

Robust Stability Of A Gyroscope Using Linear Matrix Inequalities

Research

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Abstract

In this paper linear matrix inequalities (LMIs) are applied to the real-time control of a gyroscope with two degrees of freedom. The controller is designed using routines from the LMI-toolbox for Matlab. Real-time results are presented, showing a good performance of the state-feedback controller.

Keywords: Linear matrix inequalities, robust control, convex optimization, gyroscope, quadratic stability.

Resumen

En este artículo se aplican las desigualdades lineales matriciales al problema de control en tiempo real de un giroscopio con dos grados de libertad. El controlador es diseñado utilizando rutinas de la caja de herramientas LMI de Matlab. Se presentan resultados en tiempo real, mostrando un buen desempeño del controlador de estado retroalimentado.

Palabras clave: Desigualdades lineales matriciales, control robusto, optimización convexa, giroscopio, estabilidad cuadrática.

Introduction

In recent years linear matrix inequalities (LMIs) have emerged as a powerful tool to approach control problems that appear hard if not impossible to solve in an analytic fashion. Although the history of LMIs goes back to the forties with a major emphasis of their roles in control in the sixties by Kalman, Yakubovich, Popov, Willems, only recently powerful numerical interior point techniques have been developed to solve LMIs in a practically efficient manner (Nesterov, Nemirovskii [1,2,3]). Several Matlab [4] software packages are available that allow a simple coding of general LMI problems and provide efficient tools to solve typical control problems (LMI Control Toolbox, LMI-tool).

Boosted by the availability of fast LMI solvers, research in robust control has experienced a paradigm shift-instead of arriving at an analytical solution the intention is to reformulate a given problem to

verifying whether an LMI is solvable or to optimizing functionals over LMI constraints.

The power of this approach is illustrated by several fundamental robustness and performance problems in analysis and design of linear control systems [5,6,7,8].

Optimization questions and decision making processes are abundant in daily life and invariably involve the selection of the best decision from a number of options or a set of candidate decisions. Many examples of this theme can be found in technical sciences such as electrical, mechanical and chemical engineering, in architecture and in economics, but also in the social sciences, in biological and ecological processes and organizational questions. For example, production processes in industry are more and more market driven and require an ever increasing flexibility of product changes and product specifications due to customer demands in quality, price and specification. Products need to be manufactured within strict product specifications, with large variations of input quality, against competitive prices, with minimal waste of resources, energy and valuable production time, with a minimal time-to-market and, of course, with maximal economical profit. Important economical benefits can therefore only be realized by making proper decisions in the operating conditions of production processes. Due to increasing requirements on the safety and flexibility of production processes, there is a constant need for further optimization, efficiency and improvement of production processes.

In view of the optimization problems just formulated, we are interested in finding conditions for optimal solutions to exist. It is therefore natural to resort to a branch of analysis which provides such conditions: convex analysis cited in [8].

In this paper we found robust stability of a gyroscope with two axes. For this, we worked with a linear-time invariant system and using the LMI-toolbox for MATLAB. First, we defined a Linear Matrix Inequality Problem (LMIP) to find a solution of the quadratic stability problem. Later, if the problem is feasible, it is possible to find a state-feedback that gives stability, performance and robustness to the closed-loop system.

Preliminaries

Linear Matrix Inequalities (LMIs)

A linear matrix inequalities (LMIs) have the form

$$F(x) = F_0 + \sum_{i=1}^m x_i F_i > 0, \quad (1)$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T \in R^{n \times n}$, $i=0, \dots, m$, are given. The inequality symbol in (1) means that $F(x)$ is positive-definite, i.e., $u^T F(x) u > 0$ for all nonzero $u \in R^n$. Of course, the LMI (1) is equivalent to a set of n polynomial inequalities due to F_i is a $n \times n$ matrix and $x \in R^m$, i.e., the leading principal minors of $F(x)$ must be positive.

Also exists nonstrict LMIs, which have the form

$$F(x) \geq 0. \quad (2)$$

The LMI (1) is a convex constraint on x , i.e., the set $\{x/F(x) > 0\}$ is convex. Although the LMI (1) may seem to have a specialized form, it can represent a wide variety of convex constraints on x . In particular, linear inequalities, (convex) quadratic inequalities, matrix norm inequalities, and constraints that arise in control theory, such as Lyapunov and convex quadratic matrix inequalities, can all be cast in the form of an LMI.

Given an LMI $F(x) > 0$, the corresponding LMI Problem (LMIP) is to find x^{feas} such that $F(x^{feas}) > 0$ or determine that the LMI is infeasible (the LMI $F(x) > 0$ is infeasible means the affine set $\{F(x)/x \in R^m\}$ does not intersect the positive-definite cone).

Of course, this is a convex feasibility problem. We say “solving the LMI $F(x) > 0$ ” to mean solving the corresponding LMIP.

As an example of an LMIP, consider the “simultaneous Lyapunov stability problem”: We are given $A_i \in R^{n \times n}$, $i=1, \dots, L$, and need to find P satisfying the LMI

$$P > 0, \quad A_i^T P + P A_i < 0, \quad i = 1, \dots, L \quad (3)$$

or determine that no such P exist. Determining that no such P exist is equivalent to finding Q_0, \dots, Q_L , not all zero, such that

$$Q_0 \geq 0, \dots, Q_L \geq 0, \quad (4)$$

$$Q_0 = \sum_{i=1}^L (Q_i A_i^T + A_i Q_i)$$

which is another (nonstrict) LMIP.

Linear differential inclusions

A linear differential inclusions (LDI) is given by

$$\dot{x} \in \Omega x, \quad x(0) = x_0 \quad (5)$$

where Ω is a subset of $R^{n \times n}$. We can interpret the LDI (5) as describing a family of linear time-varying systems. Every trajectory of the LDI satisfies

$$\dot{x} = A(t)x, \quad x(0) = x_0, \quad (6)$$

for some $A: R_+ \rightarrow \Omega$. Conversely, for any $A: R_+ \rightarrow \Omega$, the solution of (6) is a trajectory of the LDI (5). In the language of control theory, the LDI (5) might be described as an “uncertain time-varying linear system”, with the set Ω describing the “uncertainty” in the matrix $A(t)$.

Linear time-invariant systems

When Ω is a singleton, the LDI reduces to the linear time-invariant (LTI) system

$$\dot{x} = Ax + B_u u + B_w w, \quad x(0) = x_0, \quad (7)$$

$$z = C_z x + D_{zu} u + D_{zw} w,$$

where $x: R_+ \rightarrow R^n$, $u: R_+ \rightarrow R^{nu}$, $w: R_+ \rightarrow R^{nw}$, $z: R_+ \rightarrow R^{nz}$, x is referred to as the state, u is the control input, w is the exogenous input signal and z is the output.

The matrices in (7) satisfy

$$\Omega = \begin{bmatrix} A & B_u & B_w \\ C_z & D_{zu} & D_{zw} \end{bmatrix}. \quad (8)$$

where $\Omega \in R^{(n+nz) \times (n+nu+nw)}$.

Quadratic stability

We first study stability of the LDI

$$\dot{x} = A(t)x, \quad A(t) \in \Omega \quad (9)$$

that is, we ask whether all trajectories of system (9) converge to zero as $t \rightarrow \infty$. A sufficient condition for this is the existence of a quadratic function $V(\xi) = \xi^T P \xi$, $P > 0$ that decreases along every nonzero trajectory of (9). If there exists such a P , we say the LDI (9) is quadratically stable and we call V a quadratic Lyapunov function.

Since

$$\frac{d}{dt} V(x) = x^T \left(A(t)^T P + P A(t) \right) x, \quad (10)$$

a necessary and sufficient condition for quadratic stability of the system (9) is

$$P > 0, \quad A^T P + P A < 0 \text{ for all } A \in \Omega. \quad (11)$$

Multiplying the second inequality in (11) on the left and right by P^{-1} , and defining a new variable $Q = P^{-1}$, we may rewrite (11) as

$$Q > 0, \quad A Q^T + P A < 0 \text{ for all } A \in \Omega. \quad (12)$$

This dual inequality is an equivalent condition for quadratic stability. We now show that conditions for quadratic stability for LTI systems can be expressed in terms of LMIs.

Condition (11) becomes

$$P > 0, \quad A^T P + P A < 0 \quad (13)$$

Therefore, checking quadratic stability for an LTI system is an LMIP in the variable P . This is precisely the (necessary and sufficient) Lyapunov stability criterion for LTI systems. In other words, a linear system is stable if and only if it is quadratically stable. Alternatively, using (12), stability of LTI systems is equivalent to the existence of Q satisfying the LMI

$$Q > 0, \quad A Q + Q A^T < 0. \quad (14)$$

Of course, each of these LMIPs can be solved analytically by solving a Lyapunov equation.

State-Feedback for LTI

In this section we present the general form of the state-feedback for LTI [1].

Let $K \in R^{m \times n}$, and suppose that $u = Kx$. Since the control input is a linear function of the state, this is called state-feedback, and the matrix K is called the state-feedback gain. This yields the closed-loop LDI

$$\begin{aligned} \dot{x} &= (A(t) + B_u(t)K)x + B_w(t)w, \\ z &= (C_z(t) + D_{zu}(t)K)x + D_{zw}(t)w, \end{aligned} \quad (15)$$

The system (8) is said to be quadratically stabilizable (via linear state-feedback) if there exist a state-feedback gain K such that the closed-loop system (15) is quadratically stable (hence, stable). Quadratic stabilizability for LTI can be expressed as an LMIP as follow.

Consider a LTI system (7) without exogenous input signal, therefore this system is (quadratically) stable if and only if there exists $P > 0$ such that

$$(A + B_u K)^T P + P(A + B_u K) < 0, \quad (16)$$

or equivalently, there exists $Q > 0$ such that

$$Q(A + B_u K)^T + (A + B_u K)Q < 0. \quad (17)$$

Neither of these conditions is jointly convex in K and P or Q , but by a simple change of variables we can obtain an equivalent condition that is an LMI.

Define $Y = KQ$, so that for $Q > 0$ we have $K = YQ^{-1}$. Substituting into (11) yields

$$A Q + Q A^T + B_u Y + Y^T B_u^T < 0, \quad (18)$$

which is an LMI in Q and Y . Thus, the system is quadratically stabilizable if and only if there exists $Q > 0$ and Y such that the LMI (18) holds. If this LMI is feasible (in the feasibility problem, we consider any feasible point as being optimal), then the quadratic function $V(\xi) = \xi^T Q^{-1} \xi$ proves (quadratic) stability of system with state-feedback $u = YQ^{-1}x$.

Description of the system

Gyroscopes are used to measure the angular movement with respect to a fixed structure, and are

a key component of plane automatic pilots, rockets guidance system, spatial vehicles altitude system, navigation gyrocompasses, etc. [2].

The system considered in this work is a gyroscope with two axes, shown schematically in Figure 1, which is a lab experiment developed by Quanser Inc., [9]. The gyroscope consists basically of the following components: a support plate holding the gyro module with a rotor which rotates a constant speed, its movement being produced by a DC motor, sensors for the angles α and ψ , and a data acquisition card connecting the gyroscope to a computer.

Angle α defines the angular position of structure with the rotor, with respect to the gyro module, while angle ψ is located between the gyro module and the support plate.

The purpose of the controller to be designed for this system is to maintain the direction at which the gyroscope is pointing, while the support plate rotates relative to the base plate. In other words, the gyro module must keep its position in the presence of perturbations or any movement of the base plate. This mimics the problem of a ship on which a radar is mounted, and it is desired to maintain the direction in which the radar beam points independent of the unknown yaw of the ship due to disturbances and steering. The control input of the system is the voltage $\delta(t)$ applied to the DC motor, and the output will be considered to be the angle $\psi(t)$.

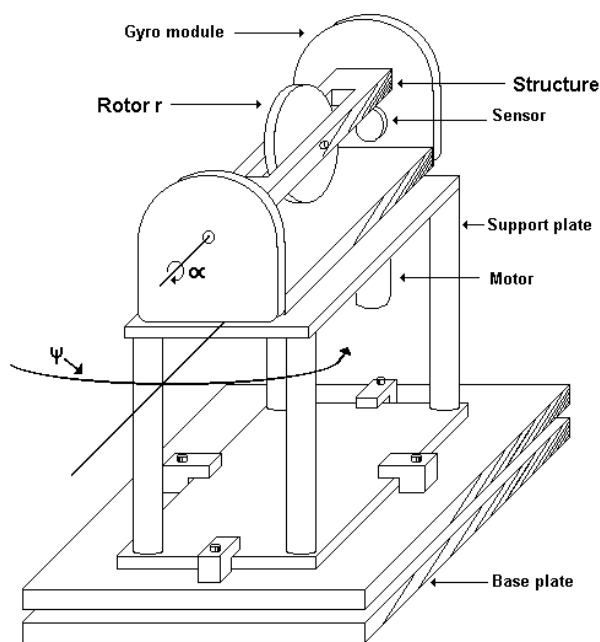


Figure. 1. Scheme of the gyroscope

Using the D'Alembert method [2] to get a mathematical representation of the gyroscope, we found the dynamic equations of the system, later using the information given by Quanser in table 1 we can have a representation of the system.

Table 1. Information from Quanser Inc.

Parameter	Symbol	Value	Unit
Gyroscope module inertia	J_Z^o	0.002	Kgm ²
Rotor mass	M_r	0.8	Kg
Rotor ratio	r_r	0.0508	m
Springs constant	K_s	1908.9	Nm
Rotational constant	K_r	2.4631	Nm/rad
Gyroscope sensitivity to θ	G_g	5.2205	(°/seg)/°
Rotor speed	η	457	Rad/seg

Linearizing the dynamic equations with the Jacobian, we have the following equations

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= 4524.831171x_1 - 866.6133774x_4 \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= 165.69769x_2 - 1.728537x_4 + 39.57912851u \\
 y &= x_3
 \end{aligned}
 \tag{19}$$

The state-space representation, is given by

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4524.831171 & 0 & 0 & -866.6133774 \\ 0 & 0 & 0 & 1 \\ 0 & 165.697690 & 0 & -1.728537 \end{bmatrix} \\
 B &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 39.57912851 \end{bmatrix} \\
 C &= [0 \quad 0 \quad 1 \quad 0]
 \end{aligned}
 \tag{20}$$

The transfer function of the gyroscope, is given by

$$P(s) = \frac{\psi(s)}{\delta(s)} = \frac{39.6s^2 - 1.8 \times 10^5}{s^4 + 1.7s^3 + 1.4 \times 10^5 s^2 - 7.8 \times 10^3 s} \tag{21}$$

where the input $\delta(t)$ is the voltage applied to the motor, and the output is the angle $\psi(t)$, which corresponds to the angle located between the gyro module and the support plate.

The zeros of the system are given by $\{67.42, -67.42\}$, and the poles of the system are $\{0, 0.06, -0.88 \pm 374.16i\}$, from which it can be seen that the system is non-minimum-phase and unstable.

Results

In this section, we present the controller to be designed. This controller should be such that the gyro module keeps its position in presence of perturbations or movements of the base plate, while providing stability, performance and robustness to the closed-loop system.

Using the matrices of (20) to find a solution of (18), the state-feedback calculated using the LMI-toolbox for the gyroscope of two axes is

$$K_{lmi} = YQ^{-1} \quad (22)$$

where

$$Q = \begin{bmatrix} 0.027889 & -0.187019 & -0.153568 & 0.150715 \\ -0.187019 & 4.526938 & -0.043523 & -0.973459 \\ -0.153568 & -0.043523 & 4.548524 & -0.802193 \\ 0.150715 & -0.973459 & -0.802193 & 1.256369 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.808564 & -8.681569 & 0.143648 & 4.05588 \end{bmatrix}$$

Since there exists $Q > 0$ and Y such that the LMI (18) holds, the system is quadratically stabilizable.

The state-feedback calculated for the gyroscope with two axes is

$$K_{lmi} = \begin{bmatrix} 34.447 & -0.63977 & 1.0611 & -0.72221 \end{bmatrix}$$

Real-Time results

To test the robustness and performance of the obtained controller in real time, we introduce a perturbation to the system by moving manually the support plate. The angle $\psi(t)$ caused by this input perturbation is shown in Figure. 2. The corresponding behavior of the angle $\alpha(t)$ is shown in Figure. 3, where it can be seen that $\alpha(t)$ opposes to the movement of the gyro module until the perturbation is rejected. The control signal $\delta(t)$ is shown in Figure. 4.

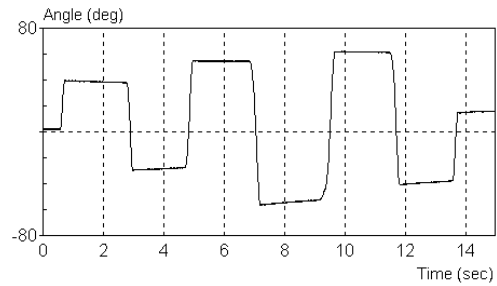


Figure. 2. Angle $\psi(t)$.

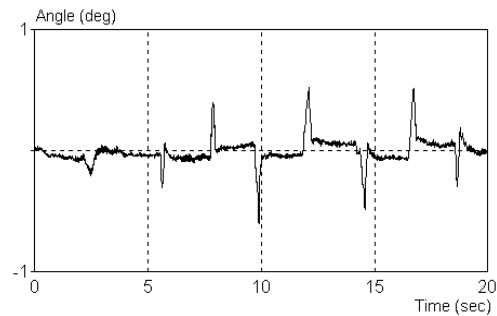


Figure. 3. Angle $\alpha(t)$.

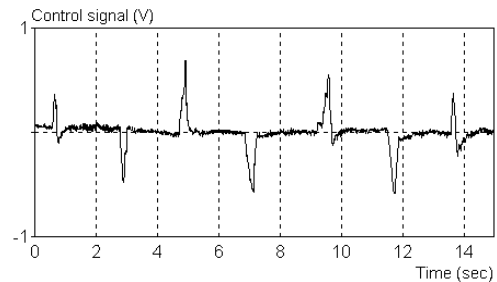


Figure. 4. Control signal $\delta(t)$.

The gyro module remains practically without movement with respect to its initial position in the presence of the introduced perturbation, showing a good performance of the designed controller.

Conclusions

In this paper robust stability using LMIs have been applied to the real-time control of a gyroscope with two degrees of freedom. The controller was designed using routines from LMI Control Toolbox. Real-time results show an excellent robustness and performance of the state-feedback.

Future work has to be concerned with testing the robustness of the controller, for instance introducing variations on the parameters of the system.

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