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Nilcompactifications and Finite Characteristic

Nilcompactaciones y característica finita

Lorenzo Acosta G.[1,a], **I. Marcela Rubio P.**[1,b]

Abstract. Nilcompactification is a method of compactification for prime spectra of commutative rings. In this paper we study the behaviour of nilcompactification in the context of rings of finite characteristic.

Keywords: Prime ideal, prime spectrum, spectral compactness, compactification, characteristic.

Resumen. La nilcompactación es un método de compactación de espectros primos de anillos conmutativos. En este artículo estudiamos el comportamiento de la nilcompactación, en el contexto de los anillos de característica finita.

Palabras claves: Ideal primo, espectro primo, compacidad espectral, compactación, característica.

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1. Introduction

In this paper we only consider commutative rings, but not necessarily with identity. The prime spectrum of a commutative ring S, usually denoted $SpecS$, is the set of its prime ideals endowed with the Zariski topology. In this topology the basic open sets are defined by:

$$D(a) = \{I \in SpecS : a \notin I\},$$

where $a \in S$. The closed sets of the space $SpecS$ are:

$$V(I) = \{P : P \text{ is a prime ideal of } S \text{ and } P \supseteq I\},$$

where I is an ideal of S (see [7]).

A ring whose prime spectrum is compact is a *spectrally compact ring.* We say that R is an *i-extension* of S, if R is a ring that contains S as ideal. For

[1]Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
[a]lmacostag@unal.edu.co
[b]imrubiop@unal.edu.co

instance $\mathbb{R}$ is an extension of $\mathbb{Z}$, but it is not an i-extension; whereas $\mathbb{Z}$ is an i-extension of $2\mathbb{Z}$.

Theorem 2.2 of [4] says that if R is an i-extension of the ring S, the function $\psi_R : SpecS \to Spec_S R$ defined by $\psi_R(I) = \{x \in R : xS \subseteq I\}$ is a homeomorphism with inverse φ defined by $\varphi(J) = J \cap S$, where $Spec_S R$ is the subspace of $SpecR$ whose elements are the prime ideals of R that not contain S. The function ψ_R can be defined for every ideal of S, not only for prime ideals; and ψ_R respects the inclusion order (see [5]).

As a consequence of this homeomorphism, if R is spectrally compact, then naturally $SpecR$ contains a compactification of $SpecS$, namely $\overline{Spec_S R}$. Example 4.1 of [4] shows that in general, this compactification does not coincide with $SpecR$.

In [2] we obtained that if R is a spectrally compact i-extension of S and $N(S)$ is the nilradical of S, then $\overline{Spec_S R}$ and $Spec(R/\psi_R(N(S)))$ are homeomorphic. We say that $Spec(R/\psi_R(N(S)))$ is the $R-$**nilcompactification** of $SpecS$. So the $R-$nilcompactification of $SpecS$ is a spectral space (see [10]). In general, the points of $\overline{Spec_S R} \setminus Spec_S R$ are the prime ideals of R that contain $S \cup \psi_R(N(S))$.

Given a ring S, in order to obtain a nilcompactification of $SpecS$ we need to find a spectrally compact i-extension R of S. Thus, in particular, it is enough to find a unitary i-extension R of S, because it is well known that every unitary ring is spectrally compact. With this in mind, we define the category $\mathcal{E}$ whose objects are pairs (S, R), where R is a unitary i-extension of the ring S and whose morphisms from (S, R) to (S_1, R_1) are homomorphisms of unitary rings $h : R \to R_1$ such that $h(S) = S_1$. It is easy to verify that these morphisms so defined, with the usual composition of functions, satisfy the conditions to determine a category.

We define a functor Q from the category $\mathcal{E}$ to the category of unitary rings. If R is a unitary i-extension of the ring S, then $Q(S, R) = R/\psi_R(N(S))$, which is a unitary ring. If $h : R \to R_1$ is a morphism of $\mathcal{E}$ between (S, R) and (S_1, R_1), then $Q(h) : Q(S, R) \to Q(S_1, R_1)$ is defined by $Q(h)(r + \psi_R(N(S))) = h(r) + \psi_{R_1}(N(S))$. So defined, Q is a covariant functor between the mentioned categories.

Now, let NC be the functor $Spec \circ Q : \mathcal{E} \to Top$. NC is a contravariant functor and if R is a unitary i-extension of the ring S, then $NC(S, R) = Spec(R/\psi_R(N(S)))$ is the $R-$nilcompactification of $SpecS$. We denote λ_R the inclusion of $SpecS$ in $NC(S, R)$.

The following result is taken from [2] and it will be useful in this work.

Theorem 1.1. *Let R and T be two unitary i-extensions of the ring S. If there exists a **surjective** morphism $\rho : R \to T$ of the category $\mathcal{E}$, that leaves fixed S, that is, $h|_S = 1_S$, then $Q(R) \cong Q(T)$ and therefore,*[1]

$$NC(R) \approx NC(T).$$

[1]With $\cong$ and $\approx$ we denote the isomorphism relationship between rings and the homeomorphism relationship between topological spaces, respectively.

2. Nilcompactifications and adjunction of identity

A natural way to obtain spectrally compact i-extensions of a given ring is through a process of adjunction of identity. Perhaps for this reason, rings without identity have been many times simply considered as ideals of unitary rings. This is one of the reasons why it is not usual to find considerations about rings without identity in the literature. Notice that, although some true results for unitary rings are true for rings without identity, the proofs that do not use the existence of identity generally require different considerations. Anderson reviewed several papers that study rings without identity, and he shows in [6] that, in general, the lack of identity is not resolved simply adjoining one. On the other hand, not every true result for unitary rings is true for rings without identity. In [9] Gilmer presents eleven conditions that, being equivalent in unitary rings, they are not when we work with rings without identity.

In this paper, we use the process of adjunction of identity in order to obtain nilcompactifications. Although there are different ways to adjoint identity to a ring, we follow the method described in [13] for $K-$algebras, which is a generalization of the way found in [8] and [11].

Definition 2.1. Let K be a commutative unitary ring. We say that S is a $K-$*algebra* if:

1. $(S,+,.)$ is a commutative ring.
2. $(S,+)$ is a $K-$module.
3. $a\,(\alpha b) = (\alpha a)\,b = \alpha\,(ab)$, for all $a, b \in S$ and all $\alpha \in K$.

If in addition, the ring $(S,+,.)$ has identity, then S is a $K-$*algebra with identity.*

If S, S_1 are two $K-$algebras, then $h : S \to S_1$ is a *homomorphism* of $K-$algebras if it is a homomorphism of rings that respects the multiplication by scalar, so $h\,(\alpha a) = \alpha h\,(a)$, for all $\alpha \in K$ and all $a \in S$.

Let K be a unitary ring and let S be a $K-$algebra. We denote by $U_K(S)$ or simply $U(S)$, if there is no confusion, the set $S \times K$ endowed with addition defined componentwise, multiplication defined by

$$(a,\alpha)(b,\beta) = (ab + \beta a + \alpha b, \alpha\beta)$$

and product by elements of K defined by

$$\beta(a,\alpha) = (\beta a, \beta\alpha).$$

It is easy to verify that $U(S)$ is a $K-$algebra with identity $(0,1)$ and that we have the following universal property.

$$\begin{tikzcd} S \arrow[r, "i_S"] \arrow[dr, "h"'] & U(S) \arrow[d, "\exists!\ \widetilde{h}"] \\ & B \end{tikzcd}$$

Proposition 2.2. *For each unitary $K-$algebra B and for each homomorphism of $K-$algebras $h : S \to B$ there exists a unique homomorphism of unitary $K-$algebras $\widetilde{h} : U(S) \to B$ such that $\widetilde{h} \circ i_S = h$, where $i_S : S \to U(S) : i_S(a) \mapsto (a, 0)$.*

Proof. It is enough to see that $\widetilde{h}(a, \alpha) = h(a) + \alpha 1$, where 1 is the unity of B. $\square$

If we denote S_0 the set $S \times \{0\}$, the homomorphism i_S of $K-$algebras, allows us to identify the ring S with S_0 in $U(S)$. Besides, clearly S_0 is an ideal of $U(S)$. Notice that for each homomorphism $g : S \to B$ in the category of $K-$algebras it is possible to define $U(g) = \widetilde{i_B \circ g}$. Through this definition U is a functor, and by the previous proposition, U is left adjoint of the inclusion functor from the category of unitary $K-$algebras to the category of $K-$algebras.

If S is a $K-$algebra, it is known that S, seen as S_0, is an ideal of $U_K(S)$, thus, as a consequence of the results of the previous section, we establish the following fact.

Corollary 2.3. *If S is a $K-$algebra then:*

1. *$NC(U_K(S))$ is the $U_K(S)-$nilcompactification of $SpecS$.*
2. *$Q(S, U_K(S))$ as a semiprime ring.*

In the conditions of this corollary, we say that $NC(U_K(S))$ is the U_K-nil-compactification of $SpecS$, for short.

In particular, there exists a standard procedure to include naturally the ring S without identity, in another one with identity. In this process we take the ring $\mathbb{Z}$ in the place of K, because every ring is a $\mathbb{Z}-$algebra. So $U_{\mathbb{Z}}(S)$ is a ring with identity $(0, 1)$ that contains the ring S as ideal. The universal property remains valid and it is expressed as follows:

Proposition 2.4. *For each unitary ring B and for each homomorphism of rings $h : S \to B$ there exists a unique homomorphism of unitary rings $\widetilde{h} : U_{\mathbb{Z}}(S) \to B$ such that $\widetilde{h} \circ i_S = h$, where $i_S : S \to U_{\mathbb{Z}}(S) : i_S(a) \mapsto (a, 0)$.*

However, the construction of $U_{\mathbb{Z}}(S)$ does not respects the characteristic of the ring S, if it is different from zero. We denote $U_{\mathbb{Z}}(S)$ as $U_0(S)$, to remember that this ring always has zero characteristic, independently of the characteristic of the ring S. In the case of rings of characteristic $n \neq 0$, it is possible to take

$\mathbb{Z}_n$ in the place of K, because those rings are $\mathbb{Z}_n-$algebras. In this case, we denote the ring $U_{\mathbb{Z}_n}(S)$ as $U_n(S)$ for short. The previous results are also true, but the ring $U_n(S)$ has characteristic n and the universal property is restricted to rings of characteristic n.

The following proposition shows a particular case of spectrally compact rings whose prime spectra coincide with its U_0-nilcompactification.

Proposition 2.5. *If S is a unitary ring and $N(S) = 0$, then*

$$Q(S, U_0(S)) \cong S$$

and thus,

$$NC(S, U_0(S)) \approx Spec(S).$$

Proof. As S is a unitary ring, by the universal property of U_0 (Proposition 2.4), there exists a unique homomorphism of unitary rings $h : U_0(S) \to S$, that behaves as the identity in S and it is defined by $h(a, \alpha) = a + \alpha 1$. Clearly this homomorphism is surjective and its kernel is $\psi_0(0)$. $\square$

Proposition 2.6. *If S is a ring without identity, then $NC(U_0(S))$ is a compactification of $SpecS$ with additional points.*

Proof. It is enough to consider the case $N(S) = 0$.

$NC(U_0(S))$ is homeomorphic to $\overline{Spec_S(U_0(S))}$ and

$$\begin{aligned}
& \overline{Spec_S(U_0(S))} \\
= \; & V(\psi_0(0)) \\
= \; & \{J \in Spec(U_0(S)) : J \supseteq \psi_0(0)\} \\
= \; & Spec_S(U_0(S)) \cup \{J \in Spec(U_0(S)) : J \supseteq \psi_0(0) \text{ and } J \supseteq S_0\}.
\end{aligned}$$

Consider $\pi : U_0(S) \to \mathbb{Z} : (a, \alpha) \mapsto \alpha$ and $W = \pi(\psi_0(0))$. It is clear that W is an ideal of $\mathbb{Z}$. If $W = \mathbb{Z}$, there exists $u \in S$ such that $(u, -1) \in \psi_0(0)$ and therefore, for every $a \in S$ we have that $au - a = 0$. We conclude that u is the identity of S, which is absurd. Thus, W is a proper ideal of $\mathbb{Z}$ and it is contained in at least one prime ideal P of $\mathbb{Z}$. By the Correspondence theorem, $\pi^{-1}(P)$ is a prime ideal of $U_0(S)$ that contains S_0. It is clear that $\pi^{-1}(P)$ also contains $\psi_0(0)$. Therefore, $\pi^{-1}(P) \in V(\psi_0(0)) - Spec_S(U_0(S))$ and $NC(U_0(S))$ is a compactification of $Spec(S)$ with additional points. $\square$

Remark 2.7. If $W = 0$ in the proof of the previous proposition then, the compactification has a countably infinite number of additional points. If $W = n\mathbb{Z}$, where $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, its factorization in prime numbers, the compactification has exactly m additional points.

3. Nilcompactifications and rings of non-zero characteristic

In this section A is a ring of non-zero characteristic n, where $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ is the decomposition of n in prime factors. Thus, A is a $\mathbb{Z}-$algebra and a $\mathbb{Z}_{tn}-$algebra for each natural t, so that it produces a nosegay of nilcompactifications of $SpecA$. In this section we study the relationship between these nilcompactifications.

3.1. Relationship between $U_{tn}(A)$ and $U_0(A)$

We use the notations ψ_n and ψ_0 for the inclusions of $SpecA$ in $Spec\,(U_n(A))$ and in $Spec\,(U_0(A))$, respectively. Consider $\theta_n : \mathbb{Z} \to \mathbb{Z}_n$ the canonical homomorphism and the homomorphism $\rho : U_0(A) \to U_n(A)$ defined by $\rho(a, \alpha) = (a, \theta_n(\alpha))$.

Notice that $\psi_0(N(A)) = \{(a, \alpha) \in U_0(A) \mid ax + \alpha x \in N(A),\ \forall x \in A\}$ and $\psi_n(N(A)) = \{(a, \alpha) \in U_n(A) \mid ax + \alpha x \in N(A),\ \forall x \in A\}$, so it is simple to verify that $\{0\} \times n\mathbb{Z} \subseteq \psi_0(N(A))$ and the equivalence between $(a, \alpha) \in \psi_0(N(A))$ and $(a, \alpha + kn) \in \psi_0(N(A))$ for all $k \in \mathbb{Z}$, because n is the characteristic of A. Besides, $\rho(\psi_0(N(A))) = \psi_n(N(A))$. The homomorphism ρ satisfies the conditions of Theorem 1.1. In general, a ring A of characteristic $n \neq 0$ is also a $\mathbb{Z}_{tn}-$algebra for each natural t and $\rho_t : U_0(A) \to U_{tn}(A)$ defined by $\rho_t(a, \alpha) = (a, \theta_{tn}(\alpha))$ is a surjective homomorphism of unitary rings that leaves fixed A_0. Then, by Theorem 1.1, it follows the result that we summarize in the following proposition.

Proposition 3.1. *If A is a ring of characteristic $n \neq 0$, then*

$$Q(U_0(A)) \cong Q(U_{tn}(A)), \text{ for each natural } t$$

and thus,

$$NC(U_0(A)) \approx NC(U_{tn}(A)), \text{ for each natural } t.$$

3.2. Decomposition of a ring according to its characteristic

We see that every commutative ring of characteristic non-zero, with at least two different prime divisors, can be decomposed as a product of rings with special characteristics.

Theorem 3.2. *If A is a ring of characteristic $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, where the p_i are different primes, then there exist rings $A_1, \dots, A_m$ such that $A \cong \prod_{i=1}^{m} A_i$*

and $char A_i = p_i^{\alpha_i}$, *for each* $i = 1, \dots, m$. *Besides, this decomposition is unique up to isomorphism.*[2]

Proof. It is enough to define $A_i = \{x \in A \mid p_i^{\alpha_i} x = 0\}$, for each i. $\square$

Definition 3.3. Let A be a ring of characteristic $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, where the p_i are different primes. We say that the decomposition of A described in the previous theorem is the **characteristic decomposition** of A.

Corollary 3.4. *If A is a ring of characteristic $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$, where the p_i are different primes and $\prod_{i=1}^{m} A_i$ is its characteristic decomposition, then A_i is an ideal of A, for each $i = 1, \dots, m$.*

3.3. Observations on nilcompactifications

Consider $\prod_{i=1}^{m} A_i$ the characteristic decomposition of the ring A. We have the following results.

Proposition 3.5. *If $R = \prod_{i=1}^{m} U_{p_i^{\alpha_i}}(A_i)$, then*

$$Q(U_n(A)) \cong Q(R).$$

Proof. $U_n(A)$ and R are unitary i-extensions of A. Consider the homomorphism of unitary rings $\rho : U_n(A) \to R$ defined by $\rho\left((a_i)_{i=1}^m, \alpha\right) = \left((a_i, [\alpha]_{p_i^{\alpha_i}})\right)_{i=1}^{m}$.

Using the isomorphism of unitary rings between $\mathbb{Z}_n$ and $\mathbb{Z}_{p_1^{\alpha}1} \times \cdots \times \mathbb{Z}_{p_m^{\alpha_m}}$ it is proved that ρ is surjective. Besides, $\rho\mid_A = 1_A$ thus, by Theorem 1.1 we have the desired isomorphism. $\square$

Proposition 3.6. *If $R = \prod_{i=1}^{m} U_{p_i^{\alpha_i}}(A_i)$, then*

$$Q(R) \cong \prod_{i=1}^{m} Q\left(U_{p_i^{\alpha_i}}(A_i)\right).$$

Proof. Notice that $N(A) = N\left(\prod_{i=1}^{m} A_i\right) = \prod_{i=1}^{m} N(A_i)$, because the characteristic decomposition of A has finite factors. Besides, it is clear that $\psi_R(N(A)) = \prod_{i=1}^{m} \psi_{p_i^{\alpha_i}}(N(A_i))$.

[2]The part of theorem corresponding to the decomposition of A as a product with two factors is an exercise in [12]. However, there is not observed which is the characteristic of the rings, nor that this decomposition is unique.

For each $i = 1, \ldots, m$ we define $R_i = \prod_{j=1}^{m} T_j$, where

$$T_j = \begin{cases} U_{p_j^{\alpha_j}}(A_j) & , \text{ if } j \neq i \\ \psi_{p_i^{\alpha_i}}(N(A_i)) & , \text{ if } j = i \end{cases}.$$

The rings R_i are pairwise comaximal ideals of R.

Thus, by the Chinese Remainder Theorem it follows that $R_1 \cdots R_m = \bigcap_{i=1}^{m} R_i = \prod_{i=1}^{m} \psi_{p_i^{\alpha_i}}(N(A_i))$ and

$$\begin{aligned} R/\left(\prod_{i=1}^{m} \psi_{p_i^{\alpha_i}}(N(A_i))\right) &\cong \prod_{i=1}^{m} (R/R_i) \\ &\cong \prod_{i=1}^{m} \left(U_{p_i^{\alpha_i}}(A_i)/\psi_{p_i^{\alpha_i}}(N(A_i))\right) \\ &= \prod_{i=1}^{m} Q\left(U_{p_i^{\alpha_i}}(A_i)\right), \end{aligned}$$

as we wanted to prove. □

Corollary 3.7. *If A is a ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the p_i are different primes and $\prod_{i=1}^{m} A_i$ is it characteristic decomposition, then*

$$Q(U_n(A)) \cong \prod_{i=1}^{m} Q\left(U_{p_i^{\alpha_i}}(A_i)\right),$$

and therefore,

$$\begin{aligned} NC(U_n(A)) &\approx Spec\left(\prod_{i=1}^{m} Q\left(U_{p_i^{\alpha_i}}(A_i)\right)\right) \\ &\approx \coprod_{i=1}^{m} NC\left(U_{p_i^{\alpha_i}}(A_i)\right), \end{aligned}$$

thus, if $m > 1$, this spectrum is disconnected, with at least m connected components.

Proposition 3.8. *If A is a non spectrally compact ring such that $char A = p^n$, where p is prime, then $NC(U_{p^n}(A))$ is a compactification of $SpecA$ by one point.*

Proof. It is enough to see that $\theta : U_{p^n}(A) \to \mathbb{Z}_{p^n}$ is a surjective homomorphism and to apply the Correspondence theorem. □

We have that $NC(U_n(A))$ is a compactification of $SpecA$ by at most m points. The number of additional points in this compactification is precisely m, if none of the factors A_i in the characteristic decomposition of A, is spectrally compact. That follows because the topological sum of a finite number of spaces is compact if and only if each summand space is compact. In the following section we present an example illustrating these observations.

Now we can state the following results.

Corollary 3.9. *If A is a ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the p_i are different primes and $\prod_{i=1}^{m} A_i$ is it characteristic decomposition, then $NC\left(U_n\left(A\right)\right)$ is a compactification of $SpecA$ by m points if each ring A_i is not spectrally compact, for $i = 1, \ldots, m$.*

Corollary 3.10. *If A is a non spectrally compact ring of non-zero characteristic, with at least two prime divisors, then there exists at least one disconnected compactification by finite points of $SpecA$.*

Remark 3.11. If A is a ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the p_i are different primes and $\prod_{i=1}^{m} A_i$ is it characteristic decomposition, then

1. the characteristic decomposition of $U_n\left(A\right)$ is $\prod_{i=1}^{m} U_{p_i^{\alpha_i}}\left(A_i\right)$. ($\rho$ of Proposition 3.5 also is injective);
2. $Spec\left(U_n\left(A\right)\right) \approx \coprod_{i=1}^{m} Spec\left(U_{p_i^{\alpha_i}}\left(A_i\right)\right)$;
3. if besides, A_i is not spectrally compact for each i, then
   (a) $Spec\left(U_n\left(A\right)\right)$ is a compactification of $SpecA$ by exactly m points;
   (b) $Spec\left(U_n\left(A\right)\right)$ has at least m compact connected components.

An study of these facts, in the particular case of Von Neumann regular rings, is presented in [3].

4. Some examples

In this section we present different examples that illustrate the results of the previous sections.

Example 4.1. Consider the ring $A = B \times \mathbb{Z}_3$, where B is a Boolean ring without identity. A is a ring of characteristic 6, with zero divisors and besides, A, B and $\mathbb{Z}_3$ are semiprime.

By Proposition 3.1, we have that $NC\left(U_0\left(A\right)\right) = Spec\left(U_0\left(A\right)/\psi_0\left(0\right)\right) \approx Spec\left(U_6\left(A\right)/\psi_6\left(0\right)\right)$, where $\psi_6\left(0\right) = \left\{\left(0,0,0\right), \left(0,1,2\right), \left(0,2,4\right)\right\}$. $U_6\left(A\right)$ has only two prime ideals that contain A : $A \times 2\mathbb{Z}_6$ and $A \times 3\mathbb{Z}_6$, but $A \times 3\mathbb{Z}_6$ does not contain $\psi_6\left(0\right)$, then $NC\left(U_6\left(A\right)\right) = Spec\left(U_6\left(A\right)/\psi_6\left(0\right)\right)$ is a compactification of $SpecA$ by one point and the additional point is $A \times 2\mathbb{Z}_6$.

On the other hand, as $B\times\mathbb{Z}_3$ is the characteristic decomposition of A, then, by Corollary 3.7:

$$\begin{aligned} NC\left(U_6\left(A\right)\right) &\approx NC\left(U_2\left(B\right)\right)\coprod NC\left(U_3\left(\mathbb{Z}_3\right)\right)\\ &\approx Spec\left(U_2\left(B\right)\right)\coprod Spec\left(\mathbb{Z}_3\right), \end{aligned}$$

because $\psi_2\left(0\right)=0$ and $\mathbb{Z}_3$ is semiprime and has identity (Proposition 2.5). Besides, $Spec\left(U_2\left(B\right)\right)\approx\left(SpecB\right)^*$, the Alexandroff compactification of $SpecB$ (see [1]). Thus, the additional point of $NC\left(U_6\left(A\right)\right)$ is adherent to $SpecB$.

Now we use the Boolean rings and the 3$-$rings[3] in order to present an example of a compactification by two points.

Lemma 4.2. *If T is a non null ring without identity, then for each $t\in T$, there exists $x\in T$ such that $xt+x\neq 0$.*

Proof. Let $t\in T$. Suppose that $xt+x=0$ for all $x\in T$; then $x=\left(-t\right)x$ for all $x\in T$, so $-t$ is the identity of T, which is contradictory. □

Lemma 4.3. *If T is a ring of characteristic 3 without identity, then for each $t\in T$, there exists $x\in T$ such that $xt+2x\neq 0$.*

Proof. Let $t\in T$. Suppose that $xt+2x=0$ for all $x\in T$, so that $xt=-2x$, for all $x\in T$. As T is of characteristic 3, then $xt=x$ for all $x\in T$, so t is the identity of T, which is absurd. □

Lemma 4.4. *If T is a 3$-$ring without identity, then T is semiprime and $\psi_3\left(0\right)=0$.*

Proof. As $b^3=b$, for all $b\in T$, then T is semiprime. Take $\left(t,\gamma\right)\in U_3\left(T\right)$, $\left(t,\gamma\right)\in\psi_3\left(0\right)$ is equivalent to $\left(t,\gamma\right)\left(x,0\right)=\left(0,0\right)$ for all $x\in T$, that is, $tx+\gamma x=0$ for all $x\in T$.

i) If $\gamma=0$, then $tx=0$, for all $x\in T$, so that $t=0$. If we take $x=t^2$, we conclude that $t=0$.

ii) If $\gamma=1$, then $tx+x=0$, for all $x\in T$, which is contradictory, by Lemma 4.2.

iii) If $\gamma=2$, then $tx+2x=0$, for all $x\in T$, which is contradictory, by Lemma 4.3.

□

Example 4.5. Consider $A=B\times T$, where B is a Boolean ring without identity and T is a non spectrally compact 3$-$ring.

[3]A 3$-$ring is a ring of characteristic 3 in which every element b is equal to b^3.

As A is a ring of characteristic 6 and $B \times T$ is its characteristic decomposition, then:

$$\begin{aligned} NC\left(U_6\left(A\right)\right) &\approx NC\left(U_2\left(B\right)\right) \coprod NC\left(U_3\left(T\right)\right) \\ &\approx Spec\left(U_2\left(B\right)\right) \coprod Spec\left(U_3\left(T\right)\right), \end{aligned}$$

because, $\psi_2\left(0\right) = 0$ and, by the previous lemma, $\psi_3\left(0\right) = 0$.

$Spec\left(U_2\left(B\right)\right)$ is the Alexandroff compactification of $SpecB$ and $Spec\left(U_3\left(T\right)\right)$ is a compactification of $SpecT$ by one point. Therefore, $NC\left(U_6\left(A\right)\right)$ is a compactification of $SpecA$ by two points.

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