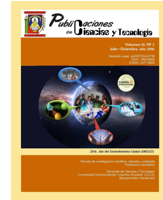
Publicaciones en Ciencias y Tecnología

Publicaciones en Ciencias y Tecnología. Vol. 10 N$^{o}$ 2, Jul-Dic (2016) 49–58



Artículo de investigación original

# Uniformly Bounded Superposition Operators in the Space of Second Bounded Variation Functions in the sense of Shiba

Operador de Superposición Uniformemente Acotado en Espacios de Funciones de Segunda Variación Acotada en el Sentido de Shiba

José Giménez[a], Nelson Merentes[b], Ebner Pineda[c], Luz Rodríguez[c]

[a]*Universidad de los Andes, Departamento de Matemáticas Mérida, Venezuela*
[b]*Universidad Central de Venezuela, Escuela de Matemáticas, Caracas, Venezuela*
[c]*Universidad Centroccidental Lisandro Alvarado, Departamento de Matemáticas, Barquisimeto, Venezuela*

Recibido: 07-04-2016 Aceptado: 18-01-2017

## Abstract

In this paper we introduce the notion of "function of second bounded variation" in the sense of Shiba and we show that if a superposition operator applies the space of all such functions on itself and it is uniformly bounded, then its generating function satisfies a Matkowski condition.

***Palabras clave:*** Bounded variation, superposition operator, Waterman-Shiba
*UNESCO Code: 1202.01*

## Resumen

En este trabajo introducimos la noción de función de segunda variación acotada en el sentido de Shiba y mostramos que si un operador de superposición aplica el espacio de todas estas funciones en sí mismo y es uniformemente acotado, entonces su función generadora satisface una condición de Matkowski.

***Keywords:*** Variación acotada, operador de superposición, Waterman-Shiba.
*Código UNESCO: 1202.01*

## 1. Introduction

The notion of function of bounded variation, or $BV$ function, was introduced by C. Jordan in 1881 [1], when he critically re-examined a faulty proof given by Dirichlet to the famous Fourier's conjecture on trigonometric series expansion of periodic functions, see [2]. By showing that functions of bounded variation are precisely those that can be expressed as the difference of two monotone functions, Jordan actually extended the Dirichlet's criterium (on convergence of the Fourier series of monotone functions) to the class of $BV$ functions. Since then, the notion has been generalized in several ways leading to many important results in mathematical analysis, among them, many concerning the nature of the composition of functions in those spaces.

If $I = [a, b]$ is a closed interval in $\mathbb{R}$, we will denote by $BV(I, \mathbb{R})$ the space of all (real valued) functions of bounded variation on the interval $I$.

Impresa: ISSN 1856-8890. Dep. Legal pp200702LA2730. Digital: ISSN 2477-9660 Dep. Legal ppi201402LA4590. 

*Email addresses:* jgimenez@ula.ve (José Giménez), nmerrucv@gmail.com (Nelson Merentes), epinedaq@ucla.edu.ve (Ebner Pineda), lrodriguez@ucla.edu.ve (Luz Rodríguez)

*Inspired by the work done in [3], we will say that a sequence of positive real numbers, $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$, is a $\mathcal{W}$-sequence if it is non-decreasing and $\sum(1/\lambda_i) = +\infty$.*

Indeed, in 1972, D. Waterman [3] introduced the class $\Lambda BV([a,b])$ of functions of $\Lambda$-bounded variation and in 1980, M. Shiba [4] generalized Waterman's notion by introducing the class $\Lambda_p BV([a,b])$ $(1 \leq p < \infty)$; that is, the class of all functions $f : [a,b] \to \mathbb{R}$ with bounded $\Lambda_p$-variation on $[a,b]$.

*Definition* 1. [5] Let $I = [a,b] \subset \mathbb{R}$ be a closed interval. Let $\Lambda$ be a $\mathcal{W}$-sequence and suppose that $p \geq 1$. A function $f : I \to \mathbb{R}$ is said to be of bounded $\Lambda_p-$variation on $I$ *($f \in \Lambda_p BV(I)$)* if

$$V_{\Lambda_p}(f) = V_\Lambda(f,p,I) = \sup_{\xi} V_\Lambda(\xi, f, p, I) < \infty,$$

where

$$V_\Lambda(\xi, f, p, I) := \left( \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{\lambda_i} \right)^{1/p},$$

and the supremum is taking over all partitions $\xi : a = x_0 < x_1 < \cdots < x_n = b$ of the interval $I$.

If $p = 1$ one obtains just the Waterman's class $\Lambda BV([a,b])$.

We will use the expression $V^I$ to denote the linear space of all functions from $I$ into a given linear space $V$. Let $\mathbb{X}$ be a subspace of $\mathbb{R}^I$. Given a function $h : I \times \mathbb{R} \to \mathbb{R}$, the superposition (*Nemytskij or composition*, see [6]) operator $H : \mathbb{X} \to \mathbb{R}^I$, generated by $h$, is defined as

$$H(f)(t) := h(t, f(t)), \quad t \in [a,b]. \tag{1.1}$$

If the function $h$ does not depend on the first variable; that is, $h : \mathbb{R} \to \mathbb{R}$, then the operator

$$H(f)(t) := h(f(t)), \quad t \in [a,b],\ f \in \mathbb{X} \tag{1.2}$$

is known as the *autonomous superposition operator* generated by $h$.

Given two linear spaces $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^I$ and a function $h : I \times \mathbb{R} \to \mathbb{R}$, a primary goal of research in nonlinear analysis is to investigate under what conditions, for the generating function $h$, the associated superposition operator maps $\mathbb{X}$ into $\mathbb{Y}$. This problem is known as *the Superposition Operator Problem*; see, e.g., [6, 7]. Following [6] we will state the Superposition Operator Problem as the following set-theoretic identity:

$$\text{SOP}(\mathbb{X}, \mathbb{Y}) := \{h : S_h(\mathbb{X}) \subset \mathbb{Y}\}$$

where $S_h(f)(t) := h(t, f(t)), \quad t \in [a,b]$ and $f \in \mathbb{X}$, also we write just $\text{SOP}(\mathbb{X})$ if $\mathbb{X} = \mathbb{Y}$.

The first work on the SOP in the space $BV(I, \mathbb{R})$ was made by Josephy in 1981 [8]. In 1974, Chaika and Waterman [9] reached a similar result for the space of functions of bounded harmonic variation $HBV(I)$.

In a seminal article of 1982, J. Matkowski, [10], showed that if a superposition operator $H$, generated by a function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, maps the space $Lip[a,b]$, of all Lipschitz functions defined on $[a,b]$, into itself and is globally Lipschitz, then $h$ has the form

$$h(x,y) = h_1(x)y + h_0(x), \quad x \in I,\ y \in \mathbb{R} \tag{1.3}$$

for some $h_1, h_0 \in Lip[a,b]$.

There are a variety of spaces besides $Lip[a,b]$ that verify this result (see [11, 12]).

In this paper we introduce the notion of second bounded variation functions, in the sense of Waterman-Shiba, $\Lambda_p^2 BV$, following the line traced by De la Vallée Poussin [13], in 1908, and M. Shiba [4], in 1980. Also, we will show that if a superposition operator applies the space $\Lambda_p^2 BV$ in itself and it is uniformly bounded then its generating function must be of the form (1.3). In order to do that, in section 2 we present the definition, properties and some auxiliary results for the new space $\Lambda_p^2 BV$.

## 2. The space $\Lambda_p^2 BV$

In these section, we introduce the notion of functions of second bounded variation *in the sense of Shiba*.

Throughout the rest of this paper we will denote by $V_{\Lambda,1,p}(f)$ or $V_{\Lambda,1,p}(f;[a,b])$ the $\Lambda_p-$variation, given in the definition 1.

In addition, $\pi = \{x_i\}_{i=0}^{n}$ will denote a partition of the interval $[a, b]$; that is, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. If such a partition includes at least three points we will write $\pi \in \Pi_3([a, b])$.

Suppose $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ be a $\mathcal{W}$-sequence.

*Definition* 2. Let $1 \le p < \infty$. The $(\Lambda, 2, p)$-th variation of $f$ on $[a, b]$ is defined as

$$V_{\Lambda,2,p}(f; [a, b]) = V_{\Lambda,2,p}(f) = \sup_{\pi} \left( \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i} \right)^{1/p},$$

where $Q_1(f; \beta, \alpha) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ and the supremun is taken over all the partitions $\pi = \{x_i\}_{i=0}^{n} \in \Pi_3([a, b])$.

The sum in the definition 2, is called an approximate sum to $V_{\Lambda,2,p}(f; [a, b])$.

When $V_{\Lambda,2,p}(f; [a, b]) < \infty$ we say that $f$ has $(\Lambda, 2, p)-$th bounded variation on $[a, b]$. We denote by $\Lambda^2_p BV([a, b])$ the space containing such functions.

*Lemma* 1. The space $\Lambda^2_p BV([a, b])$ contains all affine functions.

*Proof.* Let $\Lambda$ be a $\mathcal{W}$-sequence and suppose $\pi = \{x_i\}_{i=0}^{n} \in \Pi_3([a, b])$.

Let us define $\varphi(x) = c\,x + d, \quad x \in [a, b], \quad c, d \in \mathbb{R}$. Then

$$\begin{aligned}
& |Q_1(\varphi; x_{i+2}, x_{i+1}) - Q_1(\varphi; x_{i+1}, x_i)|^p \\
= & \left| \frac{\varphi(x_{i+2}) - \varphi(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{\varphi(x_{i+1}) - \varphi(x_i)}{x_{i+1} - x_i} \right|^p \\
= & \left| \frac{c\,x_{i+2} + d - c\,x_{i+1} - d}{x_{i+2} - x_{i+1}} - \frac{c\,x_{i+1} + d - c\,x_i - d}{x_{i+1} - x_i} \right|^p \\
= & \left| \frac{c\,(x_{i+2} - x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{c\,(x_{i+1} - x_i)}{x_{i+1} - x_i} \right|^p = 0 \\
\Rightarrow & \sum_{i=0}^{n-2} \frac{|Q_1(\varphi; x_{i+2}, x_{i+1}) - Q_1(\varphi; x_{i+1}, x_i)|^p}{\lambda_i} = 0.
\end{aligned}$$

Therefore, $V_{\Lambda,2,p}(\varphi; [a, b]) = 0$ and so $\varphi \in \Lambda^2_p BV([a, b])$. □

## 3. Main Results

This section is dedicated to establish some properties of the space $\Lambda^2_p BV([a, b])$. We start showing that $\Lambda^2_p BV([a, b])$ is a vector space.

*Lemma* 2. If $1 \le p < \infty$, $f, g \in \Lambda^2_p BV([a, b])$, and $\lambda \in \mathbb{R}$, then

- $V_{\Lambda,2,p}(\lambda f) = |\lambda| V_{\Lambda,2,p}(f)$ and
- $V_{\Lambda,2,p}(f + g) \le V_{\Lambda,2,p}(f) + V_{\Lambda,2,p}(g)$.

*Proof.* Let $\Lambda$ be a $\mathcal{W}$-sequence and and suppose $\pi = \{x_i\}_{i=0}^{n} \in \Pi_3([a, b])$. Then

$$\begin{aligned}
& \left\{ \sum_{i=0}^{n-2} \frac{|Q_1(\lambda f; x_{i+2}, x_{i+1}) - Q_1(\lambda f; x_{i+1}, x_i)|^p}{\lambda_i} \right\}^{1/p} \\
= & \left\{ \sum_{i=0}^{n-2} |\lambda|^p \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i} \right\}^{1/p} \\
= & |\lambda| \left\{ \sum_{i=0}^{n-2} | \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i} \right\}^{1/p}.
\end{aligned}$$

So, $V_{\Lambda,2,p}(\lambda f) = |\lambda| V_{\Lambda,2,p}(f)$. Moreover,

$$\begin{aligned}
& \left\{\sum_{i=0}^{n-2} \frac{|Q_1(f+g; x_{i+2}, x_{i+1}) - Q_1(f+g; x_{i+1}, x_i)|^p}{\lambda_i}\right\}^{1/p} \\
= & \left\{\sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i) + Q_1(g; x_{i+2}, x_{i+1}) - Q_1(g; x_{i+1}, x_i)|^p}{\lambda_i}\right\}^{1/p} \\
\leq & \left\{\sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i}\right\}^{1/p} \\
+ & \left\{\sum_{i=0}^{n-2} \frac{|Q_1(g; x_{i+2}, x_{i+1}) - Q_1(g; x_{i+1}, x_i)|^p}{\lambda_i}\right\}^{1/p}.
\end{aligned}$$

Therefore, $V_{\Lambda,2,p}(f+g) \leq V_{\Lambda,2,p}(f) + V_{\Lambda,2,p}(g)$.

□

Now, it is easy to prove that

$$\|f\|_{\Lambda,2,p} := \|f\|_\infty + V_{\Lambda,2,p}(f; [a,b])$$

defines a norm in $\Lambda_p^2 BV([a,b])$.

The following proposition shows that the variation $V^p_{\Lambda,2,p}(f; [a,b])$ is semi-additive.

*Theorem* 1. If $a < c < b$, $1 \leq p < \infty$, and $f \in \Lambda_p^2 BV([a,b])$, then $f \in \Lambda_p^2 BV([a,c])$, $f \in \Lambda_p^2 BV([c,b])$ and

$$(V_{\Lambda,2,p}(f; [a,b]))^p \geq (V_{\Lambda,2,p}(f; [a,c]))^p + (V_{\Lambda,2,p}(f; [c,b]))^p.$$

*Proof.* Consider $c \in (a,b)$, and fix two partitions in $\Pi_3([a,c])$ and $\Pi_3([c,b])$ respectively; say

$$\pi_1 : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = c$$

and

$$\pi_2 : c = x_m < x_{m+1} < \cdots < x_{n-1} < x_n = b.$$

Put

$$\begin{aligned}
A &:= \sum_{i=0}^{m-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i}, \\
B &:= \sum_{j=m}^{n-2} \frac{|Q_1(f; x_{j+2}, x_{j+1}) - Q_1(f; x_{j+1}, x_j)|^p}{\lambda_j}
\end{aligned}$$

and consider the partition $\pi = \pi_1 \vee \pi_2$ which is in $\Pi_3([a,b])$.

Then,

$$\begin{aligned}
A + B &\leq A + B + \frac{|Q_1(f; x_{m+1}, x_m) - Q_1(f; x_m, x_{m-1})|^p}{\lambda_{m-1}} \\
&= \sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i}
\end{aligned}$$

thus

$$\begin{aligned}
(A+B)^{1/p} &\leq \sup_{\pi \in \Pi_3([a,b])} \left(\sum_{i=0}^{n-2} \frac{|Q_1(f; x_{i+2}, x_{i+1}) - Q_1(f; x_{i+1}, x_i)|^p}{\lambda_i}\right)^{1/p} \\
&= V_{\Lambda,2,p}(f; [a,b]) \\
\text{which implies } (A+B) &\leq (V_{\Lambda,2,p}(f; [a,b]))^p.
\end{aligned}$$

Therefore,

$$A \leq (V_{\Lambda,2,p}(f;[a,b]))^p - B$$

and taking supremum over all partitions $\pi_1$ of $[a,c]$ we obtain:

$$(V_{\Lambda,2,p}(f;[a,c]))^p = \sup_{\pi_1} A \leq (V_{\Lambda,2,p}(f;[a,b]))^p - B < \infty,$$

and so

$$f \in \Lambda_p^2 BV([a,c]).$$

Hence,

$$B \leq (V_{\Lambda,2,p}(f;[a,b]))^p - (V_{\Lambda,2,p}(f;[a,c]))^p.$$

Now, taking supremum over all partitions $\pi_2$ of $[c,b]$ we have:

$$(V_{\Lambda,2,p}(f;[c,b]))^p = \sup_{\pi_2} B \leq (V_{\Lambda,2,p}(f;[a,b]))^p - (V_{\Lambda,2,p}(f;[a,c]))^p < \infty,$$

thus

$$f \in \Lambda_p^2 BV([c,b])$$

and hence

$$(V_{\Lambda,2,p}(f;[a,c]))^p + (V_{\Lambda,2,p}(f;[c,b]))^p \leq (V_{\Lambda,2,p}(f;[a,b]))^p.$$

□

*Lemma* 3. If $1 \leq p < \infty$ and $f \in \Lambda_p^2 BV([a,b])$, then $Q_1(f;\cdot,\cdot)$ is bounded on $[a,b]\times[a,b]$.

*Proof.* We first note that if $a \leq x_0 < x_1 < x_2 < x_3 \leq b$, then, by the definition of $V_{\Lambda,2,p}(f;[a,b])$, it follows that:

$$\begin{aligned}
& |Q_1(f;x_3,x_2) - Q_1(f;x_1,x_0)|^p \\
= \; & |Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1) + Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p \\
\leq \; & 2^p[|Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1)|^p + |Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p].
\end{aligned}$$

On the other hand, if $a < x_0 < x_1 < x_2 < x_3 \leq b$, let us take $\lambda_1$ and $\lambda_2$ with $\lambda_1 \leq \lambda_2$. Then

$$\begin{aligned}
& |Q_1(f;x_3,x_2) - Q_1(f;x_1,x_0)|^p \\
\leq \; & 2^p\left[\lambda_2 \frac{|Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1)|^p}{\lambda_2} + \lambda_1 \frac{|Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p}{\lambda_1}\right] \\
< \; & 2^p\left[\lambda_2 \frac{|Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1)|^p}{\lambda_2} + \lambda_2 \frac{|Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p}{\lambda_1}\right] \\
\leq \; & 2^p \lambda_2 (V_{\Lambda,2,p}(f;[a,b]))^p.
\end{aligned}$$

Similarly, if $a = x_0 < x_1 < x_2 < x_3 \leq b$, let us take $\lambda_0$ and $\lambda_1$ with $\lambda_0 \leq \lambda_1$, so

$$\begin{aligned}
& |Q_1(f;x_3,x_2) - Q_1(f;x_1,x_0)|^p \\
\leq \; & 2^p\left[\lambda_1 \frac{|Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1)|^p}{\lambda_1} + \lambda_0 \frac{|Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p}{\lambda_0}\right] \\
< \; & 2^p\left[\lambda_1 \frac{|Q_1(f;x_3,x_2) - Q_1(f;x_2,x_1)|^p}{\lambda_1} + \lambda_1 \frac{|Q_1(f;x_2,x_1) - Q_1(f;x_1,x_0)|^p}{\lambda_0}\right] \\
\leq \; & 2^p \lambda_1 (V_{\Lambda,2,p}(f;[a,b]))^p.
\end{aligned}$$

Now, let $c$ be an arbitrary point in $(a,b)$, $y_0, y_1 \in [a,b]$ and $A = |Q_1(f;c,a)|^p$.
The proof of the Lemma depend on how $y_0, y_1$ are located with respect to $a, b$ and $c$.

**Case 1**. Suppose that $a < y_0 < c < y_1 < b$. If $y_2$ is a point such that $y_1 < y_2 < b$ then $a < y_0 < c < y_1 < y_2 < b$ and this implies that

$$\begin{aligned}
& |Q_1(f; y_1, y_0)|^p \\
= \; & |Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1) + Q_1(f; y_2, y_1) - Q_1(f; c, a) + Q_1(f; c, a)|^p \\
\leq \; & 2^p |Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1)|^p + 2^{2p} |Q_1(f; y_2, y_1) - Q_1(f; c, a)|^p \\
+ \; & 2^{2p} |Q_1(f; c, a)|^p \\
= \; & 2^p \lambda_1 \frac{|Q_1(f; y_2, y_1) - Q_1(f; y_1, y_0)|^p}{\lambda_1} + 2^{2p} |Q_1(f; y_2, y_1) - Q_1(f; c, a)|^p \\
+ \; & 2^{2p} |Q_1(f; c, a)|^p \\
\leq \; & 2^p \lambda_1 (V_{\Lambda,2,p}(f; [a, b]))^p + 2^{3p} \lambda_1 (V_{\Lambda,2,p}(f; [a, b]))^p + \lambda_1 2^{2p} \frac{A}{\lambda_1}.
\end{aligned}$$

Thus,

$$|Q_1(f; y_1, y_0)|^p < \lambda_1 \left[ (2^p + 2^{3p})(V_{\Lambda,2,p}(f; [a, b]))^p + \frac{2^{2p}}{\lambda_1} A \right] = K_1.$$

**Case2**. Suppose that $a < y_0 < c < y_1 = b$. If $y_2$ is a point such that $c < y_2 < y_1 = b$ then $a < y_0 < c < y_2 < y_1 = b$ and this implies that

$$\begin{aligned}
& |Q_1(f; y_1, y_0)|^p \\
= \; & |Q_1(f; y_1, y_0) - Q_1(f; y_0, a) + Q_1(f; y_0, a) - Q_1(f; y_1, y_2) \\
+ \; & Q_1(f; y_1, y_2) - Q_1(f; c, a) + Q_1(f; c, a)|^p \\
\leq \; & 2^p |Q_1(f; y_1, y_0) - Q_1(f; y_0, a)|^p + 2^{2p} |Q_1(f; y_0, a) - Q_1(f; y_1, y_2)|^p \\
+ \; & 2^{3p} |Q_1(f; y_1, y_2) - Q_1(f; c, a)|^p + 2^{3p} |Q_1(f; c, a)|^p \\
\leq \; & 2^p \lambda_0 \frac{|Q_1(f; y_1, y_0) - Q_1(f; y_0, a)|^p}{\lambda_0} + 2^{2p} \lambda_1 2^p (V_{\Lambda,2,p}(f; [a, b]))^p \\
+ \; & 2^{3p} \lambda_1 2^p (V_{\Lambda,2,p}(f; [a, b]))^p + \lambda_1 2^{3p} \frac{A}{\lambda_1} \\
\leq \; & 2^p \lambda_0 (V_{\Lambda,2,p}(f; [a, b]))^p + \lambda_1 (2^{3p} + 2^{4p})(V_{\Lambda,2,p}(f; [a, b]))^p + \lambda_1 \frac{2^{3p}}{\lambda_1} A \\
< \; & \lambda_1 \left[ (2^p + 2^{3p} + 2^{4p})(V_{\Lambda,2,p}(f; [a, b]))^p + \frac{2^{3p}}{\lambda_1} A \right].
\end{aligned}$$

Thus

$$|Q_1(f; y_1, y_0)|^p < \lambda_1 \left[ (2^p + 2^{3p} + 2^{4p})(V_{\Lambda,2,p}(f; [a, b]))^p + \frac{2^{3p}}{\lambda_1} A \right] = K_2.$$

**Case 3**. Suppose that $a < y_0 < y_1 \leq c < b$. If $y_2$ is a point such that $c < y_2 < b$ we have that $a < y_0 < y_1 \leq c < y_2 < b$ and then

$$\begin{aligned}
& |Q_1(f; y_1, y_0)|^p \\
= \; & |Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1) + Q_1(f; y_2, y_1) - Q_1(f; b, y_2) \\
+ \; & Q_1(f; b, y_2) - Q_1(f; c, a) + Q_1(f; c, a)|^p \\
\leq \; & 2^p |Q_1(f; y_1, y_0) - Q_1(f; y_2, y_1)|^p + 2^{2p} |Q_1(f; y_2, y_1) - Q_1(f; b, y_2)|^p \\
+ \; & 2^{3p} |Q_1(f; b, y_2) - Q_1(f; c, a)|^p + 2^{4p} |Q_1(f; c, a)|^p \\
\leq \; & 2^p \lambda_1 \frac{|Q_1(f; y_2, y_1) - Q_1(f; y_1, y_0)|^p}{\lambda_1} + 2^{2p} \lambda_2 \frac{|Q_1(f; b, y_2) - Q_1(f; y_2, y_1)|^p}{\lambda_2} \\
+ \; & 2^{3p} \lambda_2 2^p (V_{\Lambda,2,p}(f; [a, b]))^p + \lambda_2 \frac{2^{4p}}{\lambda_2} A \\
< \; & \lambda_2 \left[ (2^p + 2^{2p} + 2^{4p})(V_{\Lambda,2,p}(f; [a, b]))^p + \frac{2^{4p}}{\lambda_2} A \right].
\end{aligned}$$

Therefore

$$|Q_1(f; y_1, y_0)|^p < \lambda_2 \left[ (2^p + 2^{2p} + 2^{4p})(V_{\Lambda,2,p}(f; [a,b]))^p + \frac{2^{4p}}{\lambda_2} A \right] = K_3.$$

These cases are typical. The other 3 cases:

- $a = y_0 < c < y_1 < y_2 < b$,
- $a = y_0 < c < y_2 < y_1 = b$,
- $a < c \leq y_0 < y_1 < y_2 < b$,

it can be shown analogously that $Q_1(f; \cdot, \cdot)$ is bounded. □

In the next result we prove that any $f \in \Lambda^2_p BV([a,b])$ is Lipschitz and therefore absolutely continuous on $[a,b]$.

Denote by $L^b_a(f)$ the Lipschitz constant of a function $f : [a,b] \to \mathbb{R}$; i.e.,

$$L^b_a(f) := \sup \left\{ \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| : x_1, x_2 \in [a,b], x_1 \neq x_2 \right\}.$$

*Lemma* 4. If $f \in \Lambda^2_p BV([a,b])$, where $1 \leq p < \infty$, then $f$ is Lipschitz continuous on $[a,b]$.

*Proof.* By Lemma 3, there exist $K > 0$ such that

$$|Q_1(f; y, x)|^p \leq K, \quad \forall x, y \in [a,b],$$

thus,

$$|Q_1(f; y, x)| \leq K^{1/p}, \quad \forall x, y \in [a,b],$$

or equivalently:

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq K^{1/p}, \quad \forall x, y \in [a,b].$$

This means that $f \in Lip([a,b], \mathbb{R})$, with $L^b_a(f) \leq K^{1/p}$. □

## 4. Superposition Operators in $\Lambda^2_p BV([a,b])$

Below we present a characterization of the action of the composition operator (Nemytskij) on the space $\Lambda^2_p BV([a,b])$, showing that if a superposition operator $H$ applies this space into itself and it is uniformly bounded, then its generating function $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, satisfies a, so called, Matkowski's condition, that is,

$$h(x,y) = h_1(x)y + h_0(x), \quad x \in [a,b], \; y \in \mathbb{R}$$

for some $h_1, h_0 \in \Lambda^2_p BV([a,b])$.

*Theorem* 2. Let $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function such that, for any $x \in [a,b]$ fixed, the function $h(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second variable. If the superposition operator $H$, generated by $h$, acts on the space $\Lambda^2_p BV([a,b])$ and satisfies the inequality

$$\|H(\varphi) - H(\psi)\|_{\Lambda,2,p} \leq \gamma(\|\varphi - \psi\|_{\Lambda,2,p}), \quad \varphi, \psi \in \Lambda^2_p BV([a,b]), \tag{4.1}$$

for some function $\gamma : [0, +\infty) \to [0, +\infty)$, then $H$ satisfies a Matkowski's condition.

*Proof.* Fix $y_1, y_2 \in \mathbb{R}$, with $y_1 \neq y_2$. For $\alpha, \beta \in [a,b]$ such that $\alpha < \beta$, consider the function

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0, & \text{if } a \leq t \leq \alpha \\ \dfrac{t - \alpha}{\beta - \alpha}, & \text{if } \alpha \leq t \leq \beta \\ 1, & \text{if } \beta \leq t \leq b \end{cases}$$

Let us define $\varphi_j : [a,b] \to \mathbb{R}, \quad j = 1,2$ by:

$$\varphi_j(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y_j + y_2\right], \quad j = 1,2. \tag{4.2}$$

We note that

$$\varphi_1(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y_1 + y_2\right] \quad \text{and} \quad \varphi_2(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1 - y_2) + 2y_2\right];$$

thus

$$\varphi_1(t) - \varphi_2(t) = \frac{y_1 - y_2}{2}$$

and

$$\|\varphi_1 - \varphi_2\|_{\Lambda,2,p} = \|\varphi_1 - \varphi_2\|_\infty + V_{\Lambda,2,p}(\varphi_1 - \varphi_2; [a,b]) = \frac{|y_1 - y_2|}{2}.$$

By Lemma 1, $\varphi_1, \varphi_2 \in \Lambda^2_p BV([a,b])$. Now for any $x, y \in [a,b]$ such that $x < y$, taking $\alpha = x$ and $\beta = y$ in (4.2), we get:

$$\varphi_1(x) = \varphi_2(y) = \frac{1}{2}\left[y_1 + y_2\right], \quad \varphi_2(x) = y_2 \quad \text{and} \quad \varphi_1(y) = y_1;$$

thus,

$$\begin{aligned}
&\left|\frac{[H(\varphi_1) - H(\varphi_2)](y) - [H(\varphi_1) - H(\varphi_2)](x)}{y - x}\right| \\
&= \left|\frac{H(\varphi_1)(y) - H(\varphi_2)(y) - H(\varphi_1)(x) + H(\varphi_2)(x)}{y - x}\right| \\
&= \left|\frac{h(y,\varphi_1(y)) - h(y,\varphi_2(y)) - h(x,\varphi_1(x)) + h(x,\varphi_2(x))}{y - x}\right| \\
&= \left|\frac{h(y,y_1) - h(y,\frac{y_1+y_2}{2}) - h(x,\frac{y_1+y_2}{2}) + h(x,y_2)}{y - x}\right|.
\end{aligned}$$

On the other hand,

$$\left|\frac{[H(\varphi_1) - H(\varphi_2)](y) - [H(\varphi_1) - H(\varphi_2)](x)}{y - x}\right| \leq L_a^b(H(\varphi_1) - H(\varphi_2))$$

and

$$\begin{aligned}
L_a^b(H(\varphi_1) - H(\varphi_2)) &\leq K^{\frac{1}{p}} = \left(C_1[V_{\Lambda,2,p}(H(\varphi_1) - H(\varphi_2); [a,b])]^p + C_2 A\right)^{\frac{1}{p}} \\
&\leq \left[C_1\|H(\varphi_1) - H(\varphi_2)\|^p_{\Lambda,2,p} + C_2 A\right]^{\frac{1}{p}} \\
&\leq \left[C_1(\gamma(\|\varphi_1 - \varphi_2\|_{\Lambda,2,p}))^p + C_2 A\right]^{\frac{1}{p}} \\
&\leq C\left[(\gamma(\|\varphi_1 - \varphi_2\|_{\Lambda,2,p}))^p + A\right]^{\frac{1}{p}},
\end{aligned}$$

where $K$ and $A$ are given in the proof of Lemma 4 and 3 respectively, and $C = \max\{C_1^{1/p}, C_2^{1/p}\}$ is a positive constant. Hence

$$\left|\frac{h(y,y_1) - h(y,\frac{y_1+y_2}{2}) - h(x,\frac{y_1+y_2}{2}) + h(x,y_2)}{y - x}\right| \leq C\left[(\gamma(\|\varphi_1 - \varphi_2\|_{\Lambda,2,p}))^p + A\right]^{\frac{1}{p}},$$

which implies that

$$\begin{aligned}
&\left|h(y,y_1) - h(y,\frac{y_1+y_2}{2}) - h(x,\frac{y_1+y_2}{2}) + h(x,y_2)\right| \\
&\leq C\left(\left[\gamma\left(\frac{|y_1 - y_2|}{2}\right)\right]^p + A\right)^{\frac{1}{p}} |y - x|.
\end{aligned} \tag{4.3}$$

By hypothesis the operator $H$ takes its values on the space $\Lambda^2_p BV([a,b])$, and since the constants functions are in that space, we have, for any $y \in \mathbb{R}$, that

$$h(\cdot, y) = H(y) \in \Lambda_p^2 BV([a,b])$$

and therefore $h(\cdot, y)$ is continuous for all $y \in \mathbb{R}$. Taking limit when $y \to x$ in (4.3) we obtain:

$$\left| h(x, y_1) - h(x, \frac{y_1+y_2}{2}) - h(x, \frac{y_1+y_2}{2}) + h(x, y_2) \right| = 0 \quad \forall x \in [a,b], \quad \forall y_1, y_2 \in \mathbb{R},$$

thus,

$$2h\Big(x, \frac{y_1+y_2}{2}\Big) = h(x, y_1) + h(x, y_2), \quad \forall x \in [a,b].$$

This shows that $h(x,\cdot)$ satisfies the equation of Jensen (see [14], pág. 351), and since $h(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, by Theorem 13.2.2 of [14] (pág. 354), exist $h_0,\ h_1 : [a,b] \to \mathbb{R}$ such that

$$h(x, y) = h_1(x)y + h_0(x), \quad x \in [a,b], \quad y \in \mathbb{R}. \tag{4.4}$$

Let us note that for each $y \in \mathbb{R}$, the function $h(\cdot, y) \in \Lambda_p^2 BV([a,b])$, then using equation (4.4) we have, for $y = 0$ and $y = 1$, that

$$h_0(x) = h(x, 0) \quad \text{and} \quad h_1(x) = h(x, 1) - h_0(x), \quad \forall x \in [a,b]$$

therefore $h_0,\ h_1 \in \Lambda_p^2 BV([a,b])$, which completes the proof.

□

*Definition* 3. [15]
Let $\mathcal{Y}$ and $\mathcal{Z}$ be two metric (or normed) spaces. We say that $H : \mathcal{Y} \to \mathcal{Z}$ is *uniformly bounded* if, for all $t > 0$ there exists a nonnegative real number $\gamma(t) \geq 0$ such that for all set $B \subset \mathcal{Y}$

$$\text{diam } B \leq t \Rightarrow \text{diam } H(B) \leq \gamma(t).$$

*Theorem* 3. Suppose that $a, b \in \mathbb{R}$, with $a < b$, $n \in \mathbb{N}$ and that $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function such that $\forall x \in [a,b]$, $h(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous in the second variable. If the superposition operator $H$, generated by $h$, acts on $\Lambda_p^2 BV([a,b])$ and is uniformly bounded, then there exist functions $h_0,\ h_1 \in \Lambda_p^2 BV([a,b])$ such that

$$h(x, y) = h_1(x)y + h_0(x), \quad x \in [a,b],\ y \in \mathbb{R},$$

and

$$H(\varphi)(x) = h_1(x)\varphi(x) + h_0(x), \quad \varphi \in \Lambda_p^2 BV([a,b]), \quad (x \in [a,b]).$$

*Proof.* Let $\varphi, \psi \in \Lambda_p^2 BV([a,b])$.

If $t \geq 0$ is such that $\|\varphi - \psi\|_{\Lambda,2,p} \leq t$, then diam $\{\varphi, \psi\} \leq t$ and diam $H(\{\varphi, \psi\}) \leq \gamma(t)$, since $H$ is uniformly bounded.

Taking $t = \|\varphi - \psi\|_{\Lambda,2,p}$ we obtain

$$\text{diam } H(\{\varphi, \psi\}) \leq \gamma(t) = \gamma(\|\varphi - \psi\|_{\Lambda,2,p})$$

which implies that

$$\|H(\varphi) - H(\psi)\|_{\Lambda,2,p} = \text{diam } H(\{\varphi, \psi\}) \leq \gamma(\|\varphi - \psi\|_{\Lambda,2,p}),$$

and the result follows from Theorem 2.

□

## 5. Referencias


[1] C. Jordan, *Sur la serie de Fourier*. C. R. Acad. Sci. Paris. 2:228-230 1881.

[2] I. Lakatos, *Proofs and Refutations, The logic of Mathematical Discovery*, Cambridge University Press, 1976.

[3] D. Waterman, *On the convergence of Fourier series of functions of bounded variation*. Studia Math. 44:107-117 1972.

[4] M. Shiba, *On the absolute convergence of Fourier series of functions class $\Lambda_p BV$*. Sci. Rep. Fukushima Univ. 30:7-10 1980.

[5] R.G. Vyas, *Properties of functions of generalized bounded variation*. Mat. Vesnik, 58:91-96 2006.

[6] J. Appell; J. Banaś; N. Merentes, *Bounded Variation and Around*, De Gruyter Series in Nonlinear Analysis and Applications 17, Walter de Gruyter, Berlin/Boston 2014.

[7] G. Bourdaud; M. Lanza de Cristoforis; W. Sickel, *Superposition operators and functions of bounded p-variation*, Rev. Mat. Iberoamer. **22**, 2, 455-487 2006.

[8] M. Josephy, *Composing functions of bounded variation*, Proc. Amer. Math. Soc. **83**(2):354-356 1981.

[9] M. Chaika; D. Waterman, *On the invariance of certain classes of functions under composition*, Proceedings of the American Mathematical Society, vol. **43**:345-348 1974.

[10] J. Matkowski, *Functional equations; Nemytskij operators*, Funkc. Ekvacioj Ser. Int. **25**:127-132 1982.

[11] J. Appell; N. Guanda; N. Merentes; J.L. Sánchez, *Some boundedness and continuity properties of nonlinear composition operators: a survey*, Comm. Applied Anal. vol. **15**(2-4):153-182 2011.

[12] J. Appell; N. Guanda; M. Väth, *Function spaces with the Matkowski property and degeneracy phenomena for composition operators*, Fixed Point Theory, vol. **12**(2):265-284 2011.

[13] Ch. J. De La Vallée Poussin, *Sur l'intégrale de Lebesgue*, Trans. Amer. Math. Soc., 16:435-501 1908.

[14] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*. Second Edition. Birkhäuser Verlag AG Basel.Boston. Berlin 2009.

[15] M. Wróbel, *Uniformly Bounded Nemytskij operators between the Banach spaces of functions of bounded $n-th$ variation*. Journal of mathematical Analysis and Applications. 391:451-456 2012.