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Amortizing loans with random commencement and maturity

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Abstract

Generally speaking, loans contracted in practice are non-random, that is to say, all amounts involved in the transaction are certain as well as their respective maturities. In this paper, a new alternative loan category is introduced, based on the contingencies derived from the survival of both the borrower and a linked person. The main novelty of this paper is that these contingencies affect the amortization in each period since the first and last maturities of instalments are random. Additionally, the different parameters of such random transactions are determined, as well as several measures of profitability for the lender (or cost for the borrower). These transactions can be attractive for both the lender and the borrower, and as such it is likely that they can be implemented in practice.

Keywords:
Loan, Amortization, Profitability, Cost, Random maturity.

JEL classification:
C18, G21, G23.

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Amortización de préstamos **con origen y final aleatorios**

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Resumen
En la práctica, las operaciones de préstamo se contratan en ambiente de certeza, es decir, se pactan las cuantías que intervienen en la operación, así como sus respectivos vencimientos. En este trabajo se propone una modalidad alternativa de préstamo basado en las contingencias que suponen la supervivencia del prestatario y de una persona vinculada a él. La principal novedad que introduce este artículo es que estas contingencias afectan a la amortización en cada período, ya que el primer y el último pago son aleatorios. Además, se determinan los diferentes parámetros que intervienen en tales operaciones aleatorias, así como diferentes medidas de rentabilidad/coste que las mismas suponen para el prestamista/prestatario. Estas operaciones pueden resultar atractivas para ambas partes, prestamista y prestatario, por lo que resulta probable que puedan ser implementadas en la práctica financiera.

Palabras clave:
Préstamos, amortización, rentabilidad, coste, vencimiento aleatorio.

1. Introduction

In traditional loans, the payments made by the borrower to amortize the principal are clearly established from the beginning of the operation (Bodie *et al.*, 2004). Nevertheless, there are some loans in which the maturities of the first and the last periodic instalments are random in that they depend on two contingencies whose respective probability distributions are known. For example, this would be the case of a woman who is cared for by her son, with high costs involved. In order to cover these expenses, they request a loan subject to a double condition: on the one hand, the periodic instalments will commence once the woman dies (when the expenses, consequently, come to an end) and, on the other hand, will finish with the death of the son.

This novel loan is independent of the principal repayment method which is traditional within the loan amortization models (French method, constant principal repayment method, American method, etc.) (De Pablo, 2000; Ferruz, 1994; Van Horne, 1997; Brealey and Myers, 2002; Brealey *et al.*, 2006; Ayres, 1963). Moreover, this method can be combined with other financial characteristics, such as the existence of interest-only periods and fixed or variable interest rates (Cruz and Valls, 2014).

In a previous study, Valls and Cruz (2015) analysed some loans where the maturity of the first payment is random and the maturity of the last one is certain, as well as loan transactions where the maturity of the first payment is certain and the maturity of the last one is random. While both kinds of loans are a novelty from the financial point of view, in practice we can find a number of similar transactions. Thus, we can cite the so-called inverse mortgage[1] as an example of the first kind of loan, and as an example of the second kind of loan transaction, amortization insurance (Biehler, 2008), which covers the death or permanent disability of the borrower[2]. However, there is no known current transaction which is comparable to the third kind of loan transaction presented in this paper.

This paper is organized as follows. After this brief introduction, Section 2 presents all the characteristics of this novel financial operation. Section 3 analyses the three alternative expressions to determine the outstanding principal of this loan. Section 4 introduces the noteworthy concepts of saving and risk quotas which will be essential for defining the main measures of cost, profitability and duration derived from these amortizing loans. Finally, Section 5 summarizes and concludes.

[1] In this case, the heirs would have to repay the principal when the borrower dies in a single payment, or else have to contract a new loan. Our proposal consists in replacing this payment with a loan in which the lender assumes the risk of a lower payment by the heirs, depending on the date of death of the borrower.

[2] In this case, the risk is assumed by the insurance company and so there are two different financial transactions: loan and insurance. As before, the proposed loan is a single operation whose risk is assumed by the lender.

2. Amortizing a loan with random commencement and maturity

Consider a loan in which the borrower receives principal C_0 at instant 0 to be repaid by means of n periodic instalments a_s, maturing respectively at instant $s\,(s=1,2,\ldots n)$. If making the first and last payments is subject to different contingencies, the borrower would have to make a set of payments a_s whose amount is greater than that corresponding to the equivalent operation not affected by this source of uncertainty. Under these hypotheses, we propose the following equation of financial random equivalence at instant 0, by using exponential discounting with variable discount rate (i_h) according to the corresponding periods of interest:

$$C_0=\sum_{s=1}^{n}a_s\cdot\prod_{h=1}^{s}(1+i_h)^{-1}\cdot(1+r''_h)^{-1}, \qquad (1)$$

where r''_h represents the variable risk rate affecting the payments. As indicated, the presence of these risks implies a higher burden.

When the contingency affecting the commencement of payments is the death of a person linked to the borrower (hereinafter, the person), the risk is that the person lives and, therefore, the start of the financial transaction is delayed. In addition, we assume that the contingency which determines the last payment is the death of the borrower. In such a case, the risk is that the borrower dies as this means the premature end of the operation and that the payments finish. Obviously, delaying the commencement and shortening the maturity works in the borrower's favour and against the lender.

Thus, by considering the survival (or death) of the borrower (or the person), and under the premise that they are independent random variables, we have (see Valls and Cruz, 2013):

$$1+r''_h=\frac{1}{{}_1p_{h-1}\cdot{}_1q'_{h-1}}, \qquad (2)$$

with ${}_1p_{h-1}$ being the probability that the borrower, aged $h-1$, reaches age h, and ${}_1q'_{h-1}$ the probability that the person, aged $h-1$, dies before reaching age h:

$${}_1q'_{h-1}=1-{}_1p'_{h-1}. \qquad (3)$$

Thus,

$$(1+r''_h)^{-1}={}_1p_{h-1}\cdot(1-{}_1p'_{h-1})={}_1p_{h-1}-{}_1p_{h-1}\cdot{}_1p'_{h-1}. \qquad (4)$$

Note that the borrower and the corresponding person do not have to be of the same biological age and that the subscript h enumerates the different periods of loan payments. Thus, for instance, if the borrower was 50 and the linked person was 70 at the beginning of the loan transaction (instant 0), then ${}_1p_9$ is the probability that the borrower, aged 59, reaches the age of 60 and ${}_1q'_9$ is the probability that the person linked to the borrower, aged 79, dies before turning 80. For the sake of simplicity, we have removed the current ages of both the borrower and the person.

By considering (1) and (4), simple algebra and the relationships:

$$\prod_{h=1}^{s} {}_1p_{h-1} = p_s \text{ and } \prod_{h=1}^{s} ({}_1p_{h-1} \cdot {}_1p'_{h-1}) = p_s \cdot p'_s$$

show that:

$$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1}, \quad (5)$$

with p_s being the probability of survival at instant s. That is to say, in this novel loan the periodic instalments to repay C_0 can be obtained as the difference between two ordinary annuities: the first whose commencement is uncertain and the second whose end is uncertain. We can see that if p_s is the probability of delaying the payment at time s then

$$p_s \cdot p'_s = p''_s \quad (6)$$

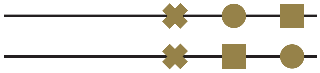
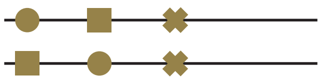
is the probability of making the periodic payment at the end of period s. This statement can be also found in Gil Peláez and Gil Luezas (1987) and Gil Peláez (1993).

3. Outstanding principal of a loan with random commencement and maturity

The outstanding principal at an intermediate instant k of the loan where the maturities of the first and the last instalment are random will depend on whether the periodic payments have commenced at instant k and whether they have finished at that instant. Thus, we can consider the following four cases (see Table 1):

I. Instant k occurs before the death of the person linked to the borrower and before the death of the borrower; its probability of occurrence is $p_k \cdot p'_k$.
II. Instant k occurs before the death of the person and after the death of the borrower, in which case its probability of occurrence is $(1-p_k) \cdot p'_k$.
III. Instant k occurs after the death of the person, but before the death of the borrower, in which case its probability of occurrence is $p_k \cdot (1-p'_k)$.
IV. Instant k occurs after the death of the person and after the death of the borrower, in which case its probability of occurrence is $(1-p_k) \cdot (1-p'_k)$.

Table 1. Probabilities according to instant *k*

Case	Graphic representation	Probability
I		$p_k \cdot p'_k$
II		$(1-p_k)\cdot p'_k$
III		$p_k \cdot (1-p'_k)$
IV		$(1-p_k)\cdot(1-p'_k)$

■ *Instant of borrower's death.*
● *Instant of death of the person linked to the borrower.*
✖ *Calculation instant of the outstanding principal.*

Thus, the amount the borrower has to pay to cancel the operation at an intermediate instant can be calculated using three different methods (Dhaene *et al.*, 2012):

- **Prospective method** (based on the future periodic payments pending, that is, from instant k to the last maturity n):

  In cases II and IV, the death of the borrow means that periodic payments have finished, and so the outstanding principal will be zero in both cases.

  In case I, the outstanding principal, C'_k, as determined by the prospective method is:

$$C'_k = \sum_{s=k+1}^{n} a_s \cdot \prod_{h=k+1}^{s} (1+i_h)^{-1} \cdot (1+r''_h)^{-1}, \tag{7}$$

  from where simple algebra shows that:

$$C'_k = \frac{1}{p_k} \cdot \sum_{s=k+1}^{n} a_s \cdot p_s \cdot \left(1-\frac{p'_s}{p'_k}\right) \cdot \prod_{h=k+1}^{s} (1+i_h)^{-1} \quad . \tag{8}$$

  In case III, the death of the person means that periodic payments have started and so the risk is the death of the borrower, that is to say:

$$(1+r''_h)^{-1} = {}_1p_{h-1},$$

  so that the outstanding principal, C''_k, as determined by the prospective method is:

$$C''_k = \frac{1}{p_k} \cdot \sum_{s=k+1}^{n} a_s \cdot p_s \cdot \prod_{h=k+1}^{s} (1+i_h)^{-1} \quad . \tag{9}$$

Nevertheless, the situation at instant *k* will evidently and *a priori* be uncertain, so it is only the average value of the outstanding principal that can be determined. Bearing in mind that, as indicated, for cases II and IV the outstanding principal is zero, the average expected value of the outstanding principal is then:

$$C_k = C'_k \cdot p_k \cdot p'_k + C''_k \cdot p_k \cdot (1-p'_k)\,. \tag{10}$$

Definitively, by considering (8), (9) and (10), the average value of the outstanding principal by the prospective method is:

$$C_k = p'_k \cdot \sum_{s=k+1}^{n} a_s \cdot p_s \cdot \left(1-\frac{p'_s}{p'_k}\right) \prod_{h=k+1}^{s}(1+i_h)^{-1} + (1-p'_k)\sum_{s=k+1}^{n} a_s \cdot p_s \prod_{h=k+1}^{s}(1+i_h)^{-1}\,. \tag{11}$$

- **Retrospective method** (based on the periodic payments made by the borrower from the loan commencement to instant *k*):

In cases II and IV, the death of the borrower means that periodic payments have finished and the outstanding principal is zero in both cases.

In case I, the periodic payments have not started yet. As the survival of the person means that loan commencement can be delayed then the outstanding principal, C'_k, as determined by the retrospective method is:

$$C'_k = \frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h)\,, \tag{12}$$

and its probability of occurrence is $p_k \cdot p'_k$ (see Table 1).

In case III, the death of the person means that periodic payments have started and the outstanding principal is conditional on the survival of the borrower at that instant. It is possible that:

1. The periodic payments started in the first period, meaning that the outstanding principal, denoted by $C''_{k,1}$, is equal to:

$$C''_{k,1} = \frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h) - \sum_{s=1}^{k-1} a_s \cdot p_s \cdot \prod_{h=s+1}^{k}(1+i_h) - a_k \cdot p_k\,, \tag{13}$$

and its probability of occurrence is ${}_1f'_0 = p'_0 - p'_1 = 1 - p'_1$.

2. The periodic payments started in the second period, meaning that the outstanding principal, denoted by $C''_{k,2}$, is equal to:

$$C''_{k,2} = \frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h) - \sum_{s=2}^{k-1} a_s \cdot p_s \cdot \prod_{h=s+1}^{k}(1+i_h) - a_k \cdot p_k\,, \tag{14}$$

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and its probability of occurrence is ${}_1f'_1 = p'_1 - p'_2$.

⋮

k. The periodic payments started in the k-th period, meaning that the outstanding principal, denoted by $C'''_{k,k}$, is:

$$C'''_{k,k} = \frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h) - a_k \cdot p_k \,, \tag{15}$$

and its probability of occurrence is ${}_1f'_{k-1} = p'_{k-1} - p'_k$.

By considering expressions (13) to (15), we have, succesively:

$$\begin{aligned} &C'''_k \cdot (1-p'_k) = \\ &= \sum_{j=1}^{k-1} \left[\frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h) - \sum_{s=j}^{k-1} a_s \cdot p_s \cdot \prod_{h=s+1}^{k}(1+i_h) - a_k \cdot p_k \right] \cdot (p'_{j-1} - p'_j) \; + \\ &+ \left[\frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h) - a_k \cdot p_k \right] \cdot (p'_{k-1} - p'_k) . \end{aligned} \tag{16}$$

Therefore, the average value of the outstanding principal as determined by the prospective method will be:

$$C_k = C'_k \cdot p_k \cdot p'_k + C'''_k \cdot p_k \cdot (1-p'_k), \tag{17}$$

where C'_k and $C'''_k \cdot (1-p'_k)$ can be displayed by using (12) and (16). Finally, taking into account that:

$$\sum_{j=1}^{k-1} (p'_{j-1} - p'_j) = 1 - p'_{k-1} \,,$$

and simplifying, we have:

$$C_k = C_0 \cdot \prod_{h=1}^{k}(1+i_h) - \sum_{s=1}^{k-1} a_s \cdot p_s \cdot \prod_{h=s+1}^{k}(1+i_h) \cdot (1-p'_s) - a_k \cdot p_k \cdot (1-p'_k). \tag{18}$$

An easy calculation shows that expressions (11) and (18) lead to the same result of the outstanding principal.

- **Recursive method** (based on the outstanding principal calculated at a former maturity):

On the one hand, if at instant k the payments have not started yet, then $a_s = 0$ for $s = 1,2,\ldots,k$. This is because the person is alive, and the probability of this occurring is p'_k . In that case, possible scenarios include:

• The payments cannot start because the borrower has died and the outstanding principal is zero.

• The payments can start in the future because the borrower is still alive, and the outstanding principal is:

$$C'_k = C'_{k-1} \cdot (1+i_k) \cdot \frac{1}{p_k} . \tag{19}$$

On the other hand, if the payments have already started in the k-th period or a previous period, this means the person has died, and the probability of this occurring is $1-p'_k$. Here, we can consider the following subcases:

• The death of the borrower means the payments have finished and the outstanding principal is zero.

• The payments continue because the borrower is still alive, and the outstanding principal is:

$$C''_k = C_{k-1} \cdot (1+i_k) \cdot \frac{1}{p_k} - a_k . \tag{20}$$

Accordingly, and supposing that the borrower is alive with a probability p_k, by considering (19) and (20), the average expected outstanding principal as determined by the recursive method is:

$$C_k = \left[C_{k-1} \cdot (1+i_k) \cdot \frac{1}{p_k} \cdot p'_k + \left(C_{k-1} \cdot (1+i_k) \cdot \frac{1}{p_k} - a_k \right) \cdot (1-p'_k) \right] \cdot p_k , \tag{21}$$

from where, using suitable algebraic operations and simplifying, we have:

$$C_k = C_{k-1} \cdot (1+i_k) - a_k \cdot p_k \cdot (1-p'_k) . \tag{22}$$

4. Main parameters of a loan with random commencement and maturity

As indicated in Section 2, if the amortization is subject to the aforementioned double contingency, the borrower will have to make the payments a_s involved in the equation of financial equivalence (1). However, if the repayment is not affected by such eventualities, the new instalments, denoted by a''_s, would verify the following equation of financial equivalence at the beginning of the operation:

$$C_0 = \sum_{s=1}^{n} a''_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} . \tag{23}$$

By comparing equations (1) and (23), it can obviously be deduced that $a''_s < a_s$. Evidently, the surplus between the two payments is exclusively due to the risk, and so this difference, denoted by a'_s, will be called the *risk quota*. Therefore,

$$a'_s := a_s - a''_s. \tag{24}$$

Thus, starting from (22) and taking into account that every periodic instalment is the sum of the risk quota and the saving quota, given by:

$$a''_s = C_{s-1} \cdot (1+i_s) - C_s\ , \tag{25}$$

we have:

$$C_s = C_{s-1} \cdot (1+i_s) - \left[a'_s + C_{s-1} \cdot (1+i_s) - C_s\right] \cdot p_s \cdot (1-p'_s), \tag{26}$$

from where, simple algebraic operations show that the additional amount that the borrower has to pay to the lender in each period for the assumed risk is:

$$a'_s = \left[C_{s-1} \cdot (1+i_s) - C_s\right] \cdot \frac{1 - p_s \cdot (1-p'_s)}{p_s \cdot (1-p'_s)}\ . \tag{27}$$

Observe that a''_s is the payment corresponding to a riskless loan and, consequently, it is put aside to pay the interest of period s (I_s) and the repayment (A_s) of a part of the principal:

$$a''_s = I_s + A_s = a_s - a'_s\ . \tag{28}$$

These random operations can be agreed with constant or variable interest rates. Moreover, the amount of the periodic payment (a_s) can be set and then the problem is to determine the principal (C_0) and, reciprocally, given C_0, we can determine the periodic payments a_s. Alternatively, given C_0 and having determined the saving quota (a''_s) in the context of a certain traditional loan, the amount of the risk quota (a'_s) corresponding to each period can be determined. So, even in the case in which a''_s is constant, this procedure leads to a variable payment a_s.

EXAMPLE 1. Assume that, in 2013, a 55-year-old man contracts a loan to care for his 80-year-old father. He agrees with a financial entity that the periodic payments will start once his father (the person) dies and will finish with the death of the borrower. It is also assumed that the principal of the loan is €60,000, the periodic instalments will be constant and that the operation has been agreed at an annual constant rate of 7%. In order to determine the risk, the financial entity considers the probability of survival corresponding to individuals of the same age and gender (Tables PERM/F-2000P). The different characteristic parameters of the loan can be observed in Table 2.

Table 2. **Amortization schedule (Example 1)**

Year	i	a_s	p_s	p'_s	a'_s	a''_s	I_s	A_s	C_s	M_s
0	–	–	1.000	1.000	–	–	–	–	60,000.00	–
1	0.07	6,852.83	0.955	0.553	3,927.71	2,925.12	4,200.00	−1,274.88	61,274.88	−1,274.88
2	0.07	6,852.83	0.952	0.512	3.671.11	3,181.72	4,289.24	−1,107.52	62,382.40	−2,382.40
3	0.07	6,852.83	0.949	0.471	3,412.11	3,440.72	4,366.77	−926.05	63,308.45	−3,308.45
4	0.07	6,852.83	0.945	0.434	3,189.38	3,663.45	4,431.59	−768.14	64,076.59	−4,076.59
5	0.07	6,852.83	0.943	0.403	2,995.98	3,856.86	4,485.36	−628.50	64,705.09	−4,705.09
6	0.07	6,852.83	0.936	0.354	2,710.81	4,142.02	4,529.36	−387.34	65,092.43	−5,092.43
7	0.07	6,852.83	0.931	0.298	2,373.34	4,479.49	4,556.47	−76.98	65,169.41	−5,169.41
8	0.07	6,852.83	0.927	0.253	2,110.30	4,742.53	4,561.86	180.67	64,988.74	−4,988.74
9	0.07	6,852.83	0.922	0.197	1,781.31	5,071.52	4,549.21	522.31	64,466.43	−4,466.43
10	0.07	6,852.83	0.915	0.131	1,406.65	5,446.19	4,512.65	933.54	63,532.89	−3,532.89
11	0.07	6,852.83	0.906	0.061	1,021.47	5,831.36	4,447.30	1,384.06	62,148.84	−2,148.84
12	0.07	6,852.83	0.900	0.000	682.72	6,170.12	4,350.42	1,819.70	60,329.14	−329.14
13	0.07	6,852.83	0.892	0.000	738.20	6,114.64	4,223.04	1,891.60	58,437.54	1,562.46
14	0.07	6,852.83	0.883	0.000	800.91	6,051.92	4,090.63	1,961.29	56,476.25	3,523.75
15	0.07	6,852.83	0.875	0.000	858.77	5,994.06	3,953.34	2,040.72	54,435.53	5,564.47
16	0.07	6,852.83	0.867	0.000	913.13	5,939.70	3,810.49	2,129.22	52,306.31	7,693.69
17	0.07	6,852.83	0.856	0.000	986.75	5,866.08	3,661.44	2,204.64	50,101.67	9,898.33
18	0.07	6,852.83	0.842	0.000	1,084.68	5,768.15	3,507.12	2,261.04	47,840.64	12,159.36
19	0.07	6,852.83	0.827	0.000	1,187.60	5,665.23	3,348.84	2,316.39	45,524.25	14,475,75
20	0.07	6,852.83	0.810	0.000	1,300.99	5,551.84	3,186.70	2,365.14	43,159.11	16,840.89
21	0.07	6,852.83	0.793	0.000	1,418.42	5,434.41	3,021.14	2,413.28	40,745.83	19,254.17
22	0.07	6,852.83	0.776	0.000	1,533.98	5,318.85	2,852.21	2,466.64	38,279.19	21,720.81
23	0.07	6,852.83	0.758	0.000	1,656.00	5,196.84	2,679.54	2,517.29	35,761.90	24,238.10
24	0.07	6,852.83	0.739	0.000	1,791.31	5,061.53	2,503.33	2,558.19	33,203.70	26,796.30
25	0.07	6,852.83	0.716	0.000	1,949.34	4,903.50	2,324.26	2,579.24	30,624.47	29,375.53
26	0.07	6,852.83	0.693	0.000	2,106.20	4,746.64	2,143.71	2,602.92	28,021.54	31,978.46
27	0.07	6,852.83	0.665	0.000	2,296.59	4,556.24	1,961.51	2,594.73	25,426.81	34,573.19
28	0.07	6,852.83	0.636	0.000	2,492.31	4,360.53	1,779.88	2,580.65	22,846.16	37,153.84
29	0.07	6,852.83	0.611	0.000	2,663.97	4,188.86	1,599.23	2,589.63	20,256.53	39,743.47
30	0.07	6,852.83	0.590	0.000	2,812.22	4,040.61	1,417.96	2,622.66	17,633.87	42,366.13
31	0.07	6,852.83	0.556	0.000	3,040.43	3,812.40	1,234.37	2,578.03	15,055.84	44,944.16
32	0.07	6,852.83	0.518	0.000	3,306.21	3,546.62	1,053.91	2,492.72	12,563.13	47,436.87
33	0.07	6,852.83	0.487	0.000	3,516.96	3,335.87	879.42	2,456.45	10,106.67	49,893.33
34	0.07	6,852.83	0.448	0.000	3,780.09	3,072.74	707.47	2,365.28	7,741.40	52,258.60
35	0.07	6,852.83	0.403	0.000	4,091.77	2,761.07	541.90	2,219.17	5,522.23	54,477.77
36	0.07	6,852.83	0.355	0.000	4,422.07	2,430.76	386.56	2,044.21	3,478.02	56,521.98
37	0.07	6,852.83	0.306	0.000	4,756.94	2,095.89	243.46	1,852.43	1,625.60	58,374.40
38	0.07	6,852.83	0.193	0.000	5,530.44	1,322.39	113.79	1,208.60	417,00	59,583.00
39	0.07	6,852.83	0.065	0.000	6,406.64	446,19	29.19	417,00	0.00	60,000.00
TOTAL		267,260.49	–	–	96,725.84	170,534.65	110,534.65	60,000.00	–	–

Assume that instalments a_s are intended for the traditional repayment of C_0 and that:

- n denotes the number of agreed years,
- $\ddot{n}$ is the number of years during which the periodic instalments are zero, and
- n' is the actual number of periodic payments, with $n' \le n - \ddot{n}$.

Observe that n' could be zero if payments do not commence because the borrower dies before the person, or even after the person but before the payment of the first instalment.

In this case, the profitability would be higher for the lender since the operation is now certain (non-random), as long as:

$$C_0 \cdot \prod_{h=1}^{\ddot{n}} (1+i_h) < \sum_{s=\ddot{n}+1}^{\ddot{n}+n'} a_s \cdot \prod_{h=\ddot{n}+1}^{s} (1+i_h)^{-1} . \tag{29}$$

In the particular case of a fixed instalment and a constant interest rate, then inequality (29) results in:

$$C_0 \cdot (1+i)^{\ddot{n}} < a \cdot a_{\overline{n'}|\, i} . \tag{30}$$

In Example 1, if $\ddot{n} = 3$ years, then:

$$60{,}000 \cdot 1.07^3 < 6{,}852.83 \cdot a_{\overline{n'}|\, 0.07} ,$$

where $n' = 20.537$. Therefore, in the 24th year after loan commencement, the profit obtained by the lender would be higher, which implies that the borrower should live to at least 78 years old.

In this particular case, if the periodic payments commence later than the 5th year, then neither the principal nor the 7% interest would be recovered. Therefore, no later than the 5th year, the borrower should start making periodic payments if he is to fully amortize the principal by the 35th year, with the agreed interest rate.

However, if the borrower lives to be 88 years old, all extra payments from then on would be additional earnings.

Observe that, if the periodic payments commence in the 5th year, the outstanding principal would increase to €84,153.10 and it would not fall below €60,000.00 until the 20th year of the loan, when the borrower is 74; the principal will then be amortized over the following 15 years.

Moreover, if from the loan commencement onwards the entire amounts were put towards the traditional amortization of the principal (that is to say, without considering the risk quotas), the principal would be repaid in 15 years. Thus, at the end of the 14th year there would only be €177.32 left to amortize, so that $C_{15}= -€6{,}663.10$.

In this case, the lender would achieve a greater profit which could offset the losses generated by other transactions that end with the early death of the borrower. These earnings alone make the implementation of these transactions in practice more attractive (Brigham and Daves, 2007). Finally, these loans can be also considered with a variable interest rate.

EXAMPLE 2. Consider the same data as Example 1, but assuming that the applicable annual interest rate will be 7% for the first five years, with a 0.2% increase every five years. The different parameters of the loan can be observed in Table 3.

In this kind of random loan, the following average interest rates can be defined:

1. Average interest rate: if the loan is agreed with a variable interest rate, it is interesting to know the average interest rate, i_m, resulting from the contract. By definition, this average is the rate that, applied to all periods, enables the financial equivalence between the principal and the periodic payments which amortize it. Thus, considering equation (5) and given the rest of parameters, the average interest rate can be deduced from the following equation:

$$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot (1+i_m)^{-s} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot (1+i_m)^{-s}. \quad (31)$$

In Example 2, if

$$60{,}000.00 = \sum_{s=1}^{39} 7{,}071.86 \cdot \ p_s \cdot (1+i_m)^{-s} - \sum_{s=1}^{n} 7{,}071.86 \cdot p_s \cdot p'_s \cdot (1+i_m)^{-s},$$

the average interest rate of the loan transaction is 7.2669%.

2. Net true interest rate: once the maturity of the first instalment, n_0 , and the last one, n_f , are known, the exact amounts of the payments can be determined, and the net true interest rate, $\bar{i}_{n_0,n_f}$, can be calculated by solving the following equation:

$$\sum_{s=n_0}^{n_f} a_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} = \sum_{s=n_0}^{n_f} a_s \cdot (1+\bar{i}_{n_0,n_f})^{-s}. \quad (32)$$

The result of the transaction (profit or loss), measured in monetary units corresponding to the first instant, for maturities of instalments between n_0 and n_f , denoted by R_{0,n_0,n_f}, is given by the following difference:

$$R_{0,n_0,n_f} = \sum_{s=n_0}^{n_f} a_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} - C_0. \quad (33)$$

Table 3. Amortization schedule (Example 2)

Year	i	a_s	p_s	p'_s	a'_s	a''_s	I_s	A_s	C_s	M_s
0	–	–	1.000	1.000	–	–	–	–	60,000.00	–
1	0.070	7,071.86	0.955	0.553	4,053.25	3,018.62	4,200.00	−1,181.38	61,181.38	−1,181.38
2	0.070	7,071.86	0.952	0.512	3,788.45	3,283.41	4,282.70	−999.28	62,180.67	−2,180.67
3	0.070	7,071.86	0.949	0.471	3,521.17	3,550.69	4,352.65	−801.96	62,982.62	−2,982.62
4	0.070	7,071.86	0.945	0.434	3,291.32	3,780.54	4,408.78	−628.24	63,610.87	−3,610.87
5	0.070	7,071.86	0.943	0.403	3,091.73	3,980.13	4,452.76	−472.63	64,083.50	−4,083.50
6	0.072	7,071.86	0.936	0.354	2,797.46	4,274.40	4,614.01	−339.61	64,423.11	−4,423.11
7	0.072	7,071.86	0.931	0.298	2,449.20	4,622.66	4,638.46	−15.80	64,438.91	−4,438.91
8	0.072	7,071.86	0.927	0.253	2,177.75	4,894.11	4,639.60	254.51	64,184.40	−4,184.40
9	0.072	7,071.86	0.922	0.197	1,838.24	5,233.62	4,621.28	612.34	63,572.06	−3,572.06
10	0.072	7,071.86	0.915	0.131	1,451.60	5,620.26	4,577.19	1,043.07	62,528.99	−2,528.99
11	0.074	7,071.86	0.906	0.061	1,054.12	6,017.74	4,627.14	1,390.60	61,138.39	−1,138.39
12	0.074	7,071.86	0.900	0.000	704.54	6,367.33	4,524.24	1,843.08	59,295.31	704.69
13	0.074	7,071.86	0.892	0.000	761.79	6,310.07	4,387.85	1,922.22	57,373.09	2,626.91
14	0.074	7,071.86	0.883	0.000	826.51	6,245.35	4,245.61	1,999.74	55,373.34	4,626.66
15	0.074	7,071.86	0.875	0.000	886.22	6,185.64	4,097.63	2,088.01	53,285.33	6,714.67
16	0.076	7,071.86	0.867	0.000	942.32	6,129.55	4,049.69	2,079.86	51,205.47	8,794.53
17	0.076	7,071.86	0.856	0.000	1,018.29	6,053.57	3,891.62	2,161.96	49,043.51	10,956.49
18	0.076	7,071.86	0.842	0.000	1,119.35	5,952.51	3,727.31	2,225.21	46,818.31	13,181.69
19	0.076	7,071.86	0.827	0.000	1,225.56	5,846.30	3,558.19	2,288.11	44,530.20	15,469.80
20	0.076	7,071.86	0.810	0.000	1,342.58	5,729.29	3,384.29	2,344.99	42,185.20	17,814.80
21	0.078	7,071.86	0.793	0.000	1,463.76	5,608.11	3,290.45	2,317.66	39,867.54	20,132.46
22	0.078	7,071.86	0.776	0.000	1,583.01	5,488.85	3,109.67	2,379.18	37,488.36	22,511.64
23	0.078	7,071.86	0.758	0.000	1,708.93	5,362.94	2,924.09	2,438.84	35,049.52	24,950.48
24	0.078	7,071.86	0.739	0.000	1,848.56	5,223.30	2,733.86	2,489.44	32,560.08	27,439.92
25	0.078	7,071.86	0.716	0.000	2,011.64	5,060.22	2,539.69	2,520.54	30,039.54	29,960.46
26	0.080	7,071.86	0.693	0.000	2,173.51	4,898.35	2,403.16	2,495.18	27,544.36	32,455.64
27	0.080	7,071.86	0.665	0.000	2,370.00	4,701.87	2,203.55	2,498.32	25,046.04	34,953.96
28	0.080	7,071.86	0.636	0.000	2,571.96	4,499.90	2,003.68	2,496.21	22,549.82	37,450.18
29	0.080	7,071.86	0.611	0.000	2,749.11	4,322.75	1,803.99	2,518.76	20,031.06	39,968.94
30	0.080	7,071.86	0.590	0.000	2,902.10	4,169.76	1,602.48	2,567.27	17,463.79	42,536.21
31	0.082	7,071.86	0.556	0.000	3,137.61	3,934.25	1,432.03	2,502.22	14,961.57	45,038.43
32	0.082	7,071.86	0.518	0.000	3,411.88	3,659.98	1,226.85	2,433.13	12,528.44	47,471.56
33	0.082	7,071.86	0.487	0.000	3,629.37	3,442.49	1,027.33	2,415.16	10,113.27	49,886.73
34	0.082	7,071.86	0.448	0.000	3,900.91	3,170.95	829.29	2,341.66	7,771.61	52,228.39
35	0.082	7,071.86	0.403	0.000	4,222.55	2,849.32	637.27	2,212.04	5,559.57	54,440.43
36	0.084	7,071.86	0.355	0.000	4,563.41	2,508.45	467.00	2,041.45	3,518.12	56,481.88
37	0.084	7,071.86	0.306	0.000	4,908.99	2,162.88	295.52	1,867.36	1,650.76	58,349.24
38	0.084	7,071.86	0.193	0.000	5,707.21	1,364.65	138.66	1,225.99	424.77	59,575.23
39	0.084	7,071.86	0.065	0.000	6,611.41	460.45	35.68	424.77	0.00	60,000.00
TOTAL		275,802.63	–	–	99,817.38	175,985.26	115,985.26	60,000.00	–	–

By considering (32) and (33), we have

$$C_0 + R_{0,n_0 n_f} = \sum_{s=n_0}^{n_f} a_s \cdot (1+\bar{i}_{n_0 n_f})^{-1}, \qquad (34)$$

and for this reason $\bar{i}_{n_0 n_f}$ is labelled "net".

This rate is variable according to the true commencement and end of the operation. In effect, at the beginning of the loan one can only determine its average expected value, taking into account all alternatives with their corresponding probabilities (see Table 4).

Table 4. Alternatives for determining the net true interest rate

Payments		Net true interest rate	
Finish	Start	Notation	Probability
Before instant 0	Never	$\bar{i}_{0,0}$	$p'_0 \cdot (p_0 - p_1)$
Period 1	Never	$\bar{i}_{0,1}$	$p'_1 \cdot (p_1 - p_2)$.
	Period 1	$\bar{i}_{1,1}$	$(p'_0 - p'_1) \cdot (p_1 - p_2)$
Period 2	Never	$\bar{i}_{0,2}$	$p'_2 \cdot (p_2 - p_3)$.
	Period 1	$\bar{i}_{1,2}$	$(p'_0 - p'_1) \cdot (p_2 - p_3)$
	Period 2	$\bar{i}_{2,2}$	$(p'_1 - p'_2) \cdot (p_2 - p_3)$
⋮	⋮	⋮	⋮

Thus, simple algebra shows that:

$$\bar{i} = E\left[\bar{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n} \left(\sum_{n_0=1}^{n_f} \bar{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \bar{i}_{0,n_f} \cdot p'_{n_f} \right) \cdot {}_1f_{n_f} \,. \tag{35}$$

3. Gross true average interest rate: once the financial transaction has finished and the last maturity is thus known, the gross true average interest rate, denoted by $\hat{i}_{n_0,n_f}$, is the parameter which allows us to write the financial equivalence between the loan principal and the periodic instalments:

$$C_0 = \sum_{s=n_0}^{n_f} a_s \cdot (1+\hat{i}_{n_0,n_f})^{-s}, \tag{36}$$

with $n_f - n_0$ being the real number of periodic payments. This rate is variable according to n_0 and n_f and, *a priori*, that is to say, at the loan commencement, only its mathematical expectation can be calculated:

$$\hat{i} = E\left[\hat{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n} \left(\sum_{n_0=1}^{n_f} \hat{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \hat{i}_{0,n_f} \cdot p'_{n_f} \right) \cdot {}_1f_{n_f} \,. \tag{37}$$

Observe that $\hat{i}_{n_0,n_f}$ can take negative values if the loan finishes before the amortization of the principal, that is to say:

$$C_0 > \sum_{s=n_0}^{n_f} a_s \,, \tag{38}$$

where the most extreme case is if the loan expires before the start of payments; in such a case the lender does not get any money back, and so $\hat{i}_{0,n_f} = -1$. The probability of this situation occurring is $p'_{n_f} \cdot (p_{n_f} - p_{n_f+1})$. Reciprocally, maximum profitability is achieved when $n_0 = 1$ and $n_f = n$; the probability of this scenario occurring is $(p'_0 - p'_1) \cdot p_n$.

4. Average interest rate due to randomness: a relationship between the net and gross average rates can be deduced through the so-called average rate due to randomness, denoted by $\tilde{i}_{n_0,n_f}$, which is defined by the following equation:

$$\tilde{i}_{n_0,n_f} = \frac{\hat{i}_{n_0,n_f} - \bar{i}_{n_0,n_f}}{1 + \bar{i}_{n_0,n_f}} . \tag{39}$$

This rate is variable according to the duration of the financial transaction, so that its *a priori* expected average value can be calculated as follows:

$$\tilde{i} = E\left[\tilde{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n} \left(\sum_{n_0=1}^{n_f} \tilde{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \tilde{i}_{0,n_f} \cdot p'_{n_f} \right) \cdot {}_1f_{n_f} . \tag{40}$$

At the beginning, when the contract has been formalized, the number of payments is uncertain, but its probability distribution is known, and so we can determine its expected value, $\bar{d}$, in the following way:

$$\bar{d} = E\left[n_f - n_0\right] = \sum_{n_f=0}^{n} \left(\sum_{n_0=1}^{n_f} \left(n_f - (n_0 - 1)\right) \cdot {}_1f'_{n_0-1} \right) \cdot {}_1f_{n_f} . \tag{41}$$

Finally, we assign the terms *financial commencement* and *end* of the random loan to the commencement or end of a certain (non-random) transaction, so that its net present value is equal to that of the random transaction, that is to say, the values n_0 and n_f satisfy the following equation:

$$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} = \sum_{s=n_0}^{n_f} a_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} . \tag{42}$$

This equation does not have an (integer) solution for pairs of values (n_0, n_f), but it is obvious that we can find a value:

$$n_f \in \left[\alpha, \alpha + 1\right], \text{ where } \alpha \in N,$$

such that:

$$\sum_{s=n_0}^{\alpha} a_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} < C_0 < \sum_{s=n_0}^{\alpha+1} a_s \cdot \prod_{h=1}^{s} (1+i_h)^{-1} . \tag{43}$$

5. Conclusions

In this paper we have introduced a new category of loan based on the uncertainty of the periodic payments that the borrower has to make to amortize the loan principal. This randomness is associated with the life expectancy of the borrower and a person linked to him/her. Nevertheless, any other risk can also be considered because our approach is general enough to apply to another contingency. In this loan, the maturity of both the first and the last periodic payment are random.

In addition, we have deduced the parameters which allow us to determine the evolution of these transactions and also those which represent a difference from the traditional loan agreed in practice. Thus, we start with the equation of financial equivalence for determining the risk quota (which is the additional amount that the borrower must pay with respect to a traditional loan transaction to compensate the lender for the assumed risk) and the outstanding principal at each instant. Likewise, if these transactions are to be agreed as usual with variable interest rates, it is necessary to calculate several average interest rates as measure of the profitability for the lender (or cost to the borrower). Table 5 shows a summary of the expressions obtained for each of the described parameters.

A possible extension of this paper could be to use the probability of Spanish companies' failure to make debt payments instead of the probability of death of individuals. Take, for example, an investment bank which is willing to fund a company's innovative activities. Both parties may agree that the company will start to repay the principal once it has reached a certain level of profitability. On the other hand, the investment bank may decide to abandon the project if it believes that the investment becomes unfeasible.

Table 5. Summary of the financial expressions

Financial equivalence	$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1}$
C_k prospective method	$C_k = C'_k \cdot p_k \cdot p'_k + C''_k \cdot p_k \cdot (1-p'_k)$ being: $C'_k = \frac{1}{p_k} \cdot \sum_{s=k+1}^{n} a_s \cdot p_s \cdot \left(1-\frac{p'_s}{p'_k}\right) \cdot \prod_{h=k+1}^{s}(1+i_h)^{-1}$ $C''_k = \frac{1}{p_k} \cdot \sum_{s=k+1}^{n} a_s \cdot p_s \cdot \prod_{h=k+1}^{s}(1+i_h)^{-1}$

C_k retrospective method	$C_k = C'_k \cdot p_k \cdot p'_k + C''_k \cdot p_k \cdot (1-p'_k)$ being: $C'_k = \frac{1}{p_k} \cdot C_0 \cdot \prod_{h=1}^{k}(1+i_h)$ $C''_k = C_0 \cdot \prod_{h=1}^{k}(1+i_h) - \sum_{s=1}^{k-1} a_s \cdot p_s \cdot \prod_{h=s+1}^{k}(1+i_h)\cdot(1-p'_s) - a_k \cdot p_k \cdot (1-p'_k)$
C_k recursive method	$C_k = C_{k-1}\cdot(1+i_k) - a_k \cdot p_k \cdot (1-p'_k)$
Risk quota	$a'_s = \left[C_{s-1}\cdot(1+i_s) - C_s\right] \cdot \frac{1-p_s\cdot(1-p'_s)}{p_s\cdot(1-p'_s)}$
Number of years from which the lender will get profit	$C_0 \cdot \prod_{h=1}^{\ddot{n}}(1+i_h) < \sum_{s=\ddot{n}+1}^{\ddot{n}+n'} a_s \cdot \prod_{h=\ddot{n}+1}^{s}(1+i_h)^{-1}$ $C_0 \cdot (1+i)^{\ddot{n}} < a \cdot a_{\overline{n'}\mid i}$, being n': payment years ($n' \leq n-\ddot{n}$)
Average interest rate	$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot (1+i_m)^{-s} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot (1+i_m)^{-s}$
Net true interest rate	$\sum_{s=n_0}^{n_f} a_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1} = \sum_{s=n_0}^{n_f} a_s \cdot (1+\bar{i}_{n_0,n_f})^{-s}$ $\bar{i} = E\left[\bar{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n}\left(\sum_{n_0=1}^{n_f} \bar{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \bar{i}_{0,n_f} \cdot p'_{n_f}\right) \cdot {}_1f_{n_f}$
Gross true interest rate	$C_0 = \sum_{s=n_0}^{n_f} a_s \cdot (1+\hat{i}_{n_0,n_f})^{-s}$ $\hat{i} = E\left[\hat{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n}\left(\sum_{n_0=1}^{n_f} \hat{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \hat{i}_{0,n_f} \cdot p'_{n_f}\right) \cdot {}_1f_{n_f}$
Average interest rate due to randomness	$\tilde{i}_{n_0,n_f} = \frac{\hat{i}_{n_0,n_f} - \bar{i}_{n_0,n_f}}{1+\bar{i}_{n_0,n_f}}$ $\tilde{i} = E\left[\tilde{i}_{n_0,n_f}\right] = \sum_{n_f=0}^{n}\left(\sum_{n_0=1}^{n_f} \tilde{i}_{n_0,n_f} \cdot {}_1f'_{n_0-1} + \tilde{i}_{0,n_f} \cdot p'_{n_f}\right) \cdot {}_1f_{n_f}$
Expected duration	$\bar{d} = E\left[n_f - n_0\right] = \sum_{n_f=0}^{n}\left(\sum_{n_0=1}^{n_f} (n_f-n_0) \cdot {}_1f'_{n_0-1} + n_f \cdot p'_{n_f}\right) \cdot {}_1f_{n_f}$
Financial last/first maturity	$C_0 = \sum_{s=1}^{n} a_s \cdot p_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1} - \sum_{s=1}^{n} a_s \cdot p_s \cdot p'_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1} = \sum_{s=n_0}^{n_f} a_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1}$ $n_f \in \left[\alpha, \alpha+1\right]$, where $\alpha \in N$ $\sum_{s=n_0}^{\alpha} a_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1} < C_0 < \sum_{s=n_0}^{\alpha+1} a_s \cdot \prod_{h=1}^{s}(1+i_h)^{-1}$

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