Reverse generalized Hölder and Minkowski type inequalities and their applications

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In this paper we give a reverse generalization of the generalized Hölder and Minkowski type inequalities and their applications to inverse source problems.

Keywords: Hölder and Minkowski type inequalities, heat equation, Weierstrass transform.

En este artículo damos una generalización de la desigualdad inversa generalizada de Hölder y Minkowski y sus aplicaciones a problemas inversos.

Palabras claves: Desigualdades de tipo Hölder y Minkowski, ecuación del calor, transformada de Weierstrass.

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1 Introduction

It is well known that, for $0 < p < 1, f \in Lp(X), g \in Lq(X)$,

$$\int_X |f\,g|\,d\mu \geq \|f\|_p\,\|g\|_q\,. \tag{1}$$

Since q is negative in this case, we assume that $g > 0$, $\mu-$ a.e. on X. Also, if $f \in Lp(X), g \in Lp(X)$, and $0 < p < 1$, then it follows, see [1], by applying the result of (1), that

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p\,. \tag{2}$$

The following version of inequality (1) was proved in [3]; see also [2] and [4], pages 125–126.

Theorem 1.1. *Suppose $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are two positive function satisfying*

$$0 < m \leq \frac{f^p}{g^q} \leq M < \infty\,,$$

on a set X. Then

$$\left(\int_X f^p\,d\mu\right)^{1/p}\left(\int_X g^q\,d\mu\right)^{1/q} \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_X f\,g\,d\mu\,, \tag{3}$$

if the right hand side integral converges.

Under appropriate conditions, we prove a generalized version of inequalities (2) and (3). Our estimates are based on Theorem 1.1.

2 Main results

Theorem 2.1. *Suppose $p, q, r > 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. If f, g and h are positive functions such that*

i) $0 < m \leq \frac{f^{p/s}}{g^{q/s}} \leq M < \infty$ *for some $s > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$, and*

ii) $0 < m \leq \frac{(fg)^s}{h^r} \leq M < \infty$ *for some $s > 0$, on a set X.*

Then

$$\left(\int_X f^p\,d\mu\right)^{1/p}\left(\int_X g^q\,d\mu\right)^{1/q}\left(\int_X h^r\,d\mu\right)^{1/r}$$
$$\leq \left(\frac{m}{M}\right)^{-\left[\frac{1}{rs}+\frac{s^2}{pq}\right]}\int_X f\,g\,h\,d\mu\,, \tag{4}$$

if the right hand side integral converges.

Proof. Let $\frac{1}{p}+\frac{1}{q}=\frac{1}{s}$ for some $s>0$, thus $\frac{s}{p}+\frac{s}{q}=1$ and $\frac{1}{s}+\frac{1}{r}=1$. Using ii) and applying Theorem 1.1 to $H=fg$ and h we have

$$\left(\int_X H^s\,d\mu\right)^{1/s}\left(\int_X h^r\,d\mu\right)^{1/r}\leq\left(\frac{m}{M}\right)^{-\frac{1}{sr}}\int_X H\,h\,d\mu\,,$$

which is equivalent to

$$\left(\int_X f^s\,g^s\,d\mu\right)^{1/s}\left(\int_X h^r\,d\mu\right)^{1/r}\leq\left(\frac{m}{M}\right)^{-\frac{1}{sr}}\int_X f\,g\,h\,d\mu\,. \tag{5}$$

Now, using once more i) and the fact that $\frac{s}{p}+\frac{s}{q}=1$, we can apply Theorem 1.1 to f^s and g^s to obtain

$$\left(\int_X f^p\,d\mu\right)^{1/p}\left(\int_X g^q\,d\mu\right)^{1/q}\leq\left(\frac{m}{M}\right)^{-\frac{s^2}{pq}}\int_X f^s\,g^s\,d\mu\,. \tag{6}$$

Combining (5) and (6) we obtain (4). Thus we have obtained the result. □

Theorem 2.2. *Suppose $p,q>0$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are two positive functions such that*

i) $0<m\leq\frac{(f+g)^{p-1}}{f}\leq M<\infty$.

ii) $0<m\leq\frac{(f+g)^{p-1}}{g}\leq M<\infty$ *on a set X .*

Then

$$\left(\int_X f^p\,d\mu\right)^{1/p}+\left(\int_X g^p\,d\mu\right)^{1/p}\leq\left(\frac{m}{M}\right)^{-\frac{1}{pq}}\left(\int_X (f+g)^p\,d\mu\right)\,. \tag{7}$$

if the right hand side integral converges.

Proof. Observe that, invoking Theorem 1.1, we have

$$\begin{aligned}
&\left(\frac{m}{M}\right)^{-\frac{1}{pq}}\int_X (f+g)^p\,d\mu\\
=\;&\left(\frac{m}{M}\right)^{-\frac{1}{pq}}\left[\int_X \left(f\,(f+g)^{p-1}+g\,(f+g)^{p-1}\right)\,d\mu\right]\\
\geq\;&\left[\left(\int_X f^p\,d\mu\right)^{1/p}+\left(\int_X g^p\,d\mu\right)^{1/p}\right]\left(\int_X (f+g)^p\,d\mu\right)^{1/q}.
\end{aligned}$$

Thus

$$\left(\int_X f^p\,d\mu\right)^{1/p} + \left(\int_X g^p\,d\mu\right)^{1/p} \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}} \left(\int_X (f+g)^p\,d\mu\right)^{1/p},$$

which is precisely (7). □

Next, without using Theorem 1.1 and with aslight variation of the hypotheses of Theorem 2.2 we have the following.

Theorem 2.3. *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f}{g} < M\,,$$

on a set X. Then

$$\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\,d\mu\right)^{\frac{1}{p}} \leq C\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}},$$

if the right hand side integral converges, where

$$C = \frac{M(m+1)+M+1}{(m+1)(M+1)}\,.$$

Proof. Since $\frac{f}{g} \leq M$, then $f \leq M(f+g) - Mf$, thus

$$(M+1)^p\,f^p \leq M^p\,(f+g)^p\,,$$

and

$$\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} \leq \frac{M}{M+1}\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}}. \tag{8}$$

On the other hand, since $mg \leq f$, we have $g \leq \frac{1}{m}(f+g) - \frac{1}{m}g$. From this, we obtain

$$\left(\frac{1}{m}+1\right)^p g^p \leq \left(\frac{1}{m}\right)^p (f+g)^p\,.$$

Hence,

$$\left(\int_X g^p\,d\mu\right)^{\frac{1}{p}} \leq \frac{1}{m+1}\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}}. \tag{9}$$

Finally, adding (8) and (9) we get

$$\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}$$

$$\leq \quad \frac{M(m+1)+M+1}{(m+1)(M+1)} \left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{p}} .$$

□

Theorem 2.4. *Let F and G be positive functions satisfying*

$$0 < m^{\frac{1}{p}} \leq F(\zeta)\,G(x-\zeta) \leq M^{\frac{1}{p}}\,, \tag{10}$$

with $p > 1$, $x \in [c,d]$ and $\zeta \in \mathbb{R}$. Then, for any positive function ρ, we have

$$\int_c^d \left(\int_{-\infty}^{\infty} F(\zeta)\,\rho(\zeta)\,G(x-\zeta)\,d\zeta\right)^p dx$$

$$\geq \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_{-\infty}^{\infty} \rho(\zeta)\,d\zeta\right)^{p-1} \int_{-\infty}^{\infty} F^p(\zeta)\,\rho(\zeta)\,d\zeta \int_{c-\zeta}^{d-\zeta} G^p(x)\,dx\,. \tag{11}$$

Inequality (11) is especially important when $G(x-\zeta)$ is a Green's function.

The proof of Theorem 2.4 is just a straightforward application of Theorem 2.1. Inequality (4) reverses the sign if $0 < p < 1$. Hence, inequality (11) reverses the sign if $0 < p < 1$. On the other hand, also note that this kind of estimates are important in inverse problems.

3 Application to the heat equation

We consider the Weierstrass transform

$$u(x,\,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} F(\zeta)\,\rho(\zeta)\,e^{-\frac{(x-\zeta)^2}{4t}}\,d\zeta\,,$$

which gives the formal solution $u(x,t)$ of the heat equation $u_t = \Delta u$ on $\mathbb{R}_+ \times \mathbb{R}$, subject to the initial condition $u(x,0) = F(x)\rho(x)$, on $\mathbb{R}$. Take $G(x) = e^{-\frac{x^2}{4t}}$, and let $x \in [-a,a]$, $\zeta \in [-b,b]$, and $a+b \leq \sqrt{\frac{4t}{p}\log\frac{M}{m}}$. From this

$$1 \leq e^{-\frac{(x-\zeta)^2}{4t}} \leq e^{-\frac{(a+b)^2}{4t}}\,,$$

and we obtain

$$0 < m^{\frac{1}{p}} \leq F(\zeta)\,e^{-\frac{(x-\zeta)^2}{4t}} \leq M^{\frac{1}{p}}\,,$$

if $m^{\frac{1}{p}} e^{-\frac{(a+b)^2}{4t}} \leq F(\zeta) \leq M^{\frac{1}{p}}$, $\zeta \in [-b, b]$. It is not difficult to see that

$$\int_{c-\zeta}^{d-\zeta} e^{-\frac{px^2}{4t}}\, dx = \sqrt{\frac{\pi t}{p}} \left[erf\left(\frac{\sqrt{p}(d-\zeta)}{2\sqrt{t}}\right) - erf\left(\frac{\sqrt{p}(c-\zeta)}{2\sqrt{t}}\right)\right],$$

where

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,,$$

is the error function. Therefore, for $-a \leq c < d \leq a$, the inequality (11) holds

$$\begin{aligned} & \int_c^d [u(x,\, t)]^p\, dx \\ \geq\ & \frac{1}{2^p\,(\pi t)^{(p-1)/2}\,\sqrt{p}} \left(\frac{m}{M}\right)^{\frac{1}{pq}} \left(\int_b^b \rho(\zeta)\, d\zeta\right)^{p-1} \\ & \times \int_b^b F^p(\zeta)\,\rho(\zeta) \left[erf\left(\frac{\sqrt{p}(d-\zeta)}{2\sqrt{t}}\right) - erf\left(\frac{\sqrt{p}(c-\zeta)}{2\sqrt{t}}\right)\right] d\zeta\,, \end{aligned}$$

where ρ is a positive continuous function on $[-b, b]$ and F satisfy (10).

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