{D} , and where certain indices L_v and U_v are equals and belongs to $(0, \infty)$ (see [16, pag. 242] for the definition and the properties of the index L_v and U_v). A weight v is called *essential* if there exists a $C > 0$ such that

$$v(z) \leq \tilde{v}(z) \leq C v(z),$$

for each $z \in \mathbb{D}$, where the *associated weight* (see [2] or [6]) \tilde{v} is defined by

$$\tilde{v}(z) := \frac{1}{\sup \left\{ \|f(z)\| : \|f\|_{H_v^\infty} \leq 1 \right\}}.$$

If we take \tilde{v} instead of v , the space H_v^∞ and the norm $\|\cdot\|_{H_v^\infty}$ do not change.

3 Other sampling-type sets for spaces of analytic functions

As was mentioned at the end of the previous section, in [16] Domański and Lindström showed that the sampling sets for v -Korenblum spaces are

certain η -lattices in the Bergman metric whose lower Seip density exceeds a certain index depending on the weight v . However, the calculation of these densities and the index mentioned above are, in general, very difficult to calculate (for some examples of the calculation of the Seip density, see [17]). For this reason, we are interested in finding other kind of sets that can be used to estimate the norm in the Korenblum–Orlicz spaces.

Our interest comes from an article by Marshall and Smith [25] where they analyzed a problem of this type for functions in the classic Bergman spaces (without weight) A^p (see [22, 17], for definitions and properties of Bergman spaces, and the references therein). The main result of [25, Theorem 1.1] ensures that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)| dA(z) > \delta \int_{\mathbb{D}} |f(z)| dA(z),$$

for any univalent function in A^1 fixing the origin, where

$$\Sigma_\varepsilon = \{w \in \mathbb{C} : |\arg(w)| < \varepsilon\}.$$

It is an *open problem* to show whether their result still holds if the hypothesis that f is univalent were omitted.

Pérez González and Ramos Fernández [26, 27, 28] have extended the results of Marshall and Smith to the widest class of weighted Bergman space A_α^p . They proved the following theorem:

Theorem 4. If $\alpha > -1$ and $p \geq 1$ satisfy $\alpha > 2p - 1$, then for all $\varepsilon > 0$, there exist a constant $\delta > 0$, depending only on p , α and ε , such that

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|^p (1 - |z|^2)^\alpha dA(z) > \delta \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z), \quad (7)$$

for any univalent function $f \in A_\alpha^p$ with $f(0) = 0$.

Also, they gave an example where the above theorem does not hold for all $\varepsilon > 0$ when $\alpha < 2p - 2$. If $p = 1$ and $\alpha \geq 0$, then Theorem 4 is true (see [27]). The case when $2p - 2 \leq \alpha \leq 2p - 1$ needs to be explored. On the other hand, if we omit the condition that the function is univalent, does the result continue to be true?

The results of Marshall and Smith were extended to the Besov-type spaces and the Bergman–Orlicz spaces by Ramos Fernández and Pérez González in [32] and [29]. But in [9], Castillo and Ramos Fernández showed that a similar result is not possible in the classical Besov spaces B_{p-2}^p (see [51] for definition and properties of Besov spaces). However, they showed the following result.

Theorem 5 ([9]). *Assume $p > 1$ and $\varepsilon > 0$. There exist a constant $K(p) > 0$ such that*

$$\int_{f^{-1}(\Sigma_\varepsilon)} (1 - |z|^2)^{p-2} |f'(z)|^p dA(z) \geq K(p)\varepsilon \frac{|f'(0)|^{p+4}}{\|f\|_{p,p-2}^4}, \quad (8)$$

for any nonconstant function $f \in B_{p-2}^p$ with $f(0) = 0$.

On Bloch–type spaces, in [31] Ramos Fernández showed that the result of Marshall and Smith cannot be extended to the Bloch space, however he obtained the following result.

Theorem 6 ([31]). *Assume $\alpha > -1$. Then there exist a constant $K(\alpha) > 0$ depending only on α such that*

$$\sup_{z \in f^{-1}(\Sigma_\varepsilon)} (1 - |z|^2)^\alpha |f'(z)| \geq K(\alpha)\varepsilon \frac{|f'(0)|^4}{\|f\|_\alpha^3}, \quad (9)$$

for any nonconstant function $f \in \mathcal{B}^\alpha$ with $f(0) = 0$.

The above result implies that the result of Marshall and Smith in [25] can be extended to α –Bloch spaces for $\alpha \geq 3$. More precisely

Corollary 2. *If $\alpha \geq 3$, then for all $\varepsilon > 0$, there exist a constant $\delta > 0$, depending only on α and ε , such that*

$$\sup_{f^{-1}(\Sigma_\varepsilon)} (1 - |z|^2)^\alpha |f'(z)| \geq \delta \|f\|_\alpha, \quad (10)$$

for any univalent function $f \in \mathcal{B}^\alpha$ with $f(0) = 0$.

Since Corollary 2 is not true for $\alpha = 1$, it is natural to ask the following.

Problem 2. *Is it true that Corollary 2 holds for $\alpha \in (1, 3)$? In Corollary 2, can one drop the condition that f be univalent? Corollary 2 will be true for another Bloch type spaces?*

About the last question, recently Castillo and Ramos Fernández in [10] show that a result similar to Corollary 2 is also true for the Korenblum–Orlicz spaces; more precisely, they show the following result.

Theorem 7 ([10]). *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be an \mathcal{N} –function satisfying the global Δ_2 –condition. Suppose that $\alpha > 2K_\Delta$ and let $\varepsilon > 0$. Then there exists a constant $\delta > 0$, depending only on α and φ , such that*

$$\sup_{z \in f^{-1}(\Sigma_\varepsilon)} (1 - |z|^2)^\alpha \varphi(|f(z)|) \geq \delta \varepsilon \|f\|_{\alpha, \varphi}, \quad (11)$$

for all conformal mappings $f \in \mathcal{A}_\varphi^{-\alpha}$ such that $f(0) = |f'(0)| - 1 = 0$.

For the definition and properties of \mathcal{N} -function and the Δ_2 -condition, we refer to [35]. Here $f \in \mathcal{A}_\varphi^{-\alpha}$ if $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha \varphi(|f(z)|) < \infty$ and

$$\|f\|_{\alpha, \varphi} = \sup_{z \in \mathbb{D}} \frac{|f(z)|}{\varphi^{-1}((1 - |z|^2)^{-\alpha})}.$$

An outstanding problem is to establish whether Theorem 7 is true for $\alpha \in (0, 2K_\Delta]$ and also if f Theorem 7 remain valid if we drop the conformal condition.

4 Composition operators on Bloch-type spaces

Let \mathcal{H}_1 and \mathcal{H}_2 be two linear subspaces of $H(\mathbb{D})$. If ϕ is a holomorphic self-map of \mathbb{D} , such that $f \circ \phi$ belongs to \mathcal{H}_2 for all $f \in \mathcal{H}_1$, then ϕ induces a linear operator $C_\phi : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ defined by

$$C_\phi(f) := f \circ \phi,$$

called the *composition operator* with *symbol* ϕ . The aim of this section is to update the results on continuity, compactness and closed range of the composition operator when \mathcal{H}_1 and \mathcal{H}_2 are Bloch-type spaces.

The study of the properties of the composition operator on Bloch space began in 1995 when Madigan and Matheson in [24] characterized continuity and compactness of the composition operator on Bloch spaces. Among other results that they obtained we shall point out the following

- C_ϕ is always continuous \mathcal{B} .
- C_ϕ is compact on \mathcal{B} if and only if

$$\lim_{|\phi(z)| \rightarrow 1^-} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0.$$

The results of Madigan and Matheson have been extended by Xiao [45] in 2001 to the α -Bloch spaces and by Yoneda [47] in 2002 to the

Log–Bloch space. Since then, several extensions and generalizations of the results due to Madigan and Matheson to another Bloch type spaces have appeared. For instance on logarithmic Bloch type spaces, one can cite the works of Stević [40, 41]. On Bloch–Orlicz the extension is due to Ramos Fernández in [33]. Also, in [48], Zhang and Xiao have characterized boundedness and compactness of weighted composition operators that act between μ –Bloch spaces on the unit ball of \mathbb{C}^n . In this case it is required that μ be a *normal* function. The results of Zhang and Xiao have been extended by Chen and Gauthier [13] to the μ –Bloch spaces being μ a positive and non–decreasing continuous function such that $\mu(t) \rightarrow 0$ as $t \rightarrow 0$ and $\mu(t)/t^\delta$ is decreasing for small t and for some $\delta > 0$. Recently, Chen, Stević and Zhou (see [14]) have studied composition operators between Bloch type spaces in the polydisc. While Giménez, Malavé and Ramos Fernández [20] have extended those results to certain μ –Bloch spaces, where the weight μ can be extended to non vanishing complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1, 1)$. In all cases on the weights mentioned before one obtain the following.

1. $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is continuous if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| < \infty.$$

2. $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and

$$\lim_{|\phi(z)| \rightarrow 1^-} \frac{\mu_2(z)}{\mu_1(\phi(z))} |\phi'(z)| = 0.$$

An outstanding problem is if the same is true for any couple of weights μ_1 y μ_2 on the disk \mathbb{D} . This latter result has been recently studied by Wolf in [43] and by Ramos Fernández in [34]. They showed the following results.

Theorem 8. *Let μ_1 and μ_2 two weights defined on \mathbb{D} .*

- (see [43, 34]) *The operator $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is continuous if and only if*

$$\sup_{z \in \mathbb{D}} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))} |\phi'(z)| < \infty.$$

- (see [34]) The operator $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ is compact if and only if $\phi \in \mathcal{B}^{\mu_2}$ and

$$\lim_{|\phi(z)| \rightarrow 1^-} \frac{\mu_2(z)}{\tilde{\mu}_1(\phi(z))} |\phi'(z)| = 0..$$

Here $\tilde{\mu}_1$ is the weight associated to the weight μ_1 defined at the end of the first section.

Others authors have found another characterization for the continuity and compactness of the composition operator in term of certain expression related with the n -th power of the symbol ϕ . On this theme we have to point it out the work of Wulan, Zheng and Zhu (see [44]). They proved the following result.

Theorem 9. [44] *Let ϕ be an analytic self-map of \mathbb{D} . Then C_ϕ is compact on the Bloch space \mathcal{B} if and only if*

$$\lim_{n \rightarrow \infty} \|\phi^n\|_{\mathcal{B}} = 0.$$

Giménez, Malavé and Ramos Fernández [20] showed that the composition operator $C_\phi : \mathcal{B} \rightarrow \mathcal{B}^\mu$ is compact if and only if $\lim_{n \rightarrow \infty} \|\phi^n\|_\mu = 0$, where the weight μ can be extended to non vanishing, complex valued holomorphic functions, that satisfy a reasonable geometric condition on the Euclidean disk $D(1, 1)$. Ramos Fernández in [33] has extended the previous result to the Bloch–Orlicz space. Recently, Zhao in [49] gives a formula for the essential norm of the composition operator $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$. In particular the operator $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if

$$\lim_{n \rightarrow \infty} n^{\alpha-1} \|\phi^n\|_\beta = 0.$$

This last result has been extended recently by Castillo and Ramos Fernández in [11] to the composition operator $C_\phi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\mu$, where μ is any weight defined on \mathbb{D} . We also left as an open problem to find a similar formula (in term of n -th power of the symbol ϕ) to characterize the compactness of the composition operator when it acts on another Bloch-type spaces; such as logarithmic Bloch spaces.

The problem when the composition operator act on Bloch spaces, with a closed range was resolved in 2004 by Gathage, Zheng and Zorboska [19] in term of the sampling sets for the Bloch spaces. More precisely if $\varepsilon > 0$, we denote by Ω_ε the set

$$\Omega_\varepsilon := \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| \geq \varepsilon \right\},$$

then, the result due to Ghatage, Zheng y Zorboska can be announced in the following way.

Theorem 10 ([19]). *The bounded composition operator C_ϕ has a closed range on \mathcal{B} if and only if there exists $\varepsilon > 0$ such that $G_\varepsilon = \phi(\Omega_\varepsilon)$ is a sampling set for \mathcal{B} .*

This result has been extended by Chen and Gauthier [12] to α -Bloch spaces. Ramos Fernández in [33] have extended all the results mentioned above to the Bloch–Orlicz spaces. More recently, Ramos Fernández has characterized the composition operators $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ with closed range in terms of the sampling sets for \mathcal{B}^μ spaces, where μ_1 and μ_2 are two arbitrary weights defined on \mathbb{D} . The following problem is outstanding:

Problem 3. Characterize the composition operator $C_\phi : \mathcal{B}^{\mu_1} \rightarrow \mathcal{B}^{\mu_2}$ with closed range in terms of the n -th power of the symbol ϕ .

As far as the authors know, this latter problem have not been considered not even for the case that both space are Bloch spaces. For more details on composition operator with closed range on Bloch type space, see a recently work of Zorboska in [52].

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