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A new probabilistic model for the PERT method: Application to investment cash flows

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Abstract

The first aim of this work is to use a new probability distribution successfully employed in hydrology as the underlying probability model to verify its advantages and disadvantages with respect to the beta distribution in the PERT method. The distribution was introduced by Kumaraswamy $K(a, b, p, q)$ and, in principle, is tetra-parametric. Therefore, the three typical expert estimates of the minimum, maximum, and modal values of this method are insufficient to estimate the four parameters. Hence, we first start from the standardized Kumaraswamy distribution $K(p, q)$ (1980), where the minimum and maximum values of the variable are zero and one, respectively, and from a relation through the modal value between the two parameters that remain unspecified. In the next step, we restrict the family by setting one of its parameters to a uni-parametric distribution. Second, we use a simulation process to estimate the parameters to reach a better distribution behavior and improve the average and variance values according to Taha's (1981) proposal related to the beta distribution, the second objective of the work. We illustrate the analysis with an investment analysis example.

Keywords:

PERT method, Net present value, Kumaraswamy distribution.

JEL classification:

C10, C52, C61.

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Un nuevo modelos probabilístico para el método PERT: **aplicación a los flujos de inversión**

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Resumen

El primer objetivo del trabajo que se presenta es usar una nueva distribución de probabilidad, utilizada con éxito en Hidrología, como modelo probabilístico subyacente en el método PERT y comprobar sus ventajas y desventajas respecto a la distribución beta utilizada en la metodología PERT. La distribución fue introducida por Kumaraswamy $K(a, b, p, q)$ y es tetra paramétrica, en principio, por lo que son insuficientes las tres típicas estimaciones periciales, sobre el valor mínimo, máximo y modal, de esta metodología, para estimar los cuatro parámetros. Por ello, en primer lugar, se parte de la distribución de Kumaraswamy estandarizada $K(p, q)$ (1980), donde los valores mínimo y máximo de la variable son cero y uno respectivamente, y de una relación a través del valor modal entre los dos parámetros que quedan sin especificar, en el paso siguiente se restringe la familia, mediante la fijación de uno de sus parámetros, a una distribución uniparamétrica. En segundo lugar, se utiliza un proceso de simulación para la estimación de los parámetros, para conseguir el segundo objetivo del trabajo que es lograr que la distribución tenga un comportamiento, en media y varianza que mejore, en el sentido propuesto por Taha (1981), al de la distribución beta. Todo ello se ilustra con un ejemplo de Análisis de Inversiones.

Palabras clave:

Método PERT, valor actual neto (VAN), distribución de Kumaraswamy.

1. Introduction

The families of tetra-parametric beta distributions, B (a, b, p, q) and two-sided power (TSP) distribution (a, b, m, n), from van Dorp and Kotz (2002) and Kumaraswamy $K(a, b, p, q)$ (1980) within the interval (a, b) are closely related and include the same distributions in particular cases. In addition, their tremendous flexibility and ability to adapt data to campanoid graphs in L, J, or U shapes makes them especially useful as probabilistic models in problems related to risk or uncertain environments, which require asymmetric distributions and different kurtosis. The other distributions used for these types of problems are the bi-parabolic and bi-cubic distributions introduced by García (2007) and López (2010), respectively, in their doctoral theses.

The problem with these tetra-parametric distributions is that they rely on estimating four parameters, usually with little information and the tendency to condense three typical maximum, minimum, and modal values in the PERT method. Thus, we use elicitation processes, request more information about the problem from an expert, restrict distributions to families of a certain probability, such as those of constant variance, null kurtosis, or mesokurtic, or introduce one or several relationships between the last two parameters. This was followed by Caballer (1998) or Herrerías and Herrerías (2013) with the beta distribution used in the PERT method.

In this work, for Kumaraswamy's distribution, we first define the value of the p parameter to obtain q. This is set according to p through the analytical expression of the population mode, which we replace by a subjective estimate provided by the expert. Second, we perform a simulation of the p parameter to determine this distribution, fulfilling Taha's (1981) conservatism criterion with respect to the distribution variance.

The germ of Kumaraswamy's distribution (1980) is the "sinepower" probability distributions and their improvement (Kumaraswamy, 1976, 1978). These appeared first as probabilistic models of typical random variables in hydrology: the daily flow of a river current, daily rainfall, daily storage volume of a swamp, and so on, which cannot be adjusted faithfully by other distributions such as the Gaussian, normal logarithmic, betas, and Johnson's empirical distribution, among others.

All of these hydrological variables share the fact that they are doubly bounded by an upper finite bound b and an inferior bound of less than ≥ 0, as it can be seen with the beta and TSP distributions.

The work proceeds as follows. Section 2 presents Kumaraswamy's $K(x/p, q, a, b)$ and $K(x/p, q, 0, 1)$ distributions. The study is restricted to the latter by means of their density, distribution, and quantile functions, as well as to obtaining the general formula of its ordinary moments and the determination of its media, mode, and median. Section 3 highlights the relationships between Kumaraswamy's distribution and the beta, potential, reflected potential, uniform, exponential, and TSP distributions.

In Section 4, we first choose the family of Kumaraswamy's distribution $K\left(2, \frac{m^2+1}{2\,m^2}\right)$ to facilitate the operation with the distribution and manifested difficulty to obtain explicit estimates of the p and q parameters in a general case. Second, we propose a simulation procedure to estimate such parameters while fulfilling Taha's (1981) conservatism criterion.

Section 4 presents an application of the selected distribution, while Section 5 provides a typical example from the investment analysis literature, in which we verify the appropriateness of Kumaraswamy's distribution with respect to the beta distribution as a model in the PERT method. Finally, the bibliographical references are listed.

2. Stochastic characteristics of Kumaraswamy's distribution

If X is a random variable defined in (a, b), the data will follow a Kumaraswamy's $K(x/p, q, a, b)$ distribution with the following density function:

$$f(x) = \frac{pq\,(x-a)^{p-1}}{(b-a)^{p+q-1}}\,[(b-a)^p - (x-a)^p]^{q-1}, \text{ for } 0 \le a < x < b \text{ and } p > 0, q > 0 \qquad (1)$$

We can simplify expression (1), standardizing the path of the variable X to the interval (0, 1), by changing the variable:

$$Z = \frac{X-a}{b-a} \qquad (2)$$

in the following expression:

$$f(z) = p\,q\,z^{p-1}(1-z^p)^{q-1}, \text{ for } 0 < z < 1 \text{ and } p > 0, q > 0, \qquad (3)$$

which is the density function presented by Kumaraswamy (1980).

We can easily prove that the moments in relation to the origin of (3) are obtained through the expression

$$\alpha_k = q\,B\left(1+\frac{k}{p}, q\right) = \frac{q\Gamma(q)\Gamma\left(1+\frac{k}{p}\right)}{\Gamma\left(q+1+\frac{k}{p}\right)}, \qquad (4)$$

where $B\left(1+\frac{k}{p}, q\right)$ is the Eulerian integral of the first kind $\int_0^1 t^{\frac{k}{p}}(1-t)^{q-1}dt$.

In fact, by changing the variable $z^p = t$ in the integral that determines $\alpha_k = \int_0^1 pq\, z^{k+p-1}$ $(1-)^{qz^{p-1}}dz$, we obtain (4) directly.

Then the mean of (3) is:

$$\alpha_1 = q\,B\left(1+\frac{1}{p}, q\right) = \frac{q\Gamma(q)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(q+1+\frac{1}{p}\right)} = \frac{\Gamma(q+1)}{\left(q+\frac{1}{p}\right)\left(q-1+\frac{1}{p}\right)\left(q-2+\frac{1}{p}\right).....\left(1+\frac{1}{p}\right)} \quad (5)$$

and its variance is:

$$\sigma^2 = \alpha^2 - \alpha_1^2 = q\,B\left(1+\frac{2}{p}, q\right) - \{q\,B\left(1+\frac{1}{p}, q\right)\}^2. \quad (6)$$

The distribution function of (3) is:

$$F(z; p, q) = 1-(1-z^p)^q, \quad (7)$$

which is easily invertible to obtain the quantiles function (Jones, 2009), through the expression:

$$Q(y) = \mathrm{F}^{-1}(y) = \left[1-(1-y)^{1/q}\right]^{1/p} \quad (8)$$

and we can obtain the quantile of probability α through:

$$z_\alpha = \left[1-(1-\alpha)^{1/q}\right]^{1/p}. \quad (9)$$

Given the desirability of this invertibility property, we adopt the evaluation method of the two distribution functions introduced by Ballestero (1973) and shown in Herrerías and Herrerías (2017). From (8), we obtain the median, taking $\alpha = 0.5$:

$$m_e = \left[1-2^{-1/q}\right]^{1/p} \quad (10)$$

We obtain the mode by deriving (3) and setting the derivative equal to zero.

$$m_o = \left(\frac{p-1}{pq-1}\right)^{1/p} \text{ for } p \geq 1, q \geq 1 \text{ and } (p, q) \neq (1, 1) \quad (11)$$

3. Relations with other probability distributions

Kumaraswamy's distribution is strongly related to other distributions used as probabilistic models in risk and uncertain environment problems. These are, respectively denoted as $K(p, q)$, $U(a, b)$, $B(p, q)$, $P(p)$, $PR(q)$, $E(p)$, and $TSP(p, m)$ for Kumaraswamy's, uniform, beta, potential, reflected potential, exponential and, "Two Sided Power" distributions of van Dorp and Kotz (2002). First, we obtain the following results directly.

1. If Z is a random variable that follows a Kumaraswamy distribution $K(1, 1)$, we can prove easily that Z follows a uniform distribution $U(0, 1)$ that coincides with the Beta distribution $B(1, 1)$. In other words, if $Z \longrightarrow K(1, 1) \Rightarrow Z \longrightarrow U(0, 1) \equiv B(1, 1)$.

Indeed, if (3) is characterized by $p = q = 1$, the density function of the uniform distribution is obtained in $(0, 1)$, which coincides with the Beta distribution $B(1, 1)$, the potential distribution $P(1)$, and the TSP distribution $(1, 1)$.

As a corollary to this point, $K(1, 1) \equiv U(0, 1) \equiv B(1, 1) \equiv P(1) \equiv TSP(1, 1)$.

2. If Z is a random variable that follows a Kumaraswamy distribution $K(p, 1)$, we can prove easily that Z follows a potential distribution $P(p)$, with a density function of $f(z) = p\, z^{p-1}$, to $0 < z < 1$. In other words, if $Z \longrightarrow K(p, 1) \Rightarrow Z \longrightarrow P(p) \equiv B(p, 1)$.

Indeed, if we take $q = 1$ in (3), then $(z) = p\, z^{p-1}$, which is the density function of a potential distribution $P(p)$, that in turn matches the beta $B(p, 1)$ and $TSP(p, 1)$ distributions.

As a corollary to this point, $K(p, 1) \equiv B(p, 1) \equiv P(p) \equiv TSP(p, 1)$. If we set $p = 1$, we obtain the corollary to point 1 above.

3. If Z is a random variable that follows a Kumaraswamy distribution $K(1, q)$, we can prove easily that Z follows a beta $B(1, q)$ distribution. In other words, if $Z \longrightarrow K(1, q) \Rightarrow Z \longrightarrow$ Beta $B(1, q)$.

Indeed, if we take $p = 1$ in (3), then $f(z) = q\,(1-z)^{q-1}$, which is the density function of a beta $B(1, q)$ distribution or reflected potential (RP) function (q).

As a corollary to this point, $K(1, q) \equiv B(1, q) \equiv PR(q)$.

4. If Z is a random variable that follows a Kumaraswamy distribution $K(p, 1)$, we can easily prove that $(1-Z)$ follows a Kumaraswamy distribution $K(1, p)$. In other words, if $Z \longrightarrow K(p, 1) \Rightarrow (1-Z) \longrightarrow K(1, p) \equiv PR(p) \equiv B(1, p)$.

Indeed, if we take $q = 1$ in (3), this results in $f(z) = p\,z^{p-1}$, and replacing z with $(1-z)$, we obtain $f(1-z) = p(1-z)^{p-1}$, which is the density function of a Kumaraswamy distribution $K(1, p)$ that matches the RP distribution (p) or Beta $B(1, p)$ distribution.

5. If Z is a random variable that follows a Kumaraswamy distribution $K(1, q)$, we can easily prove that $(1-Z)$ follows a Kumaraswamy distribution $K(q, 1)$. In other words, if $Z \longrightarrow K(1, q) \Rightarrow (1-Z) \longrightarrow K(q, 1)$.

Indeed, if we take $p = 1$ in (3), this results in $f(z) = q\,(1-z)^{q-1}$, and replacing z by $(1-z)$, we obtain $f(1-z) = q\,z^{q-1}$, which is the density function of a Kumaraswamy distribution $K(q, 1)$. This matches the Potential $P(q)$ distribution or Beta $B(1, q)$ distribution.

Second, we obtain the following results through simple changes in a variable:

6. If Z is a random variable that follows a Kumaraswamy distribution $K(p, 1)$, we can prove easily that $-\ln Z$ follows an exponential $E(p)$ distribution. In other words, if $Z \longrightarrow K(p, 1) \Rightarrow -\ln Z \longrightarrow E(p)$.

Indeed, if we change the variable $Y = -\ln Z$ en (7), we obtain the distribution function $F_Y(y) = P(Y \le y) = F_Y(-\ln Z \le y) = F_Y(Z \ge e^{-y}) = 1 - F_Y(Z \le e^{-y}) = 1 - e^{-py}$, which corresponds to the exponential $E(p)$ distribution function.

7. If Z is a random variable that follows a Kumaraswamy distribution $K(1, q)$, we can prove easily that $-\ln(1-Z)$ follows an exponential $E(q)$ distribution. In other words, if $Z \longrightarrow K(1, q) \Rightarrow -\ln(1-Z) \longrightarrow E(q)$.

Indeed, if we change the variable $Y = -\ln(1-Z)$ in (7), we obtain the distribution function $F_Y(y) = P(Y \le y) = F_Y\{-\ln(1-Z) \le y\} = P\{\ln(1-Z) \ge -y\} = P(Z \le 1-e^{-y}) = F_Z(1-e^{-y})$ $= 1-e^{-qy}$, which corresponds to the distribution of an exponential function $E(q)$.

8. If Z is a random variable that follows a uniform distribution $U(0, 1)$, we can prove easily that $[1-(1-Z)^{1/q}]^{1/p}$ follows a Kumaraswamy distribution $K(p, q)$. In other words, if $Z \longrightarrow U(0, 1) \Rightarrow [1-(1-Z)^{1/q}]^{1/p} \longrightarrow K(p, q)$.

Note that this result matches the quantile function (8).

Indeed, if in (7) we change the variable $Y = [1-(1-Z)^{1/q}]^{1/p}$, we obtain the resulting distribution function $F_Y(y) = P(Y \le y) = P\{1-(1-Z)^{1/q} \le y^p\} = P\{(1-Z)^{1/q} \ge 1-y^p\} =$ $P\{Z \le 1-(1-y^p)^q\} = 1-(1-y^p)^q$, which corresponds to a Kumaraswamy distribution function $K(p, q)$.

9. If Z is a random variable that follows a Beta $B(1, q)$ distribution, we can prove easily that $Z^{1/p}$ follows a Kumaraswamy distribution $K(p, q)$. In other words, if $Z \longrightarrow B(1, q) => Z^{1/p} \longrightarrow K(p, q)$.

Indeed, $F_K(z) = P[K(p, q) \leq z)] = \int_0^z pq\, t^{p-1}(1-t^p)^{q-1}\, dt = \{\text{taking } t^p = y\} = \int_0^{z^p} q(1-y)^{q-1}\, dy$ $= P(Z \leq z^p) = P(Z^{1/p} \leq z) = F_{Z^{1/p}}(z)$. The result follows.

We can introduce a generalized Kumaraswamy distribution using a procedure that generalizes the result of the previous point. If Z is a variable that follows a Beta $B(\alpha, q)$ distribution, then the variable $Z^{1/\Upsilon}$, where $\Upsilon > 0$, follows a generalized Kumaraswamy distribution, with a few moments about the origin, given by the expression:

$$\alpha_k = \frac{\Gamma(\alpha+q)\Gamma(\alpha+\frac{K}{\Upsilon})}{\Gamma(\alpha)\Gamma(\alpha+q+\frac{K}{\Upsilon})}. \tag{12}$$

Note that if in (12) $\alpha = 1$ and $\Upsilon = p$, we obtain (4).

4. Estimates of the distribution parameters

First, if we know $p > 1$, we can obtain q easily by matching a subjective estimate of the modal value, m, to expression (11) for the mode, m_o. Indeed, from (11), we can obtain the following relation between parameters p and q

$$q = \frac{1}{p} + \frac{p-1}{pm^p} = \frac{m^p+p-1}{pm^p}. \tag{13}$$

Since we consider the solution to the two equations, using the subjective estimation of two specific quantiles, as Berny (1989) describes, does not have an easy solution for the p and q parameters, the two unknowns. We first use these to restrict the family of distributions to that satisfying Roy's (1971) system, which we obtain for $p = 2$ since the families that satisfy the Pearson system, that is, those with $p = 1$ or $q = 1$ (Elderton and Johnson, 1969), are not Campannoid unimodal. Moreover, if $p = 2$, the expressions of the first moments and probability functions of the Kumaraswamy distribution are significantly simplified and become more operational.

For $p = 2$, we obtain from (13) that $q = \frac{m^2+1}{2m^2}$. Then, the recommended Kumaraswamy distribution for the application is $K\left(2, \frac{m^2+1}{2m^2}\right)$, which only requires a subjective estimation of the modal value.

For this family, the expressions for the average (5) and variance (6) reduce to

$$\alpha_1 = \frac{m^2+1}{2m^2} B\left(\frac{3}{2}, \frac{m^2+1}{2m^2}\right) \text{ and } \sigma^2 = \frac{m^2+1}{2m^2} B\left(2, \frac{m^2+1}{2m^2}\right) - \left\{\frac{m^2+1}{2m^2} B\left(\frac{3}{2}, \frac{m^2+1}{2m^2}\right)\right\}^2. \quad (14)$$

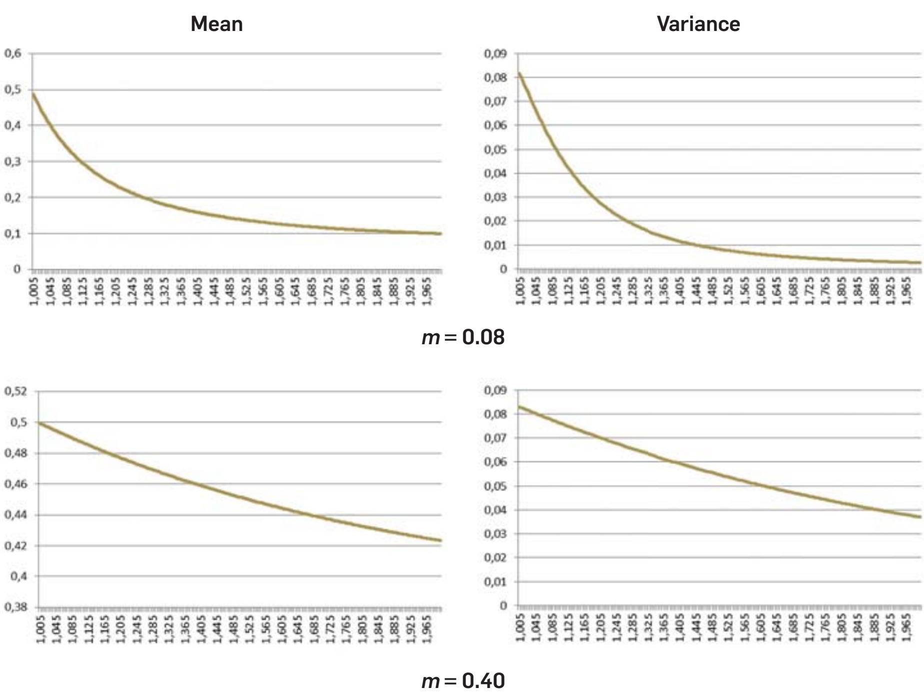
Second, to obtain a more general estimate of p and q, we use a simulation of the p parameter, making it vary from 1.01 to 2.00. We calculate the q parameter from (13) and choose the pair of p and q values according to Taha's (1981) conservatism criterion with respect to the variance.

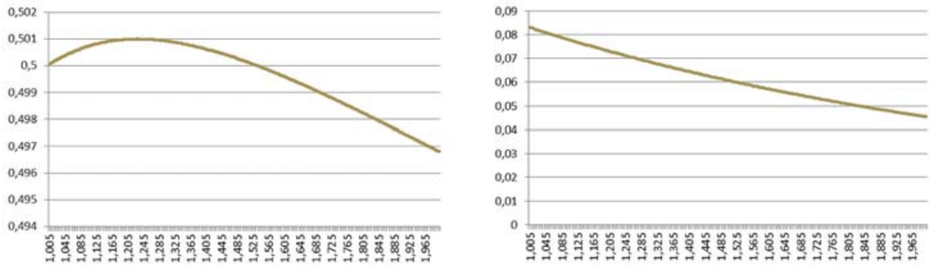
The procedure described above requires the following steps. First, we start by estimating the standardized modal value, then determine the values of the q parameter from expression (13), which is calculated for possible values of p located in the interval (1.005; 2.00). Third, we determine the mean (5) and variance (6) with the pairs of resulting p and q values.

Finally, we choose the pair of parameters that achieves the greatest variance, with which the Kumaraswamy distribution is determined and whose behavior corresponds to the most conservative probabilistic model.

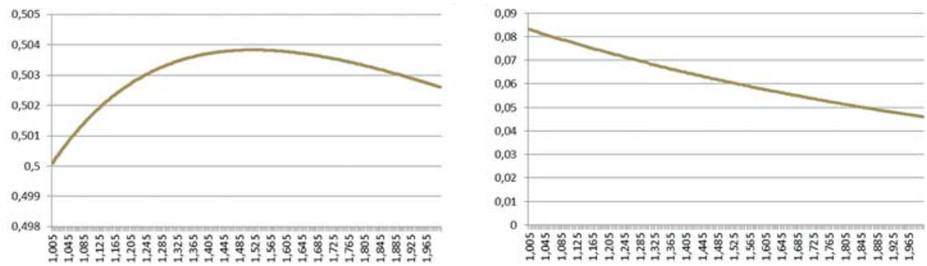
As an example, we present some means and variances of Kumaraswamy distributions for different values of m (0.08; 0.4; 0.51; 0.52; 0.99) according to different values of $p \in (1, 2)$.

Figure 1. Mean and variance graphs of the Kumaraswamy distribution

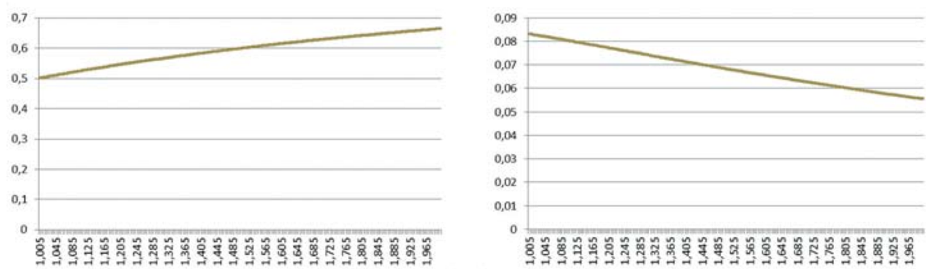




***m* = 0.51**



***m* = 0.52**



***m* = 0.99**

Thus, we observe the following:

a) for $p = 1$, the mean is 0.5 and the variance 0.08333333, regardless of the value of m;
b) for $m \leq 0.5$, the mean is inversely proportional to p;
c) if $m = 0.51$, the mean increases up to $p = 1.23$, and then decreases subsequently;
d) if $m \geq 0.52$, the mean is directly proportional to p;
e) the variance is inversely proportional to p; and
f) for $m \geq 0.6$, the representations of the mean and variance are practically linear.

5. Investment cash flow application

We consider an example from page 157 in Suárez (1980), which uses beta distributions to model the initial payment and cash flows of an investment. We consider these independent and updated at a rate of 7%. They follow the structure in Table 1.

● Table 1. **Initial payment and net cash flows of an investment**

Initial payment (A)			Cash flow of year 1 (Q_1)			Cash flow of year 2 (Q_2)		
Pessimist	More likely	Optimist	Pessimist	More likely	Optimist	Pessimist	More likely	Optimist
25,000	30,000	35,000	20,000	20,000	20,000	15,000	20,000	32,000

Since the Net Present Value, NPV, of the investment and its two main stochastic characteristics when A and Q_t are independent variables:

$$NPV = -A + \sum_1^n \frac{Q_t}{(1+k)^t}\;;\; E(NPV) = -E(A) + \sum_1^n \frac{E(Q_t)}{(1+k)^t}\;;\; V(NPV) = V(A) + \sum_1^n \frac{V(Q_t)}{(1+k)^{2t}} \quad (15)$$

in the example, where Q_1 is constant, expression (15) of the stochastic characteristics of the NPV reduces to

$$E(NPV) = -E(A) + \frac{20{,}000}{1.07} + \frac{E(Q2)}{(1.07)^2} \text{ and } V(NPV) = V(A) + \frac{V(Q2)}{(1.07)^4}. \quad (16)$$

First, we assume that the underlying probabilistic model for standardized A and Q_2 variables, follows the Kumaraswamy distribution $K\left(2, \frac{m^2+1}{2m^2}\right)$, where m is the corresponding modal value standardized. Then, the standardized variable A is distributed as $K(2;\, 2{,}5)$; that is:

$$A^* = \frac{A-25000}{35000-25000} \longrightarrow K\left(2, \frac{1.25}{0.5}\right) = K(2;\, 2.5), \text{ since } m_A{}^* = 0.5$$

Then, $E(A^*) = \frac{2.5\,\Gamma(2.5)\Gamma(1.5)}{\Gamma(4)} = 0.4908623$ using expression (5) and the result $\Gamma(0.5) = \sqrt{\pi}$ from page 255 in Abramowitz and Stegun (1972).

Reversing the change in $E(A^*)$, we finally obtain:

$$E(A) = 25{,}000 + 10{,}000\ (0.4908623) = 29{,}908.62.$$

Similarly, $Q_2^* = \frac{Q_2-15000}{32000\text{-}15000} \longrightarrow K(2;\, 6.8)$, since $mQ_2^* = 5/17$; then $q = 6.28$.

Thus, $E(Q_2^*) = \frac{6.28\,\Gamma(6.28)\Gamma(1.5)}{\Gamma(7.78)} = \frac{(6.28)(5.28)(4.28)(3.28)(2.28)(1.28)\Gamma(1.28)}{(6.78)(5.78)(4.78)(3.78)(2.78)(1.78)\Gamma(1.78)}\Gamma(1.5) =$

$$\frac{1358.4884}{3503.8217}\;\frac{0.9007184765}{0.9262273062}\;\frac{\sqrt{\pi}}{2} = 0.3341412$$

We adopt the corresponding values of $\Gamma(1.28)$ and $\Gamma(1.78)$ from Abramowitz and Stegun (1972).

Reversing the change in $E(Q_2^*)$, we finally obtain:

$$E(Q_2^*) = 15{,}000 + 17{,}000\ (0.3341412) = 20{,}680.4.$$

To determine the variances, we use expression (4) for the corresponding α_2, which in these cases reduces to $\alpha_2 = q\,B(2, q)$. Thus, $V(A^*) = 2.5\,B(2;\,2.5) - E^2(A^*) = \frac{2.5\,\Gamma(2.5)\Gamma(2)}{\Gamma(4.5)} - (0.4908623)^2 = 0.0447685$. Reversing the change in $V(A^*)$, we obtain $V(A) = 10000^2\,V(A^*) = 4{,}476{,}850$ and $D(A) = 3{,}596.9564 \approx 3{,}596.96$.

Similarly, for Q_2^*, we have $V(Q_2^*) = 6.28\,B(2;\,6.28) - E^2(Q_2^*) = \frac{6.28\,\Gamma(6.28)\Gamma(2)}{\Gamma(8.28)} - 0.1116503 = 0.0257123$, from which we find $V(Q_2) = 17{,}000^2\,V(Q_2^*) = 7{,}430{,}854.7$ and $D(Q_2) = 2{,}725.9594 \approx 2{,}725.96$.

If we insert the determined values into expression (16), the stochastic characteristics of the NPV are:

$$E(NPV) = -29{,}908.62 + \frac{20{,}000}{1.07} + \frac{20{,}680.4}{(1.07)^2} = 6846.029;$$

$$V(NPV) = 4{,}476{,}850 + \frac{7{,}430{,}854.7}{(1.07)^4} = 10{,}145{,}813;$$

$$D(NPV) = 3{,}185.2492 \approx 3{,}185.25; \text{ and}$$

$$CV(NPV) = 0.4652696.$$

We see that these values are slightly different from those obtained by Suárez (1980) using beta distributions of $E(NPV) = 7{,}178$; $V(NPV) = 8{,}902{,}129$; $D(NPV) = 2{,}983$, and $CV(NPV) = 0.4155753$

If we use Chebychev's inequality with constant 1.75 for the random variable NPV, it must be that $P\{|NPV - E(NPV)| \leq 1.75\,D(NPV)\} \geq 1 - 0.3265306 = 0.6734694$. Then, the NPV is in the range $(1{,}957.75;\,12{,}398.25)$ with a probability greater than 0.6734694 if we use the beta distribution model, while this interval increases its amplitude if we use the Kumaraswamy K distribution model $\left(2, \frac{m^2+1}{2\,m^2}\right)$; in this case, $(1{,}271.8415;\,12{,}420.216)$. Thus, using Taha's (1981) conservatism principle in this example, the investment analyst must use this distribution as a model for variables A and Q_2.

Second, we proceed with the method to simulate the p parameter of Kumaraswamy's distribution $K(p, q)$. For this, when we fix the corresponding modal standardized value of the distribution using a Microsoft Excel spreadsheet, we find values for the p parameter and obtain the corresponding values of the q parameter using expression (13). From the resulting distribution $K(p, q)$, we obtain their means and variances using expressions (5) and (6), respectively, and end the process by selecting the pair of parameters (p, q) with which we obtain the greater variance in the distribution. According to the last paragraph of point 4, this will be $K\left(1.01;\,\frac{m^{1.01}+0.01}{1.01\,m^{1.01}}\right)$.

In the example, we have: for $m = 0.5$, that $E(A^*) = 0.499997618$ and $V(A^*) = 0.082780038$. Reversing the changes in $E(A^*)$ and $V(A^*)$, we finally find that $E(A) = 25{,}000 + 10{,}000\ (0.499997618) = 29{,}999.97618$; $V(A) = 10{,}000^2\ V(A^*) = 8{,}278{,}003.8$, and $D(A) = 2{,}877.2904$.

Similarly, for $m = 5/17 = 0.2941176$, then $E(Q_2^*) = 0.496532633$ and $V(Q_2^*) = 0.082389868$. Reversing the changes in $E(Q_2^*)$ and $V(Q_2^*)$, we finally find that $E(Q_2) = 15{,}000 + 17{,}000\ (0.496532633) = 23{,}441.054$; $V(Q_2) = 17{,}000^2\ V(Q_2^*) = 23{,}810{,}671.8351$, and $D(Q_2) = 4{,}879.6179$.

If we take the values, $E(A)$, $D(A)$, $E(Q_2)$, and $V(Q_2)$ determined based on expression (16), the stochastic characteristics of the NPV are:

$$E(NPV) = -29{,}999.98 + \frac{20{,}000}{(1.07)} + \frac{23{,}441.054}{(1.07)^2} = 9{,}165.933$$

$$V(NPV) = V(A) + \frac{V(Q_2)}{(1.07)^4} = 8{,}278{,}003.8 + \frac{23{,}810{,}671.8351}{1.310796} = 26{,}443{,}051.3735$$

$$D(NPV) = 5{,}142.2807 \approx 5{,}142.28$$

$$CV(NPV) = 0.5610209.$$

The Chebychev's interval for NPV, with the same probability as the previous ones using this distribution, turns out to be (166.943; 18,164.923).

Table 2 summarizes the stochastic characteristics of the NPV.

Table 2. Stochastic characteristics of *NPV*

	Beta	$K\left(2, \frac{m^2+1}{2m^2}\right)$	$K(p, q)$, with maximum variance
$E(NPV)$	7,178	6,846.029	9,165.933
$V(NPV)$	8,902,129	10,145,813	26,443,051.3735
$D(NPV)$	2,983	3,185.25	5,142.28
$CV(NPV)$	0.4155753	0.4652696	0.5610209
Interv. Chebychev	(1,957.75; 12,398.25)	(1,271.8415; 12,420.216)	(166.943 ; 18,164.923)

from which we can extrapolate the following conclusions:

a) the most conservative distribution is $K(p, q)$ of maximum variance, its greater variance and higher Pearson's variation coefficient highlight; and

b) the intermediate behavior of the distribution $K\left(2, \frac{m^2+1}{2m^2}\right)$, its proximity to the beta distribution, and easier parameter estimation makes it an ideal candidate in more complicated real problems. In addition, since it has a variance and one CV greater than the beta distribution, it determines a more prudent interval than that of the beta distribution, even with one $E(NPV)$ minor.

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