REVISTA DE ECONOMÍA, Vol. 24, Nº 2, Noviembre 2017. ISSN: 0797-5546

PRICING INTEREST RATE DERIVATIVES: AN APPLICATION TO THE URUGUAYAN MARKET

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Fecha de postulación: julio de 2017 Fecha de aceptación: noviembre de 2017

RESUMEN

En los últimos años, la volatilidad del sistemas financiero internacional se ha convertido en una seria amenaza para los tenedores de bonos. Esto motiva la idea de la introducción de derivados de cobertura en el mercado uruguayo. El objetivo de este trabajo es desarrollar una metodología que permita de forma sencilla la valoración de derivados de renta fija. El modelo Black-Derman-Toy (BDT) es un modelo popular de un factor para la tasa de interés que es ampliamente utilizado por los profesionales. Una de sus ventajas es que el modelo se puede calibrar tanto para la estructura de la tasa de interés actual del mercado como para la estructura actual de las volatilidades. La dinámica de las tasas de interés se aproxima a mediante un árbol binomial, en el que el precio justo de cualquier valor se calcula como el valor presente esperado libre de riesgo mediante inducción hacia atrás. Para este trabajo se utilizan las curvas de rendimiento diarias en UI proporcionadas por BEVSA para el 30/06/2017 con vencimientos de hasta 10 años. Finalmente, se presenta el precio de varios derivados financieros, tales como Swap, Swaption, Floor, Cap, Futuros, Forward y Opciones y la relación entre ellos.

Palabras clave: Modelos de tasas de interés, tasa instantánea, Black-Derman-Toy, árbol Binomial, precios Martingala, fijación de precios de derivados de tasas de interés, finanzas cuantitativas.

Clasificación JEL: G12, G13.

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1 He thanks comments done by an anonymous referee of Revista de Economia. All remaining errors are the author`s.

ABSTRACT

In recent years, the volatility of the international financial system has become a serious threat to bondholders. This motivates the idea of the introduction of hedging derivatives in the Uruguayan market. The aim of this work is to develop a methodology that allows in a simple way the valuation of fixed income derivatives. The Black-Derman-Toy (BDT) model is a popular one-factor interest rate model that is widely used by practitioners. One of its advantages is that the model can be calibrated to both the current market term structure of interest rate and the current term structure of volatilities. The dynamic of the interest rates is approximated with a binomial lattice, in which the fair price of any security is calculated as the present value of the risk-neutral expectation by backward induction. For this work is use the daily yield curves in UI released by BEVSA for the 6/30/2017 with maturities up to 10 years. Finally, it is presented the price of several financial derivatives, such as Swap, Swaption, Floor, Cap, Futures, Forward and Options and a relation between them.

Keywords: Interest Rate Models, Short-Rate, Black-Derman-Toy, Binomial Lattice, Martingale Pricing, Pricing Interest Rate Derivatives, Quantitative Finance.

JEL Classification: G12, G13.

1. INTRODUCTION

According to a report, issued by the Ministry of Economy and Finance of Uruguay in April 2017, the total amount of the Uruguayan debt was 26275 million of USD in March 2017. This debt was mainly (91%) composed by bonds issued by the government. Only the 26% of this debt is under Uruguayan jurisdiction, the remaining debt being mainly under USA jurisdiction (i.e. corresponds to bonds issued by the Uruguayan government in USD markets and through USA financial institutions). It is important to notice that the 47% of the total debt is nominated in the local currency (mainly CPI-Indexed “UI”), meanwhile the remaining 53% is nominated in foreign currencies (USD, Euros and Yens). It should be mentioned that this 53%-47% of foreign vs. local currency debt is the result of a Central Bank of Uruguay (BCU) policy to un-dollarize the debt: in 2005 this proportion was 88%-12% (foreign vs. local currencies).

Regarding the holders of this debt, the main participants of the local bond markets are the pension funds (AFAP) created by law in the country in 1996. As a result of legal regulation, these institutions, according to data of March 2017, hold a portfolio of more than 65% invested in Uruguayan bonds, from a total turnover of 13432 million of USD. In short, approximately the 30% of the Uruguayan bonds integrate the AFAP’s portfolios (for details see [18]). The market for bond hedging instrument are poorly developed, specially taking into account the dependence structure of an interconnected international financial system. In the Uruguayan derivatives market, the underlying is mainly the currency, due to the relevance of the exchange rate risk present in the economy.

As an example of possible hedging uses of derivatives in the bond market, in mid-2013, as a consequence of a FED announcement, the yield curve of inflation-indexed bonds estimated by the Electronic Stock Exchange of Uruguay (BEVSA) had a significant increase, corresponding to an important drop in all the bond prices. For instance, the movement in the 10 year UI yield bond, was larger than 2%. This situation motivated a large loss for corporate investment institutions, as pension funds, with a cost of approximately the 5% of the total portfolio. This type of movements in the yield curve induce undesired volatility with negative consequences in important areas as the social security, motivating the idea of the introduction of hedging derivatives into the bond market.

The purpose of this work is then to provide an adequate methodology for pricing interest rate derivatives in the Uruguayan market. The selected methodology is the simplest of all, the one-factor models. In the one-factor models, a single stochastic factor, the short rate, determines the future evolution of all interest rates. Then it present some result of the martingale pricing theory. The martingale pricing approach is a cornerstone of modern quantitative finance and can be applied to a variety of derivatives contracts, e.g. options, futures, interest rate derivatives, credit derivatives, etc.

In the literature [20] there are two different approaches to replicate the dynamics of the evolution of interest rates: the equilibrium methods and the no-arbitrage methods. The distinction between them comes from the input that is used to calibrate the parameters of the model. The models of equilibrium, including those proposed by Vasicek (1977) [21] and Cox, Ingersoll and Ross (1985) [7], are estimated from historical data and are assumed to be constant over time.

No-arbitrage models adjust the market price of the underlying assets to the price that the model provides. Some of these models are proposed by Ho and Lee (1986) [11], Hull and White (1990) [13], Black, Derman and Toy (1990) [2], Black and Karasinski (1991) [3]. This approach is preferred by professionals to value interest rate derivatives, since these ensure that the market prices of the bonds coincide with those provided by the model at the time of calibration. In contrast, equilibrium models do not accurately value such bonds.

The paper is structured as follows: first, it will present some fundamental financial definitions and a brief description of the Martingale Pricing Theory. After it will present the Binomial-Lattice Models and its calibration under the Black-Derman-Toy model. Then it will present the result of the study and end with the conclusion.

2. FUNDAMENTALS

Before beginning, some of the fundamentals that support the theory to be developed are defined in the next chapters.

Arbitrage – The model treated in this article assume no-arbitrage principle, in other words, the absence of arbitrage opportunities. Arbitrage opportunity means any of this two situations: (1) at the same time, a security is sold at different prices at different places; (2) with zero investment at time 0 there is no probability of loss but there is a possibility of a riskless profit at time 1.

Interest – The reward for both postponed consumption and the uncertainty of investment is usually paid in the form of interest.

Interest rate is a quantitative measure of interest expressed as a proportion of a sum of money in question (principal or present value) that is paid over a specified time period. There are two approaches, under simple interest model, only interest form principal is received at any period. Instead under compound interest model, the interest after each period is added to the previous principal and the interest for the next period is calculated from this increased value of the principal. This article assumes compound interest.

Spot Rates and Term Structure of interest rate – Spot rates are the basic interest rates that define the term structure. Defined on an annual basis, the spot rate s_t, is the rate of interest charged for lending money from today ($t = 0$) until time T. In particular, this implies that if you lend N dollars for t years from today, you will receive $N\,(1+ s_t/m)^{mt}$ dollars when the t years have elapsed. The term structure of interest rates may be defined to constitute the sequence of spot rates $\{s_k : k = 1,..,n\}$, if is a discrete-time model with n periods. Alternatively, in a continuous-time model the set $\{s_t : t \in [0,T]\}$ may be defined to constitute the term-structure. The spot rate curve is defined to be a graph of the spot rates plotted against time.

Discount Factors: there are discount factors corresponding to interest rates, one for each time t. The discount factor d_t, for period t is given by $d_t := 1 / (1 + s_t/m)^{mt}$.

Forward Rates and Short Rate – a forward rate $f_{t1,t2}$, is a rate of interest that is agreed upon today for lending money from dates t_1 to t_2 where t_1 and t_2 are future dates. It is easy using arbitrage arguments to compute forward rates given the set of spot rates. For example, if $j > i$

$$(1+s_j/m)^j = (1+s_i/m)^i\,(1+f_{i,j}/m)^{j-i}$$

The instantaneous forward rate f_{t1} describes the interest that is agreed upon today for lending money from an investment for an infinitesimal period after t_1

$$f_{t_1} = \lim_{\Delta t \to 0} f_{t_1, t_1 + \Delta t}$$

The short-rate r_t, is the instantaneous forward at time t, $r_t := f_t$. A short-rate model is developed in Section 5.

Securities – a financial security is a medium of investment in the money market or capital market like stocks, bonds, cash account, mortgages, financial derivatives like swaps, options, forwards, etc. In accounting terms, the holder (purchaser) of it has an asset while the issuer o borrower (seller) has a liability. Securities usually takes the form of an agreement (contract) between the seller and the purchaser providing an evidence of debt or of property. The holder of a security is called to be in a long position while the issuer is in a short position. The basic types of securities used in this article are listed below. See [8], [15], [17] and [22] for more details.

Cash account – A cash account is the simplest securities. It define $B(t)$ to be the value of a cash account at time $t \geq 0$. It assume $B(0) = 1$ and the cash account evolves according to the following differential equation:

$$dB(t) = r_t B(t)dt, \;\; B(0) = 1$$

where r_t is the short rate. As a consequence,

$$B(t) = e^{\left(\int_0^t r_s ds\right)}$$

The above definition tells us that investing a unit amount at time 0 yields at time t the value in $B(t)$, and r_t is the short rate at which the cash account accrues. Note that $B(t)$ is a strictly positive price process.

Coupon Bond – A coupon bond is the long-term (usually from 5 to 30 years) financial instrument issued by either central or local governments (municipals), banks or corporations. It is a debt security in which the issuer promises the holder to repay the principal or nominal value N at the maturity date and to pay (periodically, at equally spaced dates up to and including the maturity date) a fixed amount of interest C called coupon for historical reasons [8].

A typical period for the coupon payment is semiannual, rarely annual, but both the coupon and coupon rate are expressed on the annual base. The number of periods in a year *m*, is called frequency. In case of semiannually paid coupon, the frequency is 2. For example, the price in time *t* of a bond having face value of $100 carrying coupon rate of 6% to be paid semiannually and maturing in exactly 10 years is

$$P(t, 10, 6\%) = \sum_{i=1}^{i=20} 3\, d_{i/2} + 103\, d_{10}$$

Zero Coupon Bonds – or discount bond pays only the principal or nominal value *N* at maturity. A coupon bond may be considered as a series of zero coupon bonds, all with nominal value equal to the coupon payment, but the last with nominal equal to the coupon payment plus the nominal of the underlying coupon bond. For example, the price in time *t* of a zero coupon bond having face value of $100 and maturing in exactly 2 years is $P(t,2) = 100\, d_2$.

Financial Derivatives – financial derivative securities or contingent claims are the instruments where the payment of either party depends on the value of an underlying asset or assets. The underlying assets in question may be of a rather general form, e.g. stocks, bonds, commodities, currencies, stock exchange index, interest rate, and even derivatives themselves. The basic types of financial derivatives used in this article are listed below.

Swap – is an agreement between two parties to exchange cash flows in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated. Usually the calculation of the cash flows involves the future value of an interest rate, an exchange rate, or other security. The most common type of swap is a plain vanilla interest rate swap. In this swap a party agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time. The principal itself is not exchanged. The principal is the same for both the fixed and floating payments. If the principal were exchanged at the end of the life of the swap, the nature of the deal would not be changed in any way. For both parties this agreement is an obligation. Swap are typically a privately negotiable agreement and it is not traded on exchanges. The swap agreement is a risky investment from two reasons, at least. First reason, since the two parties agree to exchange cash flows in the future, the loss of one party equals the profit of the counterparty and vice versa. The second reason is the default risk in which case one party is not willing to provide the cash flow to the other party.

Forwards and Futures – A forward contract is an agreement between two parties, a buyer and a seller, such that the seller undertakes to provide the buyer a fixed particular underlying security (like a bond, a amount of the currency or commodity) at a fixed future date called delivery date for a fixed price called delivery price agreed today, at the beginning of the contract. For both parties this agreement is an obligation. By fixing the price today the buyer is protected against price increase while the seller is protected against price decrease. Forward is typically a privately negotiable agreement and it is not traded on exchanges. Like Swaps, the forward contract is a risky investment from at least two reasons. First reason, since the spot price of the underlying asset generally differs from the delivery price, the loss of one party equals the profit of the counterparty and vice versa. The second reason is the default risk in which case the seller is not willing to provide the buyer with delivery.

A futures contract shortly futures, is of a similar form as the forward but it has additional features. The futures is standardized (specified quality and quantity, prescribed delivery dates dependent on the type of the underlying asset). The futures are traded on exchanges. One of the most popular is Chicago Board of Trade (CBT). To reduce the default risk to minimum, both parties in a futures must pay so called margins (initial margin, maintenance margin). These margins serve as reserves and the account of any party in the contract is daily recalculated according to the actual price of the futures, the futures price.

Options – An option is a contract giving its owner (holder, buyer) the right to buy or sell a specified underlying asset at a price fixed at the beginning of the contract (today) at any time before or just on a fixed date. The seller of an option is also called writer. It must be emphasized that an option contract gives the holder a right and not an obligation as was the case of futures. For the writer, the contract has a potential obligation. He must sell or buy the underlying asset accordingly to the holder's decision. A call option (CALL) is the right to buy and a put option (PUT) is the right to sell.

The fixed date of a possible delivery is called expiry or maturity date. The price fixed in the contract is called exercise or strike price. If the right is imposed it is said that the option is exercised. If the option can be exercised at any time up to expiration date, it speaks of an American option and if the option can be exercised only on expiration date, it speaks of a European option. These are the simplest forms of options contracts and in literature such options are called vanilla options. The right to buy/sell has a value called an option premium or option price which must be paid to the seller of the contract. Like futures, options are mostly standardized contracts and are traded on exchanges since 1973. The first such exchange was the Chicago Board Options Exchange (CBOE).

Put–Call Parity – defines a relationship between the price of a European call option and European put option, both with the identical strike price and expiration date, namely that a portfolio of a long call option and a short put option is equivalent to a single forward contract at this strike price and expiration date. This is because if the price at expiry is above the strike price, the call will be exercised, while if it is below, the put will be exercised, and thus in either case one unit of the asset will be purchased for the strike price, exactly as in a forward contract.

Swaption – is an option granting its owner the right but not the obligation to enter into an underlying swap. Although options can be traded on a variety of swaps, the term swaption typically refers to options on interest rate swaps. There are two types of swaption contract, a payer swaption gives the owner of the swaption the right to enter into a swap where they pay the fixed leg and receive the floating leg. A receiver swaption gives the owner of the swaption the right to enter into a swap in which they will receive the fixed leg and pay the floating leg. The buyer and seller of a swaption agree on the premium (price) of the swaption, the length of the option period and the terms of the underlying swap.

Caplet, Caps, Floorlets and Floors – a caplet is similar to a European call option on the interest rate. They are usually settled in arrears but they may also be settled in advance. If the maturity is t_2 $(t_2 > t_1 > t)$ and the strike is c then the payoff of a caplet (settled in arrears) at time t_2 is $(r_{t1} - \mathrm{c})^+$. That is, the caplet is a call option on the short rate prevailing at time t_1, settled at time t_2. A cap is a string of caplets, one for each time period in a fixed interval of time and with each caplet having the same strike price c. A floorlet is similar to a caplet except it has a payoff $(c - r_{t1})^{+}$ and is usually settled in arrears at time t_2. A floor is a string of floorlets, on for each time period in a fixed interval of time and with each floorlet having the same strike price c. Usually the strike is 0% for both the caps and floors. An important relationship is that if the fixed swap rate is equal to the fixed rate of the caps and floors, then the following put-call parity: Cap-Floor = Swap.

Exchange-traded derivative contract – are standardized derivative contracts such as futures and options contracts that are transacted on an organized derivatives exchanges. The contract specifications for listed derivatives are typically standardized to a relatively high degree, which facilitates trading and enhances liquidity [10]. At the same time, execution through an exchange facilitates price discovery and transparency and affords anonymity of trade counterparties. They are standardized and require payment of an initial deposit or margin settled through a clearing house. The following are the definitions of the concepts used in the derivatives market: Closing out positions, Conversion Factors and Cheapest-to-deliver bond

Closing out positions – the vast majority of derivatives contracts do not lead to delivery [12]. The reason is that most traders choose to close out their positions prior to the delivery period specified in the contract. Closing out a position means entering into the opposite trade to the original one. The trader total gain or loss is determined by the change in the derivatives price between settlement and the day when the contract is closed out.

Conversion Factors – Generally, in the derivative contracts the party with the short position is allowed to choose to deliver any bonds with a certain maturity. When a particular bond is delivered, a parameter known as its conversion factor defines the price received for the bond by the party with the short position. The applicable quoted price for the bond delivered is the product of the conversion factor and the most recent settlement price for the derivatives contract. Taking accrued interest into account, the cash received for each \$100 face value of the bond delivered is

(most recent settlement price × conversion factor) + accrued interest

The conversion factor for a bond is set equal to the quoted price the bond would have per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equal 3%

per annum whit semiannual compounding. The bond maturity and the times to the coupon payment dates are rounded down to the nearest 3 months for the purposes of the calculation. If, after rounding, the bond last for an exact number of 6-months periods, the first coupon is assumed to be paid in 6 months. If, after rounding, the bond does not last for an exact number of 6-month periods (i.e., there are an extra 3 months), the first coupon is assumed to be paid after 3 months and accrued interest is subtracted.

Cheapest-to-deliver bond – At any given time during the delivery month, there are many bonds that can be delivered in the derivative contract. These vary widely as far as coupon and maturity are concerned. The party with the short position can choose which of the available bond is cheapest to deliver. Because the party with the short position receives

(most recent settlement price × conversion factor) + accrued interest

and the cost of purchasing a bond is

quoted bond price + accrued interest

the cheapest-to-deliver bond is the one for which

quoted bond price - (most recent settlement price × conversion factor)

is least. Once the party with the short position has decided to deliver, it can determine the cheapest-to-deliver bond by examining each of the deliverable bonds in turn.

3. REVIEW OF MARTINGALE PRICING THEORY

These definitions are initially given in the context of discrete time models. For a continuous-time analogues see [1], [4], [8] and [19].

Consider a financial market with $N + 1$ securities and true probability measure P. It is assume that the investment horizon is $[0,T]$ and that there are a total of T trading periods. Any security $S_t\,(i)$ may therefore be purchased or sold at any date t for $t = 0,1,\ldots,T-1$.

A trading strategy is a vector, $\theta_t = (\theta_t\ (0),\ldots,\theta_t\ (N))$ of stochastic processes that describes the number of units of each security held just before trading at time t. For example, $\theta_t\ (i)$ is the number of units of the i^{th} security held between times $t-1$ and t. Note that θ_t is known at date $t-1$.

In order to respect the evolution of information as time elapses, it is necessary that θ_t be a predictable stochastic process. In this context, predictable means that θ_t cannot depend on information that is not yet available at time $t-1$. It can also be said that θ_t is F_{t-1}) -measurable where it use F_{t-1} to denote all the information in the financial market that is known at date $t-1$.

A self-financing trading strategy is a strategy, θ_t where changes in the portfolio V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. Then a strategy θ_t is a self-financing trading strategy if

$$V_{t+1} - V_t = \sum_{i=0}^{N} \theta_{t+1}(i)\ \ (S_{t+1}(i) + C_{t+1}(i) - S_t(i))$$

where $C_t\ (i)$ is the dividend or coupon paid by the i^{th} security just before trading at time t and $S_t\ (i)$ the ex-dividend price.

Arbitrage opportunity is defined as a self-financing trading strategy, θ_t such that $V_0\ (\theta) = 0$, $P\ (V_T \geq 0) = 1$, $P\ (V_T > 0) > 0$. A contingent claim X is a random variable whose value is known by time T, i.e., X is F_{t-1} -measurable. It can also be said that a contingent claim X is attainable if there exists a self-financing trading strategy θ_t, whose value process satisfices $V_T = X$. The value of the claim X must equal the initial value of the replicating portfolio if there are no arbitrage opportunities available. It can also be said the market is complete if every contingent claim is attainable. Otherwise the market is said to be incomplete.

A numeraire security is a security with a strictly positive price process. Let N_t be the time t price of a chosen numeraire security and S_t the time t price of any other security, then S_t/N_t is the deflated security price. Nota that the deflated price of the numeraire security is identically 1. The default numeraire is the cash account.

An Equivalent Martingale Measure (EMM) Q is a probability measure that is equivalent to P under which all deflated security gains process are martingales. Note that an EMM Q is specific to the chosen numeraire. In particular, an EMM-numeraire pair (Q,N) for a security that pays intermediate cash-flows between s and $s+t$ satisfies

$$\frac{S_t}{N_t} = E_t^Q \left[\sum_{i=t+1}^{t+s} \frac{C_i}{N_i} + \frac{S_{t+s}}{N_{t+s}} \middle| \mathcal{F}_t \right]$$

With these definitions two important results are presented.

Fundamental Theorem of Asset Pricing: Part 1 – There is no arbitrage if only if there exist an EMM Q.

A consequence of theorem 1 is that in the absence of arbitrage, the deflected value process V_t / N_t of any self-financing trading strategy is a Q-martingale. This implies that the deflated price of any attainable security can be computed as the Q-expectation of the terminal deflated value of the security.

Fundamental Theorem of Asset Pricing: Part 2 – Assume there exists a security with strictly positive price process and that there are no arbitrage opportunities. Then the market is complete if and only if there exists exactly one EMM Q.

4. BINOMIAL-LATTICE MODELS

These models may be viewed as models in their own right or as approximations to more sophisticated models [5]. It will be take the latter approach later in this article when explicitly construct binomial models as approximations to continuous-time short-rate models.

Constructing an Arbitrage-Free Lattice

Consider the binomial lattice in figure 1, it specify the short rate $r_{i,j}$ that will apply for the single period beginning at node $N(i,j)$. This means for example that if \$1 is deposited in the cash account at $t = i$, state j, (i.e. node $N(i,j)$), then this deposit will be worth \$$(1 + r_{i,j})$ at $t + 1$ regardless of the successor node to $N(i,j)$.

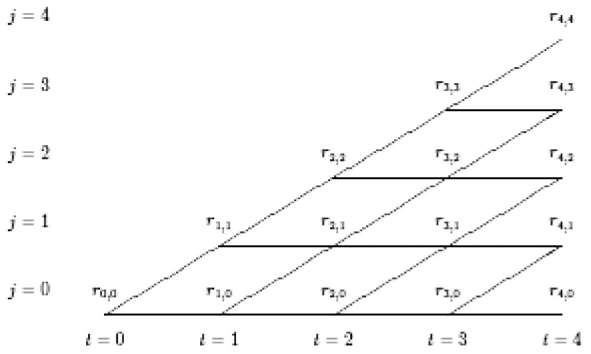

Figure 1: Constructing an Arbitrage-Free Lattice

It used martingale pricing on this lattice to compute security prices. For example, if S_i (j) is the value of a security at time i and state j, then

$$S_i(j) = C_i + \frac{1}{1+r_{i,j}} [\, +q_u S_{i+1}(j+1) + q_d S_{i+1}(j)] \qquad (1)$$

where C_i is the dividend or coupon paid just before trading at time i, q_u and q_d are the probabilities of up- and down-moves, respectively. This model is arbitrage-free by construction.

Computing bond prices from the Binomial Tree

It is easy to compute the price of a coupon-bearing bond once the EMM Q, (with the cash account as numeraire) and the short-rate lattice are specified. The short rate-lattice in figure 2 (where the short rate increases by a factor of u = 1.25 or decreases by a factor of d = 0.9 in each period), it is assume that the Q-probability of each branch is 0.5 and node-independent. We can then use the martingale-pricing to compute the prices of a coupon-bearing bond.

					18,31%
				14,65%	13,18%
			11,72%	10,55%	9,49%
		9,38%	8,44%	7,59%	6,83%
	7,50%	6,75%	6,08%	5,47%	4,92%
6,00%	5,40%	4,86%	4,37%	3,94%	3,54%

Figure 2: Short rate lattice.

For example, the price of a 6-period 10% coupon-bearing bond with face value 100 that expires at $t = 6$. Assuming the short rate lattice in figure 2 and iterating backwards, it find that the bond is worth 114.14 at $t = 0$.

						110,00
					102,98	110,00
				101,66	107,19	110,00
			104,03	108,44	110,46	110,00
		108,98	112,49	113,83	112,96	110,00
	115,83	118,55	119,27	118,00	114,84	110,00
114,14	126,14	126,27	124,57	121,16	116,24	110,00

Figure 3: 6-year 10% coupon-bearing bond.

The bond price at node (2,2) is given by

$$108.98 = 10 + \frac{1}{1 + 0.0938}\left[\frac{1}{2}104.03 + \frac{1}{2}112.49\right]$$

In an analogous way the price of a zero-coupon bond with face value 100 that expires at $t = 4$ can be calculated

				100,00
			89,51	100,00
		83,08	92,22	100,00
	79,27	87,35	94,27	100,00
77,22	84,43	90,64	95,81	100,00

Figure 4: 4-year zero-coupon bond.

Note that given the price of the 4-period zero-coupon bond, it can be find the 4-period spot rate s_4. It satisfies that $77.22 = 1/(1 + s_4)^4$ if it quote spot rate on a per-period basis. In this manner it can be construct the entire term-structure by evaluating zero-coupon bond prices for all maturities.

Pricing Interest Rate Derivatives

Now introduce and price several interest rate derivatives using the martingale pricing methodology. The following examples will be based on the short rate lattice (figure 2).

Plain Vanilla Interest Rate Swap – In a plain vanilla interest rate swap, there is a maturity date T a notional principal P and a fixed number of periods M. Every period party A makes a payment to party B corresponding to a fixed rate of interest on P. Similarly, in every period party B makes a payment to party A that corresponds to a floating rate of interest on the same notional principal P. For example, the party A receive a \$ 99.004,43 now to enter in a 6 period plain vanilla interest rate swap with 5% fixed rate an principal \$ 1.000.000,00.

					0,1125
				0,1648	0,0723
			0,1793	0,1014	0,0410
		0,1686	0,1021	0,0512	0,0172
	0,1403	0,0829	0,0400	0,0122	-0,0008
0,0990	0,0496	0,0137	-0,0085	**-0,0174**	-0,0141

Figure 5: 5% fixed rate swap with expiration t = 6.

The swap price at node (4,0) is given by

$$-0{,}0174 = (0.0394 - 0.050) + \frac{1}{1 + 0.0394}\left[-\frac{1}{2}0.0008 - \frac{1}{2}0.0141\right]$$

Futures Contracts on Bonds – let F_k be the date k price of a futures contract written on a particular underlying security in a complete market model. It will assume that the contract expires after n periods and let S_k denote the time k price of the security. Then it is know that $F_n = S_n$, i.e., at expiration the futures price and the security price must coincide. At $t = n - 1$ by recalling that anytime it enter a futures contract, the initial value of the contract is 0. Therefore the futures price, F_{n-1} at date $t = n - 1$ must satisfy

$$0 = E_{n-1}^{Q}\left[\frac{F_n - F_{n-1}}{B_n}\right]$$

where B_n is the value of the cash account at date n. Since B_n and F_{n-1} are both know at date $t = n - 1$ it therefore have $F_{n-1} = E_{n-1}^{Q}[F_n]$. By the same argument, it also have more generally that $F_k = E_k^{Q}[F_{k+1}]$. It can be then use the law of iterated expectations to see that $F_0 = E_0^{Q}[F_n]$, implying in particular that the futures price process is a martingale. Since $F_n = S_n$ it have $F_0 = E_0^{Q}[S_n]$.

Now to price a futures contract for a delivery at $t = 4$ of a 2-year 10% coupon-bearing bond where we assume that delivery takes place just after a coupon has been paid is given at node (0,0) in the figure 6.

				91,66
			95,05	98,44
		98,09	101,14	103,83
	100,81	103,52	105,91	108,00
103,22	105,64	107,75	**109,58**	111,16

Figure 6: futures price with expiration t = 4.

The futures price at node (4,0) is given by

$$109{,}58 = \left[\frac{1}{2}108{,}00 + \frac{1}{2}111{,}16\right]$$

Forward Contract on Bonds – now consider the date 0 price G_0 of a forward contract for delivery of the same security at the date $t = n$. Recall that G_0 is chosen in such a way that the contract is initially worth zero. In particular, risk-neutral pricing implies

$$0 = E_0^Q\left[\frac{S_n - G_0}{B_n}\right]$$

Rearranging terms and using the fact that G_0 is know at date $t = 0$ is obtain

$$G_0 = \frac{E_0^Q[S_n/B_n]}{E_0^Q[1/B_n]} \tag{2}$$

For example, the price of a forward contract expiring at $t = 4$ on the same coupon-bearing bond that the previous example. I then use (2) to price the contract, with the numerator given by the value at node (0,0) of figure 7 and the denominator equal to the 4-year discount factor.

$$G_0 = \frac{79{,}83}{0{,}7722} = 103.38$$

				91,66
			85,08	98,44
		81,53	93,27	103,83
	79,99	90,45	99,85	108,00
79,83	89,24	97,67	**104,99**	111,16

Figure 7: forward tree with expiration t = 4.

Note that between $t = 0$ and $t = 4$ in figure 7 coupons are ignored. The forward price at node (4,0) is given by

$$104{,}99 = \frac{1}{1 + 0.0394}\left[\frac{1}{2}108.00 + \frac{1}{2}111.16\right]$$

Options – it want to compute the price of a European CALL option on the zero-coupon bond of figure 4 that expires at $t = 2$ and has strike \$ 84. The option price of \$ 2.97 is computed by backwards induction on figure 8.

		0,00
	1,56	3,35
2,97	4,74	**6,64**

Figure 8: European CALL option with expiration t = 2.

The option price at node (2,0) is given by

$6{,}64 = (95.81 - 84.00)^{+}$

To compute the price of an American PUT option on the same zero-coupon bond when the expiration date is $t = 3$ and the strike is \$ 88. Again the price is computed by working backwards. The maximum of the continuation value and exercise value is equal to the option value at each node.

			0,00
		4,92	0,00
	8,73	0,65	0,00
10,78	3,57	**0,00**	0,00

Figure 9: American PUT option with expiration t = 3.

The option price at node (2,0) is given by

$$0{,}00 = \max\left\{(88.00 - 95.81)^{+}, \frac{1}{1 + 0.0486}\left[\frac{1}{2}0.00 + \frac{1}{2}0.00\right]\right\}$$

Swaption – is an option on a swap. Consider a swaption on the swap of the figure 5, it is assumed that the option strike is 0% and swaption expiration is at $t = 3$. Swaption value at expiration is therefore max $(0,S_3)$ where S_3 is the underlying swap price at $t = 3$. Again the price is computed by working backwards.

			0,1793
		0,1286	0,1021
	0,0908	0,0665	0,0400
0,0620	0,0406	**0,0191**	0,0000

Figure 10: Swaption with strike 0% and expiration t = 2.

The swaption price at node (2,0) is given by

$$0{,}0191 = \frac{1}{1 + 0.0486}\left[\frac{1}{2}0.0400 + \frac{1}{2}0.0000\right]$$

Caplet, Caps, Floorlets and Floors – to compute the price of a caplet with expiration at $t = 6$ and strike 2% using the short rate lattice in figure 2. The caplet price is 0.0420 and is calculated by backwards induction as shown in figure 11.

					0,1379
				0,1032	0,0988
			0,0800	0,0756	0,0684
		0,0637	0,0592	0,0528	0,0453
	0,0515	0,0471	0,0412	0,0346	0,0278
0,0420	0,0376	0,0323	0,0264	0,0206	**0,0149**

Figure 11: Caplet with strike 2% and expiration t = 6.

The caplet price at node (5,0) is given by

$$0{,}0149 = \frac{(0.0354 - 0.0200)^{+}}{(1 + 0.0354)}$$

In short, it is easy to calculate the price of any financial derivative from a binomial lattice. The next chapter will present a model for the short rate and how the model is calibrated to match the term structure of interest rates observed in the market and there volatility.

5. THE BLACK-DERMAN-TOY MODEL (BDT)

The Black-Derman-Toy one-factor model [2] is one of the most used yield-based models to price bonds and interest-rate options. The model is arbitrage-free and was developed to match the observed term structure of yields on zero coupon bonds and their corresponding volatilities. The continuous time version of the BDT model where the short-rate process has the form

$$r_t = U_t \, e^{\sigma_t W_t} \tag{3}$$

where U_t is the median of the short-rate distribution at time t, σ_t the short-rate volatility, and $\{W_t\}$ a standard Brownian motion. The dynamics of the logarithm of the short rate is given by a stochastic differential equation (see [1], [4], [9] and [14] for more details).

$$d \ln r_t = \left[\theta_t + \frac{\sigma'_t}{\sigma_t} \ln r_t\right] dt + \sigma_t dW_t \tag{4}$$

Unfortunately, due to the log-normality of r_t in (4), analytic solutions are not available for bond option prices and numerical procedures like binomial tree are required.

Calibration of the BDT model

It can be use a binomial lattice to calibrate the model parameters to fit both the current term structure of yields on zero coupon bonds, and the volatility structure. An up move in the short rate has a probability q = 0.5, so the corresponding down move has a probability 0.5. When referring to nodes in a binomial tree, the index notation is used in figure 12

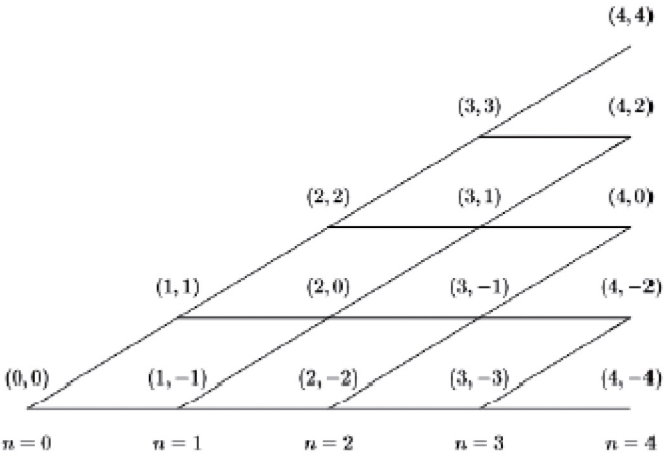

Figure 12: Example of four period binomial tree.

Let the unit time be divided into M periods of length $\Delta t = 1/M$ each. At each period n, corresponding to time $t = n/M = n\Delta t$, there are $n + 1$ states. These states range according to $j = -n, -n + 2, \ldots, n - 2, n$. At the present period $n = 0$, there is a single state $i = 0$. Let $r_{n,j}$ denote the annualized one period rate at period n and state j. Define the variable $X_k = \sum_{j=1}^{j=k} y_j$ where $yj = 1$ if an up move occurs at period j and $yj = -1$ if a down move occurs at period j. The variable X_k gives the state of the short rate at period k. At any period k, the X_k has a binomial distribution with mean zero and variance k. It is follows that $X_k\sqrt{\Delta t}$ has the same mean and variance as the Brownian motion W_t. X_k has independent increments, and the binomial process $X_k\sqrt{\Delta t}$ converges to the Brownian motion W_t as Δt approaches zero. The state of the short rate was denoted by j. Replacing X_k by j will lead to having W_t approximated as $j\sqrt{\Delta t}$. Now, replacing t by n gives the discrete version of r_t

$$r_{n,j} = U(n)e^{\sigma(n)j\sqrt{\Delta t}}$$

with $j = -n, n + 2, \ldots, n - 2, n$. To calibrate the tree to the term structure and the volatility it is use the Arrow-Debreu prices where it represents price of primitive securities and forward induction. Let $G\ (n,i,m,j)$ denote the price at period n and state i of a security that has a cash flow of unit at period m ($m \geq n$) and state j. Note that $G\ (m,j,m,j) = 1$ and that $G\ (m,i,m,j) = 0$ for $i \neq j$. It can be compute $G\ (n,i,m+1,j)$ with the forward induction:

$$\begin{cases} \frac{1}{2}d_{m,j-1}G(n,i,m,j-1) & j = m+1 \\ \frac{1}{2}d_{m,j+1}G(n,i,m,j+1) + \frac{1}{2}d_{m,j-1}G(n,i,m,j-1) & |j| < m-1 \\ \frac{1}{2}d_{m,j+1}G(n,i,m,j+1) & j = -m-1 \end{cases}$$

where $d_{m,j}$ is the one period discount factor. By intuitive reasoning, the forward induction function above states how it discount a cash flow of unity for receiving it on period later. Arrow-Debreu prices are the building blocks of all securities. The price of a zero-coupon bond which matures at period $m + 1$ can be expressed in terms of the Arrow-Debreu prices and the discount factor in period m.

$$P(0, m+1) = \sum_j G(0,0,m,j)d_{m,j}$$

The term structure $P(0,m)$, which represent the price today of a bond that pays unity at period m, can be obtained for all values of m, by the maximum smoothness criterion. One the binomial tree is calibrated, it is easy to calculate bond prices and financial derivatives using backward induction as shown previously.

6. EMPIRICAL RESULTS

To model the debt structure of the Uruguayan UI, the daily yield curves in UI released by BEVSA for the 6/30/2017 with maturities up to 10 years are used. To estimate the volatility curve it is necessary to use historical data corresponding to the last year. The respective curves for the 6/30/2017 are presented in figure 13

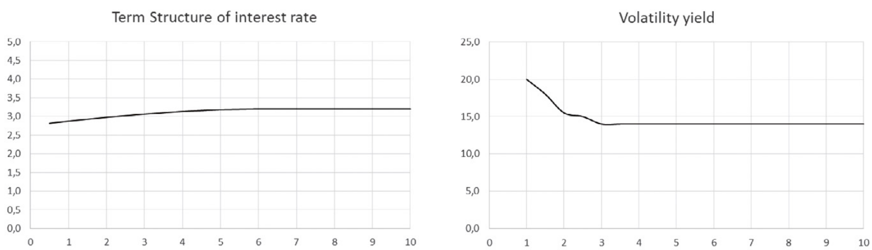

Figure 13:
Term structure of interest rate and the volatility yield for 6/30/2017.

For the elaboration of the binomial lattice $\Delta t = 0.5$ and the probabilities $q = 1 - q = 0.5$, that is, the short rate will change every six months and the probability of an up move in the short rate has equal probability to a down move.

Pricing Interest rate derivatives

Table 1 presents the results of different interest rate derivatives, assuming a 10-year expiration, a principal of 10 million UI and settled in arrears. In the case of swap the price obtained is for the long position, i.e. the buyer will pay a fixed rate and receive a floating rate on the same principal. The swaption has a maturity of 3 years and a strike of 0% on the underlying is the previous swap. In the case of Cap, the price obtained is for the long position, i.e., the buyer will receive the difference between the floating rate and the fixed rate if it is positive, otherwise it will not receive anything. The Floor is equal to the Cap, except that buyer will receive the difference between the fixed rate and the floating rate, otherwise it will not receive anything. In both Cap and Floor the buyer should not pay anything except the price at $t = 0$.

Fixed-Rate	Swap	Swaption	Cap	Floor
2,00%	1.397.138	877.706	1.464.987	67.849
2,50%	534.252	425.531	787.325	253.073
3,00%	-328.635	160.452	344.509	673.143
3,50%	-1.191.521	47.369	132.841	1.324.362
4,00%	-2.054.408	8.433	50.378	2.104.786

Table 1: Interest rate derivatives.

Note that the negative price on the swap means that the long position would receive money for entering the swap. Another aspect that can be seen in Table 1 is that you can create a cap and floor portfolio that has the same behavior as the swap. If you combine a long position of the cap with a short position of the floor you obtain the price of the swap.

Pricing Bond derivatives

The prices of the derivatives will be calculated on a fictitious coupon-bearing bond paying 3% semi-annually with face value 100 and an exact maturity of 10 years. The today price of the bond is 101.64.

First, the prices of a forward contract and a futures contract are calculated for a period of 4 years, resulting in 101.74 and 101.69 respectively. Traditionally, the futures contract allows the party with the short position to choose to deliver any bond that has a certain maturity. For this case, it is assumed that any global UI bond maturing between 6 and 20 years can be chosen. The possible bonds together with the conversion factors, the price and the interest accrued for 06/30/2017 are presented in table 2

Bond	Conversion Factors	Price	interest accrued
URUGUA UI 4.25% 2027	1,100776	109,53	1,00347
URUGUA UI 4.375% 2028	1,100080	112,13	0,18229
URUGUA UI 4% 2030	1,123409	109,66	1,88888
URUGUA UI 3.7% 2037	1,090334	111,00	0,04111

Table2:Bond,Conversionfactor,priceandinterestaccruedfor6/30/2017.

Based on the results of table 2 cheapest-to-deliver bond is URUGUA UI 4% 2030.

Table 3 shows the Call and Put prices for American options contracts with a maturity of one and a half years and different strike prices. In the same it is observed that the price of an American Call Option with a strike 105 and one and a half years maturity is 0.28.

Strike	CALL	PUT
95,00	6,75	0,07
100,00	2,57	0,93
105,00	0,28	3,88
110,00	0,00	8,50
115,00	0,00	13,36

Table 3: American option with one and a half years of maturity.

Now, the table 4 shows the Call and Put prices for European options contracts with a maturity of 2 years and different strike prices. In the same it is observed that the price of a European Put Option with a strike 100 and maturity of 2 years is 0.76.

Strike	CALL	PUT
90,00	10,91	0,00
95,00	6,24	0,05
100,00	2,23	0,76
105,00	0,16	3,40
110,00	0,00	7,97

Table 4: European option with maturity of 2 years.

Knowing that the discount factor is $d_{0,4} = 0.943822$ and the price of a forward contract is $F = 101.56$ it can be seen that the put-call parity is met, with K the strike and C and P the prices of the call and put options respectively.

$$C - P = d_{0,4}(F - K) \tag{5}$$

The left side of (5) corresponds to a portfolio of long a call and short a put, while the right side corresponds to a forward contract. The assets C and P on the left side are given in current values, while the assets F and K are given in future values (forward price of the bond, and strike price paid at expiry), which the discount factor $d_{0,4}$ converts to present values.

7. DISCUSSION

In recent years the volatility of the international financial system has become severe exposing bondholders to serious risks. Added to this is the fact that the market for bond hedging instruments is underdeveloped. The undesired volatility in the yield curve can lead to negative consequences in the portfolios of important areas such as social security, motivating the idea of the introduction of hedging derivatives in the Uruguayan market. The objective of this work was to develop a methodology for the Uruguayan market that allows in a simple way the valuation of fixed income derivatives. It is expected that the methodology presented in this paper may serve as a reference for the future development of a possible market for interest rate derivatives in Uruguay.

First, it is present some fundamental financial definitions and a brief description of the Martingale Pricing Theory. From this it is concluded that in the absence of arbitrage, the deflected value process V_t/N_t of any self-financing trading strategy is a Q-martingale. This implies that the deflated price of any attainable security can be computed as the Q-expectation of the terminal deflated value of the security. The model used to determine the short rate was the Black-Derman-Toy (BDT), this model is one-factor, i.e., it has a single source of uncertainty to determine the future evolution of all interest rates and the model is arbitrage-free. The approach used to value financial derivatives consisted in discretizing the continuous model and working with binomial trees on the short rate. Finally, it is presented the price of several financial derivatives, such as Swap, Swaption, Floor, Cap, Futures, Forward and Options and a relation between them.

A possible extension of this research is use Multi-factor models, like Longstaff-Schwartz (1992) [16] and Chen (1996) [6], instead of one-factor models. This because the assumption implicit within one-factor models is that all information about future interest rates is contained in the current short rate and hence the prices of all bonds may be represented as functions of this short rate and time only. Also, within a one-factor framework the instantaneous returns on bonds of all maturities are perfectly correlated. These characteristics are inconsistent with reality and motivate the development of multi-factor models.

Finally, I invite the readers to continue deepening in the Martingale Pricing Theory and its applications in Quantitative Finance and recommend the followings books, Dupacova et al. (2003) [8], Musiela, & Rutkowski (2005) [19], Brigo & Mercurio (2006) [4] and Bjork (2009) [1].

REFERENCES

Bjork, Tomas (2009). Arbitrage Theory in Continuous Time. Third Edition Oxford Finance.

Black, Fischer.; Derman, Emmanuel and Toy, William (1990). "A One-Factor Model of Interest Rates and Its Application to Treasury bond Options", *Financial Analysts Journal,* Jan/Feb 1990; 46, 1: 24-32.

Black, Fischer and Piotr Karasinski (1991). "Bond and Option pricing when Short rates are Lognormal", *Financial Analysts Journal,* July/Aug 1991: 52-59.

Brigo Damiano and Fabio Mercurio (2006). Interest Rate Models - Theory and Practice. Springer Finance.

Buetow, Gerald W. and James Sochacki (2001). Term-Structure Models Using Binomial Trees. The Research Foundation of AIMR.

Chen, Lin (1996). "Stochastic Mean and Stochastic Volatility - A Three-Factor Model of the Term Structure of Interest Rates and Its Application to the Pricing of Interest Rate Derivatives", *Financial Markets, Institutions & Instruments,* 5: 1–88.

Cox, John C.; Ingersoll Jonathan E. and Stephen A. Ross (1985). "A Theory of the Term Structure of Interest Rates", *Econometrica,* 53: 385-407.

Dupacova, Jitka; Hurt, Jan and Josef Stepán (2003). Stochastic Modeling in Economics and Finance. Kluwer Academic Publishers.

Filipovic, Damir (2009). Term-Structure Models. Springer Finance.

Heckinger, Richard (2013). "Understanding Derivatives: Markets and Infrastructure", Federal Reserve Bank of Chicago.
https://www.chicagofed.org/publications/understanding-derivatives/index

Ho, Thomas S. and Sang-bin Lee (1986). "Term structure movements and pricing interest rate contingent claims", *Journal of Finance,* 41, 1986.

Hull JohnC. (2012). Options, Futures, and Other Derivatives. Eighth Edition Prentice Hall.

Hull John and Alan White (1990). "Pricing interest-rate derivative securities", *The Review of Financial Studies,* Vol 3, No. 4: 573-592.

Jamshidian Farshid (1991). Forward Induction and Construction of Yield Curve Diffusion Models. *The Journal of Fixed Income,* Summer 1991, 1 (1): 62-74.

Kwok, Yue-Kuen. (1998). Mathematical Models of Financial Derivatives, Springer, Singapore.

Longstaff, Francis A. and Eduardo S. Schwartz (1992). "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model", *Journal of Finance,* 47 (4): 1259–82.

Luenberger David (1997). Investment Science. Oxford University Press.

Magnou, Guillermo (2015). Opciones Financieras sobre Bonos. Una aplicación para el Mercado Uruguayo. Master's Thesis. Facultad de Ingeniería, Universidad de la República.

Musiela, Marek and Marek Rutkowski (2005). Martingale Methods in Financial Modelling. Second Edition, Springer.

Svoboda, Simona (2004). Interest Rate Modelling. Palgrave Macmillan.

Vasicek, Oldrich (1977). "An equilibrium characterization of the term structure", *Journal of Financial Economics,* 5: 177-188.

Wilmott, Paul, Howison, Sam and Jeff Dewynne (1995). The Mathematics of Financial Derivatives. A Student Introduction. Cambridge University Press, Cambridge.