A note on zip and reversible skew PBW extensions

Una nota sobre extensiones PBW torcidas zip y reversibles

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Abstract. In this note we establish necessary and sufficient conditions to guarantee the existence of zip and reversible skew PBW extensions.

Keywords: Skew PBW extensions, zip and reversible rings.

Resumen. En esta nota establecemos condiciones necesarias y suficientes para garantizar la existencia de extensiones PBW torcidas zip y reversibles.

Palabras claves: Extensiones PBW torcidas, anillos zip y reversibles.

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1. Introduction

Faith in [6] called a ring B *right zip* provided that if the right annihilator $r_B(X)$ of a subset X of B is zero, then $r_B(Y)=\{0\}$ for a finite subset $Y\subseteq X$. Equivalently, for a left ideal I of B with $r_B(I)=\{0\}$, there exists a finitely generated left ideal $I_1\subseteq I$ such that $r_B(I_1)=\{0\}$. In this way, B is called *zip* if it is right and left zip (this concept was initiated by Zelmanowitz in [18]). Several works have been published about this topic, see [2], [3], [6], [7], [9], [18], and others. For instance, in [2] it was shown that if B is a commutative zip ring, then the polynomial ring $B[x]$ over B is zip; in [9] it was proved that if B is an Armendariz ring, then B is a right zip if and only if $B[x]$ is a right zip ring. With respect to the notion of *reversible* rings, see [4], it is said that a ring B has this property if $ab=0$ implies $ba=0$ for $a,b\in B$. For a reversible ring B, $r_B(\{a\})$ is an ideal of B for each $a\in B$, see [10].

In the case of noncommutative rings, in [1], [5] and [9], we can find some results about Ore extensions of zip rings and reversible rings. For instance,

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using a notion of *rigidness* on B, see [11], in [1] it was proved that (i) B is a right zip ring if and only if the Ore extension $B[x;\alpha,\delta]$ is right zip; (ii) B is a reversible ring if and only if the Ore extension $B[x;\alpha,\delta]$ is a reversible ring. With this in mind, and since Ore extensions of injective type (i.e., α is injective) are particular cases of a more general kind of noncommutative rings known as skew PBW extensions (PBW means Poincaré-Birkhoff-Witt), in this paper we investigate conditions under which one can obtain zip and reversible skew PBW extensions. Therefore, the results presented in this note are new for skew PBW extensions, and all of them are similar to others existing in the literature concerning Ore extensions of injective type. More exactly, we generalize the results presented in [1], Theorem 4, [9], Theorem 11, and [5], Theorem 2.9. It is important to say that skew PBW extensions were introduced in [8] and include remarkable examples of noncommutative rings as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc., see [13] and [17] for a detailed list of examples. In fact, these extensions include several algebras which can not be expressed as Ore extensions (universal enveloping algebras of finite Lie algebras, diffusion algebras, and others, see [13], [14] and [16] for more details).

The results presented in this paper use an adequate notion of *rigidness* for skew PBW extensions (Definition 3.1) considering the treatment in [16]. The techniques used are fairly standard and follow the same path as other text on the subject. The interest of studying annihilators of elements in this kind of extensions started in [15] with the question about uniform (also known as Goldie) dimension for these extensions. With this report, we continue the task of studying ring and module theoretical properties of skew PBW extensions (cf. [12], [13], [14], [15], [16], [17], and others).

2. Definitions and elementary properties

In this section we consider the definition of skew PBW extensions and present some key properties of these noncommutative rings.

Definition 2.1 ([8], Definition 1)**.** Let R and A be rings. We say that A *is a skew PBW extension of* R (also called a σ*-PBW extension of* R) if the following conditions hold:

i) $R\subseteq A$;

ii) there exist elements $x_1,\dots,x_n\in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A):=\{x^{\alpha}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\mid \alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\}$.

iii) For each $1\leq i\leq n$ and any $r\in R\setminus\{0\}$, there exists an element $c_{i,r}\in R\setminus\{0\}$ such that $x_ir-c_{i,r}x_i\in R$.

iv) For any elements $1\leq i,j\leq n$ there exists $c_{i,j}\in R\setminus\{0\}$ such that $x_jx_i-c_{i,j}x_ix_j\in R+Rx_1+\cdots+Rx_n$.

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n\rangle$.

Proposition 2.2 ([8], Proposition 3). *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Two particular cases of skew PBW extensions are considered in the following definition.

Definition 2.3 ([8], Definition 4). Let A be a skew PBW extension of R.

i) A is called *quasi-commutative* if the conditions iii) and iv) in Definition 2.1 are replaced by iii'): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; iv'): for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

ii) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

The class of skew PBW extensions contains various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. A detailed list of examples of skew PBW extensions is presented in [13] and [14].

Definition 2.4 ([8], Definition 6). Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.2.

i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we denote $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \mathrm{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1X_1 + \cdots + a_mX_m$, with $a_i \in R \setminus \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_mX_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [8], Section 3.

Skew PBW extensions are characterized in the following way.

Proposition 2.5 ([8], Theorem 7). *Let A be a polynomial ring over R with respect to $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

i) for each $x^\alpha \in \text{Mon}(A)$ *and every* $0 \neq r \in R$*, there exist unique elements* $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$*,* $p_{\alpha,r} \in A$*, such that* $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$*, where* $p_{\alpha,r} = 0$ *or* $\deg(p_{\alpha,r}) < |\alpha|$ *if* $p_{\alpha,r} \neq 0$*. If* r *is left invertible, so is* r_α*.*

ii) For each $x^\alpha, x^\beta \in \text{Mon}(A)$ *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that* $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$*, where* $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$*.*

In Proposition 2.6 and Remark 2.7 we will look more closely at the form of the polynomials $p_{\alpha,r}$ and $p_{\alpha,\beta}$ given by Proposition 2.5.

Proposition 2.6 ([16], Proposition 2.9)**.** *If* $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ *and* $r \in R$*, then*

$$
\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n-1}^{j-1} \right) x_n^{\alpha_n} \\
&+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\sigma_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j} \delta_2(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))))) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R \ \text{ for } \ 1 \leq j \leq n.
\end{aligned}
$$

Remark 2.7 ([16], Remark 2.10). If we have the expression $a_i X_i b_j Y_j$ where $X_i := x_1^{\alpha i1} \cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, then every summand of $a_i X_i b_j Y_j$ can be obtained considering products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

3. Zip and reversible rings

In [16], the first author introduced a notion of *rigidness* for skew PBW extensions with the purpose of proving that the properties of being Baer, quasi-Baer, p.p. and p.q.-Baer are stable over this kind of extensions. We will see that this notion is very important to construct zip and reversible skew PBW extensions (Propositions 3.6 and 3.9).

We recall that a ring B is *reduced* if B has no nonzero nilpotent elements, and a ring B is called *abelian* if every idempotent is central. Reduced rings are abelian and also semiprime (that is, its prime radical is trivial). It is clear that a reduced ring is also reversible.

For a ring B with a ring endomorphism $\sigma : B \to B$, and an σ-derivation $\delta : B \to B$, considering the Ore extension $B[x; \sigma, \delta]$, Krempa in [11] defined σ as a *rigid endomorphism* if the condition $b\sigma(b) = 0$ implies $b = 0$ for $b \in B$. Krempa called B σ-rigid if there exists a rigid endomorphism σ of B. Since Ore extensions of injective type are particular examples of skew PBW extensions, we consider the following notion of *rigidness* for these extensions.

Definition 3.1 ([16], Definition 3.2)**.** If B is a ring and Σ a family of endomorphisms of B, then Σ is called a *rigid endomorphisms family* if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$ for every $r \in B$ and $\alpha \in \mathbb{N}^n$. A ring B is called to be Σ-*rigid* if there exists a rigid endomorphisms family Σ of B.

Note that if Σ is a rigid family of endomorphisms, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, that is, B is reduced (note that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, see [16] for more details). With this in mind, we consider the family of injective endomorphisms Σ and the family Δ of Σ-derivations in a skew PBW extension A of a ring R (Proposition 2.2).

Lemma 3.2 ([16], Lemma 3.3)**.** *Let B be an Σ-rigid ring and $a, b \in B$. Then:*

i) If $ab = 0$ then $a\sigma^{\alpha}(b) = \sigma^{\alpha}(a)b = 0$ for $\alpha \in \mathbb{N}^n$.

ii) If $ab = 0$ then $a\delta^{\beta}(b) = \delta^{\beta}(a)b = 0$ for $\beta \in \mathbb{N}^n$.

iii) If $ab = 0$ then $a\sigma^{\alpha}(\delta^{\beta}(b)) = a\delta^{\beta}(\sigma^{\alpha}(b)) = 0$ for every $\alpha, \beta \in \mathbb{N}^n$.

iv) If $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$ for some $\theta \in \mathbb{N}^n$, then $ab = 0$.

Corollary 3.3 ([16], Corollary 3.4)**.** *Suppose that A is a skew PBW extension of a ring R. If R is Σ-rigid and $ab = 0$ for $a, b \in R$, then we obtain $ax^{\alpha}bx^{\beta} = 0$ in A for any $\alpha, \beta \in \mathbb{N}^n$.*

Proposition 3.4 ([16], Proposition 3.5)**.** *Let R be a ring. R is Σ-rigid if and only if the bijective skew PBW extension A is a reduced ring.*

For the next proposition, suppose that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R, that is, they commute with every element of R.

Proposition 3.5 ([16], Proposition 3.6)**.** *Suppose that R is an Σ-rigid ring. Let $f = a_0 + a_1X_1 + \cdots + a_mX_m$, $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ be elements of a bijective skew PBW extension A of R. Then $fg = 0$ if and only if $a_ib_j = 0$ for all $0 \leq i \leq m$, $0 \leq j \leq t$.*

Proof. Suppose that $fg = 0$. We have $fg = (a_0 + a_1X_1 + \cdots + a_mX_m)(b_0 + b_1Y_1 + \cdots + b_tY_t) = \sum_{k=0}^{m+t}\left(\sum_{i+j=k} a_iX_ib_jY_j\right)$. Note that $\mathrm{lc}(fg) = a_m\sigma^{\alpha_m}(b_t)c_{\alpha_m,\beta_t} = 0$. Since A is bijective, we have $a_m\sigma^{\alpha_m}(b_t) = 0$, and by Lemma 3.2 (iv), $a_mb_t = 0$. The idea is to prove that $a_pb_q = 0$ for $p + q \geq 0$. We proceed by induction. Suppose that $a_pb_q = 0$ for $p + q = m + t, m + t - 1, m + t - 2, \ldots, k + 1$ for some $k > 0$. By Corollary 3.3 we obtain $a_pX_pb_qY_q = 0$ for these values of $p + q$. In this way we only consider the sum of the products $a_uX_ub_vY_v$, where $u+v=k,k-1,k-2,\dots,0$. Fix u and v. Consider the sum of all terms of fg having exponent $\alpha_u+\beta_v$. By Proposition 2.6, Remark 2.7, and the assumption $fg=0$, we know that the sum of all coefficients of all these terms can be written as

$$a_u\sigma^{\alpha_u}(b_v)c_{\alpha_u,\beta_v}+\sum_{\alpha_{u'}+\beta_{v'}=\alpha_u+\beta_v} a_{u'}\sigma^{\alpha_{u'}}(\sigma'\text{s and }\delta'\text{s evaluated in}\, b_{v'})c_{\alpha_{u'},\beta_{v'}}=0. \quad (1)$$

By assumption we know that $a_pb_q=0$ for $p+q=m+t,m+t-1,\dots,k+1$. So, Lemma 3.2 (iii) guarantees that the product

$$a_p(\sigma'\text{s and }\delta'\text{s evaluated in } b_q) \qquad (\text{any order of } \sigma's \text{ and } \delta's)$$

is equal to zero. Then $[(\sigma'\text{s and }\delta'\text{s evaluated in } b_q)a_p]^2=0$ and hence we obtain the equality $(\sigma'\text{s and }\delta'\text{s evaluated in } b_q)a_p=0$ (R is reduced). In this way, multiplying (1) by a_k, and using the fact that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R,

$$a_u\sigma^{\alpha_u}(b_v)a_kc_{\alpha_u,\beta_v}+\sum_{\alpha_{u'}+\beta_{v'}=\alpha_u+\beta_v} a_{u'}\sigma^{\alpha_{u'}}(\sigma'\text{s and }\delta'\text{s evaluated in } b_{v'})a_kc_{\alpha_{u'},\beta_{v'}}=0, \quad (2)$$

whence, $a_u\sigma^{\alpha_u}(b_0)a_k=0$. Since $u+v=k$ and $v=0$, then $u=k$, so $a_k\sigma^{\alpha_k}(b_0)a_k=0$, i.e., $[a_k\sigma^{\alpha_k}(b_0)]^2=0$, from which $a_k\sigma^{\alpha_k}(b_0)=0$ and $a_kb_0=0$ by Lemma 3.2 (iv). Therefore, we now have to study the expression (1) for $0\le u\le k-1$ and $u+v=k$. If we multiply (2) by a_{k-1} we obtain

$$a_u\sigma^{\alpha_u}(b_v)a_{k-1}c_{\alpha_u,\beta_v}+\sum_{\alpha_{u'}+\beta_{v'}=\alpha_u+\beta_v} a_{u'}\sigma^{\alpha_{u'}}(\sigma'\text{s and }\delta'\text{s evaluated in } b_{v'})a_{k-1}c_{\alpha_{u'},\beta_{v'}}=0. \quad (3)$$

Using a similar reasoning as above, we can see that $a_u\sigma^{\alpha_u}(b_1)a_{k-1}c_{\alpha_u,\beta_1}=0$. Since A is bijective, $a_u\sigma^{\alpha_u}(b_1)a_{k-1}=0$, and using the fact $u=k-1$, we have $[a_{k-1}\sigma^{\alpha_{k-1}}(b_1)]=0$, which imply $a_{k-1}\sigma^{\alpha_{k-1}}(b_1)=0$, that is, $a_{k-1}b_1=0$. Continuing in this way we prove that $a_ib_j=0$ for $i+j=k$. Therefore $a_ib_j=0$ for $1\le i\le m$ and $1\le j\le t$.

The converse follows from Corollary 3.3. □

The next theorem is the important result of this paper. This result generalizes [1], Theorem 4, [9], Theorem 11, and [5], Theorem 2.9. Again, we suppose that the elements $c_{i,j}$ in Definition 2.1 (iv) are in the center of R.

Theorem 3.6. *If R is an Σ-rigid ring and A is a bijective skew PBW extension of R, then R is a right zip ring if and only if A is a right zip ring.*

Proof. Consider A a right zip ring and $Y \subseteq R$ with $r_R(Y) = \{0\}$. For an element $f \in r_A(Y)$ given by $f = a_0 + a_1X_1 + \cdots + a_mX_m$, we know that $bf = 0$, for all $b \in Y$ implies $ba_i = 0$, and hence $a_i \in r_R(Y) = \{0\}$, for every i. This means that $f = 0$ which shows that $r_A(Y) = \{0\}$. By assumption, A is right zip, so there exists a finite subset $Y_0 \subseteq Y$ such that $r_A(Y_0) = \{0\}$, and using that $r_R(Y_0) = r_A(Y_0) \cap R$, then $r_R(Y_0) = \{0\}$, which proves that R is a right zip ring.

Now, suppose that R is a right zip ring. Consider $T \subseteq A$ with $r_A(T) = \{0\}$ and let T' be the set of all coefficients of elements in T. Since $T' \subseteq R$, if $a \in r_R(T')$, then $ba = 0$ for all $b \in T'$. Then Remark 2.7 and Lemma 3.2 imply that $fa = 0$ for every $f = a_0 + a_1X_1 + \cdots + a_mX_m \in T$, whence $a \in r_A(T) = \{0\}$, i.e., $r_R(T') = \{0\}$. Using the assumption on R, there exists a finite subset $T_0 \subseteq T'$ such that $r_R(T_0) = \{0\}$. Now, note that for every $a \in T_0$, there exists an element $h_a \in T$ such that at least one of the coefficients of h_a is a. With this in mind, let Y be a minimal subset of T with $h_a \in T_0$ for each $a \in Y_0$. In this way, Y is a nonempty finite subset of T. Let Y' be the set of all coefficient elements in Y. It is clear that $T_0 \subseteq Y'$ and $r_R(Y') \subseteq r_R(T_0) = \{0\}$. If $f = a_0 + a_1X_1 + \cdots + a_mX_m \in r_A(Y)$, then $gf = 0$ for every element $g = b_0 + b_1Z_1 + \cdots + b_tZ_t \in Y$. Since R is Σ-rigid, then $b_ia_j = 0$ for all $0 \leq i \leq t,\ 0 \leq j \leq m$, by Proposition 3.5. Therefore $a_j \in r_R(Y') = \{0\}$ for all j, that is, $f = 0$, which proves that $r_A(Y) = \{0\}$, that is, A is a right zip ring. □

Remark 3.7. As consequences of Proposition 3.6, we have the following results:

- If R is an α-rigid ring in the sense of [11], then R is a right zip ring if and only if $R[x;\sigma]$ is a right zip ring ([1], Corollary 5).
- If R is a reduced ring, then R is a right zip ring if and only if $R[x]$ is a right zip ring ([1], Corollary 6).

As a final part of this paper, we consider the notion of reversible rings. Note that for any ring B, if the Ore extension $B[x;\alpha,\delta]$ is a reversible ring, then B is also invertible. However, the converse is false as the following example shows.

Example 3.8 ([1], Example 14)**.** Let $B = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and consider $\alpha : B \to B$ defined by $\alpha((a,b)) = (b,a)$ for any $a, b \in \mathbb{Z}_3$. Note that α is an automorphism of B, B is reversible and is not α-rigid. If $f = (1,0) + (0,1)x,\ g = (0,1) + (0,1)x \in B[x;\alpha]$, then

$$fg = (1,0)(0,1) + ((1,0)(0,1) + (0,1)(1,0))x + (0,1)(1,0)x^2 = (0,0)$$

and

$$gf = (0,1)(1,0) + ((0,1)(0,1) + (0,1)(0,1))x + (0,1)(1,0)x^2 = (0,1)x \neq 0$$

which shows that $B[x;\alpha]$ is not reversible.

From Example 3.8 we can see that the condition of rigidness on R is not superfluous. This motivates the following result.

Proposition 3.9. *If R is an Σ-rigid ring and A is a bijective skew PBW extension, then R is reversible if and only if A is reversible.*

Proof. If A is reversible, so is R. Conversely, since R is Σ-rigid, then A is a reduced ring (Proposition 3.4), and hence A is reversible. $\square$

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