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APPLICATIONS IN THE HISTORY OF MATHEMATICS TEACHING

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RESUMEN

La motivación de este estudio reside en el campo de la pedagogía matemática más que en la historia de las matemáticas o de su educación. Mucho se habla actualmente de la necesidad de hacer la educación matemática "relevante", de demostrar su utilidad, de iluminar su relación con el mundo real y con otros temas. Trabajos sobre la historia de la educación metemática -especialmente HOWSON, 1982- han puesto de manifiesto que existen una cantidad considerable de experiencias históricas en las que inspirarse que ilustran a la perfección no sólo la variedad de funciones pedagógicas que las aplicaciones han desempeñado en la educación matemática, sino también la existencia de problemas crónicos asociados con su uso. Este trabajo representa un primer intento de abordar el campo que espero impulsará investigaciones mucho más detalladas. Los ejemplos han sido extraídos de fuentes mayoritariamente inglesas. Sin embargo, creo que sus correspondientes réplicas pueden ser halladas en muchos otros países.

ABSTRACT

The motivation for this study lay in the field of mathematical pedogogy rather than in the history either of mathematics or of its teaching. Much is heard nowadays of the need to make mathematics education "relevant", to demonstrate its utility, to illumine its links with the "real world" and with other subjects. Work on the history of mathematics education -in particular, HOWSON, 1982- had made it clear that there was considerable historical experience on which one could draw which illustrated very well not only the very varied pedagogical rôle that applications have played in mathematics teaching, but also the existence of long-standing problems associated with their use. This paper is a preliminary attempt to describe the field and will, I hope, prompt far more detailed research and investigation. The examples are, in the main, drawn from English sources. Nevertheless, I believe that counterparts for them could be found in many other countries.

Palabras clave: Mathematics, Teaching, Applications, Utility, Applying Mathematics, Appliers of Mathematics, Recorde, Textbooks, Exercises, "Liberal education", Whewell, Innovators.

1. What is an application of mathematics?

Almost every mathematical textbook speaks of "applications" of theorems, etc. Many such "applications", however, lie entirely within mathematics itself: for example, taking a specimen text from my bookshelves, in this case Apostol's *Mathematical Analysis*, I find a section (16-17) entitled *Application of the residue theorem to the inversion formula for Laplace Transforms*. Such "applications" are not our concern. Rather, I shall define an "application of mathematics" to be:

> an attempt to set mathematics into a "non-mathematical" context or to use mathematics to describe or investigate an apparently non-mathematical situation.

As we shall see, this definition embraces several different types of application, some of which have specifically pedagogical purposes.

It must also be emphasised at this point that in this paper we are not concerned so much with how mathematics was being applied at any particular time, but at the rôle which applications played in contemporary mathematics education.

2. Beginnings

The first English educational institutions, the schools attached to the early monastic institutions, such as those at Canterbury, Westminster and York, were designed with one principal aim in view, the more widespread dissemination of Christianity. In this sense they were vocational, and it is not surprising therefore that the mathematics they taught was largely utilitarian, One clerical need was that of calculating the dates of movable feasts and it was largely for this reason that Bede (674-735) included elementary arithmetic in his *De Temporum Ratione*[1].

Such calculations provide us with an early example of what might be called "utilitarian" mathematics, i.e.

the teaching of well-defined techniques for applying mathematics to "everyday" problems and which have a high probability of being employed by the learner.

Utilitarian applications will play an important part in our story, although they form but a small part of the field. The educational goals of such applications are limited, yet their importance cannot be denied. Moreover, their range at any point in history provides interesting insights into, for example, the extent of educational provision, the aims of different types of education, and the manner in which mathematics was employed within contemporary society.

3. Applicable and applied mathematics

The education which the Church provided was extremely limited. Nevertheless, the basis of an educational system existed, which in the thirteenth century was to provide the foundation for expansion. In particular, the century saw the establishment of the great medieval universities. The place of mathematics in these varied, but generally the study of the *quadrivium*, arithmetic, geometry, music and astronomy, came to be associated with the master's degree. Now "applications" had a new aspect. For example, at Prague University (1350) candidates for the bachelor's degree were required to read Sacrobosco's *Tractabus De Sphaera*, an example of "applicable" mathematics, whilst those for the master's degree had to be acquainted, *inter alia*, with optics, hydrostatics, the theory of the lever and Ptolemaic astronomy.

Here then we have examples of "applications" being taught not because they were specifically utilitarian, but because they were included in that body of knowledge which it was thought that any educated man should know. They demonstrated how a knowledge of mathematics could increase our understanding of the physical world. There was, however, no apparent thought that by studying such applications a learner might be trained to become a creative applier of mathematics. Such an aim would have been anachronistic.

4. Recreational Mathematics

Yet another type of "application" was also well-established by this time. "Recreational mathematics" has a long history. In the words of Heinrich Hermelink (1978), these are *problems and riddles which use the language of everyday but do not much care for the circumstances of reality.* Classical examples are the wolf, goat and cabbage problem[2] which certainly dates back to Alcuin in the 9th century A.D., and those concerned with exponentiation, for example, rice on chessboards, or the cost of shoeing the horse with 100% inflation on every nail. The exponentiation problem was posed in Islamic manuscripts from the ninth century onwards, was known to the Chinese and at the court of Charlemagne, and has recently been found on a Babylonian cuneiform tablet dating from the 18th century BC (see HOYRUP, 1987). The relation of such problems to "reality" does, however, merit serious pedagogical consideration. Their presentation in a whimsical or unreal "real" context serves important purposes. First it can give "meaning" to a problem. (Try expressing the wolf, goat and cabbage problem in purely abstract terms.) Secondly, it can also give meaning to the result. Recorde makes use of the horseshoe problem in his *Ground of Artes*[3] and calculates that it would then cost £34,952, 10s $7^{1/2}$d to shoe a horse with 24 nails. This sum, astronomical though it might have appeared in 16th century England, is, nevertheless, commensurate with experience in a way that a number without units might not be.

We must be careful, then, in rejecting "unreal" examples: worse still, in deriding them because we do not recognise their mathematical and pedagogical significance. Perhaps there is also a danger in classifying them prematurely as "recreational". Kirkman's schoolgirls problem (1850)[4], based on a problem set in the *Lady's and Gentleman's Diary* and itself appearing first in that "popular" periodical, was later to be investigated by, to name but a few, Cayley, Sylvester, Steiner and Benjamin Peirce.

5. Progress

The fifteenth and sixteenth centuries saw enormous social changes. Printing had been invented, the great sea voyages were giving rise to a demand for navigational mathematics; commerce grew apace. The needs for utilitarian mathematics increased rapidly -and was met by books in different vernaculars, English, German and French. Moreover, the Reformation and the humanist movement caused a rethinking of educational goals in the rapidly expanding systems which were loosening their ties with the Church.

Yet in looking back to the distant past, if only to restructure the knowledge derived from it, the humanists ignored what was soon to become a major force, the concept of scientific progress[5]. The idea of "contributing" to continuous progress, rather than of re-arranging and re-presenting a static body of knowledge was essentially new. The implications for the applications of mathematics were spelled out explicitly by Francis Bacon in his *On the Advancement of Learning* (1605, Bk2, VIII, 2): *As for the mixed mathematics* *(of which sort are perspective, music, astronomy, cosmography, architecture, enginery* [sic] *and divers others) I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed.*

The idea of progress is a vital one for all educators and its significance for "applications" and "applied mathematics" is great. How are those who are to make "progress" to be trained? How does one help create an "applier" of mathematics? Of course, these aims were not explicitly set out in the seventeenth century. Indeed, the first specific recognition of the need for an *art of teaching the art of applying mathematics*[6] that I have chanced upon, came two centuries later. Then, Olinthus Gregory (1840, p. 3) wrote *Science makes [the learner] not a passive recipient of knowledge, but an active discoverer of it; and a successful applier of what is discovered to farther discovery, as well as to daily practice.*

Gregory, whilst clearly recognising the need for "appliers", felt that the art of discovery would be acquired naturally whilst learning about the applications of mathematics. Certainly, in his book he makes no specific suggestions as to how that acquisition might be expedited. Indeed, it is only in comparatively recent times that educational courses in modelling have been established, having the creation of "appliers" as a principal aim.

6. Robert Recorde (c. 1512-1558)

As the first major writer of mathematical works in English, Recorde can claim a special place in any history of mathematics in England. Indeed, Lilley (1985) suggests that Recorde was much more than a textbook writer, but rather the first "learned man" to give expression to the idea that potential future developments, i.e. scientific progress, was the ultimate goal of his work. For us, though, Recorde's books have a special interest. The full title of Recorde's 1551 book on geometry illustrates why:

> The Pathway to Knowledg, containing the first principles of Geometrie, as they may moste aptly be applied unto practise, bothe for the use of instrumentes Geometricall, and astronomicall and also for projection of plattes [plans] in every kinde, and therfore much necessary for all sortes of men.

Here the emphasis on applications is clear. Indeed this book was the second in a series intended to cover all the elementary mathematics of value to navigators, surveyors, merchants and craftsmen. It was based on Euclid's *Elements* (at that time unavailable in English), but the rearranged theorems were accompanied by figures and explanations rather than proofs. The first "part" also included a number of approximate constructions which would have been used by craftsmen. (Such constructions, for example, how to construct a regular n-gon using ruler and compass, persisted in texts for craftsmen until late Victorian times. What led to their disappearance was not Gauss's proof that such constructions were impossible, but the appearance of protractors. They are an interesting, but ignored, part of what is often referred to nowadays as ethnomathematics -mathematics recognised within the culture rather than within the body of "academic mathematics").[7]

Recorde's books can be seen, then, as a combination of the applicable and the utilitarian, intended for a potential user rather than a student in school or university. For it must be recalled, see, e.g. HOWSON (1982), that at that period little mathematics teaching took place in such institutions. Thus, his *Ground of Artes* (1543?) is essentially utilitarian arithmetic, although some Euclid-style work on numbers can be found in his *Whetstone of Witte* (1557). Recorde himself lists the "profit" of learning arithmetic. It can be used in astronomy, geometry, music and physics; in law a judge can use it to distribute food and debts, in the army there must be victuals, artillery, armour, ... With the help of arithmetic *you may attain to all things.* Similar strongly utilitarian arguments for the learning of mathematics are to be found in the famous 25,000 word Preface to Billingsley's *Euclid* (1570) written by John Dee (a later editor of Recorde's *Ground*). Dee whilst stressing that the learning of mathematics can have non-utilitarian benefits (it allowed *the good and pregnant English ... virtuously to occupy the sharpness of their wits*), emphasised how the *common artificer* could *by these good helps and informations ... find out and devise new works, strange engines and instruments* either for *sundry purposes in the commonwealth ... private pleasure or the better maintaining of their own estate.*

The demand for mathematics education was primarily coming from those who wished to use it, rather than from those who wished to see it play a major rôle in school or university education and this demand influenced the way in which arguments for teaching mathematics were framed.

The increasingly utilitarian view of mathematics, revealed in the many editions of Recorde's *Ground*, resulted in the almost total extinction of interest in theoretical arithmetic, what we should now call elementary number theory. *Arithmetic became so commercialised that in the seventeenth century it was neglected except by those whose way in life demanded it* (YELDHAM, 1936, p. 13). Indeed, Florence Yeldham's account of the drawn-out and largely unsuccessful battles to win back for general education what had been sacrificed to purely vocational ends is still worth reading and heeding.

7. Extra-mathematical messages

We have already referred to some of Recorde's "applications", the "recreational" horseshoe problem and the geometrical constructions. Others of his books illustrate well what we have referred to as utilitarian, applicable and applied mathematics. Yet Recorde makes use of yet another important, and common, type of "application". In the first edition of the *Ground,* Recorde, an active champion of Protestantism, was able gently to mock the established church through the question the "Master" set: *There is in a cathedrall churche 20 cannones and 30 vicars, those maie spend by yeare £2,600 but every cannon muste have to his part 5 tymes so much as every vicar hath: how much is every mans portion saie you?* (*Ground,* Sig M. v,v).

More contentiously, in the second edition he added a question concerning the contemporary vexed political problem of sheep and enclosures (see EASTON, 1967).

Here we have an example of the use of "applications" to alert/teach/indoctrinate/educate learners to/on non-mathematical manners. Sometimes the resulting questions aspired to near-perfection. If one wished to exercise specific mathematical techniques and to poke fun at Roman Catholicism simultaneously, it would be hard readily to improve upon:

> "If a cardinal can pray a soul out of purgatory by himself in an hour, a bishop in three hours, a priest in five and a friar in seven, in what time can they pray out three souls all praying together?" (BONNYCASTLE, 1808).

A recent example seems somewhat leaden in comparison: "In the USA the number of half-starved people is twice the number of unemployed and is five millions less than the number who live in slums. As one half the number of slum dwellers is eleven and a half million, what is the number of unemployed in the United States?" (Chinese text quoted in SWETZ, 1978).

Such examples demonstrate how "applications" can be used to convey extra-mathematical messages. They should also alert us to the fact that these messages may be less obvious, indeed possibly unintended. Certainly, changing views on feminism and racism would nowadays rule out of court many "applications" to be found in textbooks -even of the recent past.

Just how much non-mathematical weight examples should, or might, bear is by no means obvious. The matter is discussed in HOWSON and MELLIN-OLSEN (1986). From a historical viewpoint, Butler's *Arithmetical Questions for the Use of Young Ladies* (1795) would appear a perfect *reductio* *and absurdum* argument. Butler, in the words of a contemporary reviewer, set out to rid his book of the customary *dryness and dullness* by combining *some historical, geographical, political, or philosophical fact with every arithmetic question*. Thus, for example, a 600 word history of those *pestilential time-wasters*, playing cards, serves to introduce the question *What are 684 packs of cards worth at 3s.3d per pack?*.

The use of mathematics lessons as vehicles for moral and other education was discussed by Fitch in 1881. He decided that in general it was better when seeking such ends to *adopt some other machinery than that of the mathematics lesson*. Fitch, however, lived before subliminal advertising and mental conditioning were explicitly recognised. Questions relating to the non-mathematical impact of "applications" must still be put: the possible rewards and dangers recognised. Applications of mathematics are not politically and educationally neutral; they are not merely catalysts for mathematical activities. Yet the mathematics should never be sacrificed, as in Butler's book, for non-mathematical ends.

8. A dichotomy

Earlier Yeldham was quoted on the way in which during the seventeenth century a utilitarian view of mathematics came to swamp all others. Mathematics, perhaps, became too closely associated with its applications. In Wallis's oft-quoted words relating to his student years at Cambridge: *[Mathematics] were scarce looked upon as Academical studies but rather Mechanical; as the business of Traders, Merchants, Seamen, Carpenters, Surveyors, ...* (WALLIS, 1967).

Whilst schools and the two universities still neglected mathematics teaching, the educational needs of these users of mathematics were being met by those "mathematical practitioners" who made a living from tutoring, lecturing and instrument-making (see, e.g. TAYLOR, 1970).

The establishment of professorial chairs at Oxford and then Cambridge marked a move away from this blinkered view of mathematics, but tended to reinforce the dichotomy between utilitarian mathematics and its practitioners on the one hand, and "gentlemanly applied mathematics" and gentlemen on the other. Aubrey's story of Gunter's "interview" for the first Savilian Chair of Mathematics at Oxford tells us much (Aubrey is not the most reliable of commentators, but his tales usually serve to illustrate contemporary thought and prejudices): *[Savile] first sent for Mr. Gunter... to have been his professor of Geometry: so he came and brought with him his sector and quadrant, and* *fell to resolving of triangles and doing a great many fine things. Said the grave knight, "Do you call this reading of Geometry? This is, showing of tricks, man!" and so dismissed him... and sent for Briggs* (AUBREY, 1975, p. 274).

Years later, De Morgan (1847) was to write of his disappointment on reading Wells's *Young Gentleman's Course of Mathematics* (1712). He had hoped that Wells, an Oxford tutor, would have demonstrated the place of mathematics within a liberal education, for earlier (1698) that author had produced a book which, whilst not rejecting commercial questions, had avoided those faults which De Morgan had found in the famous Cocker and his followers. But no: *I was mistaken: it is* gentlemanly *education, as opposed to that of* the meaner part of mankind, *that he wants to provide for... . The gentlemen are those whom God has relieved from the necessity of working.*

Yet it would be very wrong to believe that applications had no place in university mathematics. Certainly the "scheme" or syllabus which David Gregory drew up and proposed at Oxford in 1700 places great emphasis on applications, for his seven "courses" were:

1. Euclid.

2. Plane trigonometry, logarithms and practical geometry including the use of instruments.

3. Algebra.

4. Mechanics.

5. Optics including instruments such as telescopes and microscopes.

6. Astronomy.

7. The theory of the planets.

Possible extensions included fortifications, hydrostatics, ballistics and navigation. (See BRAYBROKE (ed) 1828 or, for an edited version, HOWSON 1982).

This, then, was far from a "Pure" syllabus, yet one suspects that the syllabus was more a development from that taught in medieval Prague (see above) than one intended to be followed by users and appliers. They, including those *Gentleman concerned in Collieries and Lead-Mines,* would have been more likely to benefit from the series of public (fee-paying) lectures offered by James Jurin in 1711 in Newcastle upon Tyne (see HOWSON, 1982).

Jurin's prospectus is of interest on many counts. It indicates that the need for mathematics was national, not merely London-centred, and, in view of the charges for the course (about £10 in 1711 money), was real and deeply-felt, but also, and for our purposes most importantly, it indicates how new technological developments were increasing the range of applicable and applied mathematics.

1. So much of the Principles of Geometry, Arithmetic and Algebra as shall be necessary for this undertaking.

2. The general laws of Motion, and the Principles of Mechanics deduced from them.

3. The Doctrine of Percussion, or the Effects which follow from the Stroke of Bodies upon one another.

4. The Natural Motion of all heavy Bodies.

5. The Motion of Bodies upon inclined Planes.

6. The Theory of all Kinds of Engines simple and compound, with a particular Explication of the Engines used in Collieries, and the method of Examining their Advantages or Defects.

7. Hydrostatics, under which Head will be demonstrated by Experiment; the chief Properties of Water and other fluids...; The Method of Calculating the Weight or Pressure of Water against the Banks of Rivers, or Milldams... and other Surfaces, and consequently determining the strength requisite for those Bodies to support the Pressure...

8. Pneumatics, the Weight and Spring of the Air, its Rarefaction and Condensation... The Air-pump, and Condenser, together with the Barometer, Thermometer, Hygrometer... their Nature and Uses explained...

9. Hydraulics, or the Doctrine of Water and other Fluids in Motion. The Method of estimating the Swiftness of Water running in any Canals open or closed, as in Rivers, or Mill-Races,... Conduit Pipes etc, with the Quantities of Water that they discharge. A particular Application to the draining of Collieries... for clearing and keeping a Colliery clear of Water. Of the Force of Fluids, as Air, or Water to carry about the Sails or Wheels of Mills and other

Engines, and the best proportion of the Machines driven by them, or by Horses.

10. Lastly, The Important Theory of the Friction of Rubbing of Machines, with the Impediment caused by the stiffness of the Ropes, for want of which the greatest Engineers have been disappointed in their undertakings, and the best concerted Machines have been rendered useless... it will be explained in an easy manner, partly by Experiment, and the Application of it to the Calculation of Machines will be demonstrated.

9. God and applied mathematics

Nature and Nature's laws lay hid in night;
God said, Let Newton be! *and all was light.*
(Pope's epitaph for Newton)

But could a knowledge of Newton's work lead a student to God? Timothy Jollie, founder of the dissenting academy at Attercliffe (at which Nicholas Saunderson, the blind Lucasian Professor of Mathematics, studied) *forbade mathematics as tending to scepticism and infidelity* (McLACHLAN, 1931, p. 108). Yet this was not the view of other dissenters. Newton, though far from an orthodox Christian, saw his great work as providing proof of God, and others saw the learning of mathematics as a step towards the acceptance of God. One such was Isaac Watts who also clearly demonstrated the distinction between a utilitarian and a gentlemanly mathematical education:

"Several parts of mathematical learning are also necessary ornaments of the mind ... and many of these are so agreeable to the fancy that youth will be entertained and pleased in acquiring the knowledge of them.

Beside the common skill in accounts which is needful for a trader, there is a variety of pretty and useful rules... in arithmetic to which a gentleman should be no stranger. ...

The world is now grown so learned in mathematical science that this sort of language is often used in common writing and conversations, far beyond what it was in the days of our fathers. And besides, without some knowledge of this kind we cannot make any farther progress toward an acquaintance with the arts of surveying, measuring, geography and astronomy, which is so entertaining and so useful an accomplishment to persons of a polite education".

This view of mathematics as a key component of liberal education, because of the *language* it provides and the numerous applications to which it provides an entrée, is a far-sighted one and one to which our universities in particular have given insufficient cognisance in recent years.

Yet Watt's enthusiasm for mathematics was soon to wane. Studying mathematics tended to *beget a secret and refined pride, an over-weening and over-bearing vanity* and to tempt its practitioners to *presume on a kind of omniscience in respect of their fellow creatures*. Mathematics could only be *trusted in the hands of... those who have acquired a humble heart, a lowly spirit and a sober, and teachable temper*. (HOWSON, 1982, pp. 50-2).

Too much mathematics -even its applications- could make Jack an arrogant boy!

10. Exercises

A most important innovation involving mathematical text books occurred during the eighteenth century. Until then texts contained worked examples demonstrating how mathematical problems could be solved. In that century, however, it became increasingly popular to provide lists of exercises to be attempted by the reader. This was a significant move in many ways, for it clearly helped shift the emphasis from learning "what" to learning "how to". Ability to *do* mathematics came to be prized alongside that of *knowing* mathematics -of remembering theorems, proofs, etc. (Yet it must be noted that, even in the early nineteenth century, candidates for the "poll" (ordinary) degree at Cambridge were set only "bookwork", i.e. recall questions). More particularly, in our context, authors were allowed greater scope to introduce the "applications" of mathematics into their texts and, concurrently, to assist the reader to learn how to apply mathematics. As is to be expected, authors responded in very different ways.

Three "questions" from Cocker's famous and much reprinted *Arithmetic* (1677 and frequently thereafter) give a flavour of what was to be found in a typical, utilitarian, commercially-oriented work (29th Edn, undated, p. 127):

> 8 If 100 workmen in 12 days finish a piece of work or service, how many workmen are sufficient to do the same in 3 days?
>
> 9 A colonel is besieged in a town in which are 1000 soldiers with provision of victuals only for 3 months. The question is how many of his soldiers must he dismiss that his victuals may last the remaining soldiers 6 months?

10 If wine worth £20 is sufficient for the ordinary of 100 men when the tun [barrel] is sold for £30, how many men will the same £20 worth suffice when the tun is worth £24.

Perhaps the only "real life" problem here is in determining in which of these situations it is actually appropriate to apply "the single rule of three inverse". Yet such questions persist[8] and still with contradictory effects. For, whilst exercising the powers of "translation" and practising specific mathematical techniques, they treat the way in which mathematics can actually be applied in an unthinking and potentially confusing manner. One is hardly likely to learn how to *apply* mathematics through solving such examples. Yet the association with such "real life" problems can motivate the learning of mathematics, if only in a completely spurious manner.

More genuine applications can, of course, be found in other texts. It is, for example, interesting to take a particular topic in pure mathematics and to see how texts written at different times attempt to demonstrate its applicability. This also serves the purpose of showing how particular exercises and examples gradually enter texts and often are either generalised or become stock questions. Thus, looking quickly at "maxima and minima" in four different calculus texts published between 1750 and 1850 we see the gradual acceptance of exercises, a changing attitude to applications, and the introduction of what are now stock questions. The first text, SAUNDERSON (1756), was, like the following two, written with Cambridge students in mind. It is a fluxional text and has no exercises. The uses of maxima and minima are illustrated through six examples which, in modern notation and words, are:

(1) Given $x + y = a$, find x such that xy is a maximum.

(2) Given $x + y = a$, find x such that xy^2 is a maximum.

(3) Find the max and min of $x^3 - 18x^2 + 96x$.

(4) Find the greatest horizontal range of a ball shot into the air with a given velocity.

(5) To find in which position a plane turning about a given axis will be most affected by a wind.

(6) To construct the frustrum of a cone having given base and altitude which would experience least resistance in a fluid flowing in a given direction.

Here we see a genuine, if limited, attempt to link the newly learned calculus with material learned in other branches of mathematics. (Note that as late as the 1880s, subjects in Part I of the Cambridge Mathematical Tripos, including elementary statics and dynamics, hydrostatics, optics and astronomy, were *to be treated without the use of the Differential and Integral Calculus and the methods of Analytical Geometry* (see also §11, 12).)

Vince, who held the Plumian Chair at Cambridge from 1796 to 1822, published a *Fluxions* in 1795 as part of Wood's *Principles of Mathematics and Natural Philosophy*. Again, there are no exercises, but now the worked examples are more extensive:

1. Given $x + y = a$, to maximise $x^m y^n$ (an interesting generalisation of SAUNDERSON (1) and (2)).

2. To inscribe the rectangle of greatest area in a given triangle.

3. To inscribe the cylinder of greatest volume in a given cone.

4. To inscribe the greatest rectangle in a given parabolic section.

5. To cut the parabola of greatest area from a given cone.

6. Given $x + y + z = a$ and xy^2z^3 a maximum to find x, y and z.

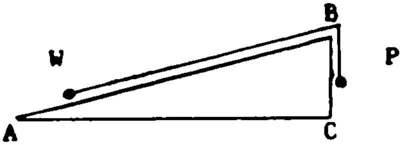
7. To find when y is a maximum given that $(x^3 + y^3)^2 = a^4x^2$.

So far, the examples have not been "applications" in our sense. However, an interesting set of applications now follows:

8. To determine at what angle the wind must strike against the sails of a windmill to maximise motion (cf. SAUNDERSON (5)).

9. *Given two elastic bodies A and C, to find an intermediate body x, so that the motion communicated from A to C through x may be the greatest possible.*

10. *Given the altitude BC of an inclined plane, to find it's length, so that a weight P acting upon another W in a line parallel to the plane may draw it up through AB in least time.*



11. *To find the position of the position of the planet Venus, when it gives the greatest quantity of light to the Earth.*

12. *Let Q be an object placed beyond the principal focus F of a convex lens, to find it's position, when it's distance Qq from it's image q is the least possible.*

13. *To find the Sun's place in the ecliptic, when that part of the equation of time which arises from the obliquity of the ecliptic is a maximum.*

14. *Given the base CB of an inclined plane AC, to find it's altitude BA, when the time of the descent of a body down the plane is the least possible.*

15. *Given the base CB, to find the perpendicular BA, such that a body descending from A to B and then describing BC with the velocity acquired, the time through AB and BC may be the least possible.*

16. *Given the base CB of the inclined plane AC, to find its altitude BA, such that the horizontal velocity of a body at C descending down AC may be the greatest possible.*

17. To find when $x^3 - 18x^2 + 96x - 20$ has max and min (cf. SAUNDERSON (3)).

18. To find the max and min of $y = x^3 - px^2 + qx - r$ (Seen as a generalisation of 17).

Here, then, we have an impressive set of examples illustrating how the newly explained results on maxima and minima can be applied to problems in *Hydrostatics, Mechanics, Optics and Astronomy,* and which raises several important pedagogical points. At one level one can consider the anomalous position of Qn. 15. This makes interesting demands on the solver's knowledge of elementary calculus and mechanics. Yet unlike 14 and 16 it is not a sensible "real-life" problem. Should we then object to its presence in the list?

Most interestingly, though, we see how the author has demonstrated to the reader a variety of ways in which the calculus can be applied to physical problems. In doing so, he revises various laws of physics. Yet it is unlikely that many of his readers would have been able to cope with these examples in the form of exercises. Here, then, we have the value of *awareness* of the power of mathematics stressed, rather than those of "knowledge for action" or "passive knowledge". Writing in 1910 (see HOWSON, 1982, p. 158), Charles Godfrey (who himself taught the future King George VI) pointed out that

> "In England we have a ruling class whose interests are sporting, athletic and literary. They do not know, or if they know do not realise, that this western civilisation on which they are parasitic is based on applied mathematics. This defect will lead to difficulties...".

The importance of demonstrating applications of mathematics in order that people might *appreciate* the use of mathematics -might be favourably influenced, rather than "filled with assessable knowledge", and so might help foster the further development of the subject, can be every bit as important as attempting the much more difficult task of turning people into appliers of mathematics. (Compare the provision of courses in musical appreciation alongside those in performance).

The third book "designed for the use of students in the [i.e. Cambridge] university", Hind's *Differential Calculus* (2nd Edn. 1831) need detain us little, for it avoided mention of any applications other than those to *curves and curve surfaces*. Thus in the use of notation and in its choice of examples it resembled a French text of the eighteenth century more than those of Saunderson and Vince. The book includes no exercises, but at that time published collections of exercises were available. (Of these, Bland's *Algebraic Problems producing Simple and Quadratic Equations* was a best-seller. Bland's questions were often of a whimsical nature which placed much emphasis on the ability to abstract a problem and set up an equation from a great deal of "noisy" verbiage[9]). Other authors, eg. Lardner 1825, also provided what we should now think of as a text in "Pure Mathematics". De Morgan's *Differential and Integral Calculus* devoted its seventeeth chapter to *Applications to Mechanics:* the reader was warned, however, that De Morgan *was not teaching [mechanics] for the sake of [its] results, but only showing how the differential calculus is applied to [it]*. Since the chapters of this work were distributed in serial fashion in the *Library of Useful Knowledge*, the applications appeared up to two years after the theory!

In this period, however, we begin to see new emphases appearing in English university text books, to accompany a new type of mathematical professionalism, and specialisation.

De Morgan's book was intended for the general reader, as was Tate's (1849): *it is highly desirable that Teachers and Practical Men should possess some knowledge of this most important branch of pure mathematics, in order to enable them to understand our best works on mechanical and experimental philosophy. The great physical laws to which it has pleased the Almighty to govern the universe... can only be duly interpreted by the aid of the symbolic language of the higher analysis* (v-vi). Tate, a schoolteacher who became one of the country's first teacher-trainers, had a strong scientific background (see, HOWSON, 1982) and placed great emphasis on utilitarian aims. His calculus examples *and* exercises comprised a blend of pure and "slightly" applied questions of a type which would not seem out of place today. Some of Vince's questions reappear (e.g. 2, 3, 4), but now we meet (for the first time in an English text?) the rectangular sheep-fold to be built alongside an existing wall, the open and closed boxes with fixed surface area whose volumes are to be maximised, the conical vessel to be constructed from a sheet of tin. Here we see the beginning of the establishment of a "convention" for the treatment of applications of maxima and minima in school texts.

Any deeper study of the rôle of "applications" in exercises should be linked with consideration of their place in examinations. The spread of written examinations was a significant feature of nineteenth century education in England (see, for example, MONTGOMERY, 1965 and ROACH, 1971). The impetus came from the redesigned Tripos examination at Cambridge. A new kind of Tripos began to take shape in the latter half of the eighteenth century. Not surprisingly, mechanics is well represented in the earliest extant examination paper (1785): for example, *Suppose a body thrown from an Eminence upon the Earth, what must be the velocity of Projection, to make it become a secondary planet to the Earth?* (It is interesting to consider when this question could first be described as a "real life" question. Clearly, in some senses it always was but if we accept that interpretation of "real-life" when did it become "relevant"?)

The influence of examinations on the teaching of applications of mathematics has been enormous. The presence and demands of examinations has led to subjects being taught; but often to their being taught in a manner which made it easy for them to be examined. Emphasis became placed on theory rather than on practice, on the model rather than on reality. Stock questions emerged which tested perhaps one line of "applied" mathematics followed by many of trigonometry, coordinate geometry, differential equations,... . Various entities of value in setting and solving examination questions were created, rods of length $2a\cos\phi$, particles of mass m_1 and m_2 joined by a light inextensible string of length l, etc. Thus, garbling, over-simplification and misrepresentation occurred as a consequence of institutionalisation, and the need to mark large numbers of papers in an "objective" manner.

Any pedagogical discussion of the place of applications in mathematics teaching must, if it is to have any lasting value, take fully into account the constraints imposed by the examination system and by the provision of exercises intended to prepare students for examinations.

11. Applications of Mathematics in a Liberal Education

On a number of occasions reference has been made to a "liberal education" and to the part that mathematics, and specifically the applications of mathematics, might play within it. To some extent such references must be confusing, for the idea of a "liberal education" is not a permanent, objective concept, but one which has evolved with time and was always highly subjective. Indeed, the historical development of the notion "mathematics within a liberal education" is deserving of detailed and in-depth study[10]. Here I can only hint at certain ideas which could be pursued. I have chosen to do this by looking at two key Victorian works on the subject: Whewell's *Of a Liberal Education* (1845) (and also the same author's *Thoughts on the Study of Mathematics as a Part of Liberal Education* (1835)), and the collection of essays edited by F.W. Farrar which appeared under the title *Essays on a Liberal Education* (1867).

Both publications must be considered in their educational context. Whewell's was particularly concerned with the two ancient universities of Oxford and Cambridge, at that time under attack from many quarters. Their continuing exclusion of Dissenters had led to the establishment of London University (what we know now as University College, London), with a different and entirely secular curriculum. But increasingly insidious comparisons came to be made with the German universities and the French Grandes Ecoles, or even, nearer to home, the Scottish system based on professorial lectures. A university education based almost entirely upon Latin, Greek and Pure Mathematics became a matter for heated criticism and debate. The discussion was to continue for many years with seminal contributions from, among others, Matthew Arnold and Cardinal Newman, but was in some sense officially answered by Royal Commissions on Oxford and Cambridge leading to the University Acts of 1854 and 1856. These Acts prefaced a growing interest in education by the state. In 1861 came the Newcastle Commission's report on elementary education, followed by reports on the "nine great" Public Schools (CLARENDON, 1864) and the endowed (grammar) schools (TAUNTON, 1868). It was partly as a reaction to these commissions, partly in an attempt to ward off that governmental control of education to be found in France and Prussia, that Farrar's book appeared: *Liberal Education in England is not controlled by the Government, nor is it entirely in the hands of tutors and schoolmasters; it is an institution of national growth and it will expand and improve only with the expansion and improvement of our national ideas of what education ought to be (p.v.).*

In his *Thoughts*, Whewell argued that some of the algebra currently taught at Cambridge might well be replaced by Mechanics and Hydrostatics, because *students whose talents and proficiency in pure mathematics are very considerable, are frequently found quite incapable of solving a very simple mechanical problem* (p. 165). However, Whewell was to place little emphasis on "experience" and "experiment" in establishing such branches of applied mathematics: *if mathematics be taught in such a manner that... its first principles be represented as borrowed from experience, in such a manner that the whole science is empirical only... [then] so studied it may... unfit the mind for dealing with other kinds of truth.*

That Whewell should publish *The Mechanical Euclid containing the Elements of Mechanics and Hydrostatics demonstrated after the manner of the Elements of Geometry* (1837) comes, then, as no surprise. It represents an extreme approach to applied mathematics -that which sees it not as modelling a physical situation but as illustrating divine, fundamental truths. *Statics, like Geometry, rests upon axioms which are neither derived directly from experience, nor capable of being superseded by definitions, nor by simpler principles. ... These axioms are not arbitrary assumptions, nor selected hypotheses; but truths which we must see to be necessarily and universally true, before we can reason on to anything else* (1837, pp. 165-6).

Although the philosophy of mathematics has advanced much since 1837 -it is interesting to think that Whewell wrote after Bolyai's and Lobatchewsky's papers on non-Euclidean geometry first appeared and, at a stroke, demolished so many of his beliefs- the spirit in which much applied mathematics is *still taught* bears a more than passing resemblance to that set out by Whewell. Yet he was to fight against one prominent feature of today's teaching: the use of analytical methods in preference to geometrical ones. Pedagogically-based opposition to analytical (algebraic and coordinate) methods arose in several European countries early in the 19th century (see, e.g. HOWSON 1984). *Analysis... has in it no clearness or light. The student* *who is led on in such a course, is immersed in a mist of symbols, in which he only here and there sees a dim twilight of meaning* (1837, 54) *The geometrical student has a firmer hold of his principles than the analytical student has. The former holds his fundamental truths by means of his conceptions; the latter, by... symbols. In the applications of Mathematics to problems of engineering and the like, the generalities in which the analyst delights are a source of embarrassment and confusion, rather than of convenience and advantage* (pp. 61-62).

Whewell saw it as very desirable that the "best mathematics" students should be made aware of how mathematics was being used in engineering: Poncelet, Willis and de Pambour would all provide them with *excellent examples of mathematical rigour, ingenuity and beauty.* Perhaps the central reason for dealing with such applications of mathematics was that, given this knowledge, *all men of education [can] understand each other in discussing mechanical questions* (1845, p. 57). Essentially, then -and this is not to be scoffed at- a "liberal education" enabled the recipients to *converse* with engineers; to be one, called for a vocational education.

The danger of such an approach to applications is clearly set out in J.M. Wilson's essay in the Farrar volume: *It is singular that the Mathematical Tripos at Cambridge is so unscientific, and the Natural Science Tripos at Oxford so unmathematical. At Cambridge a man may get the highest honours in mathematics... and have never seen... a lens, an air pump or a thermometer; and at Oxford a man may get his first in natural science without knowing the Binomial Theorem... Surely these are mistakes* (p. 254).

Scientific "applications" *divorced* from experience, then, could be essentially unscientific. This is an interesting criticism which is equally valid today. "Applications of mathematics" taken from other disciplines can well induce misunderstandings of how those disciplines actually operate.

Wilson effectively solved this problem by thinking of "Mathematics", as a time-tabled subject, in almost "pure" terms. This was *not,* however, because he valued the subject as *the key to all reasoning and... the perfection of training* (p. 253), but because arithmetic and geometry *form the language of science... and are indespensable to its Study* (p. 254). He, therefore, laid great emphasis on the teaching of science *and* the teaching of *experimental* science. Statics and dynamics were to be first studied experimentally within physics and their study was then to be enriched through the use of mathematics.

Unlike Whewell, Wilson thought mechanics should be taught under the banner of physics. The debate as to whether or not mechanics is a fit subject for the "mathematics" curriculum continues. In Britain it was traditionally seen as a *sine qua non* of specialist school mathematics, whereas in other countries this was not so. Now its position in school mathematics courses is under attack. Yet its new defenders, for example, the Mechanics in Action Project, propose that its teaching should be based on experiment and observation, a manner of which Wilson rather than Whewell would have approved[11].

The question of a "liberal education" then is a complex one. Yet Whewell and Wilson would still seem to have certain degrees of validity. From Whewell we observe the need to prepare "educated" people to understand and speak the language of scientists, engineers, and others; from Wilson we learn of the requirement to do so in a way which does not misrepresent what happens in these other disciplines.

12. The University Curriculum

In Section 3 we referred to the presence of "applied mathematics" in the curriculum of Prague and other fourteenth century universities and in later pages there have been other references to university courses.

The way in which different topics in applied mathematics entered universities, the forms their teaching took and the manner in which they were assessed, are worthy of detailed study. Here I can only touch upon certain trends.

Although, music generally ceased to be considered part of mathematics by the end of the eighteenth century[12], the other "Prague" topics of optics, hydrostatics and astronomy continued to hold their place. Mechanics, was, of course, given much greater emphasis in the years following the publication of Newton's *Principia* and, indeed, even as late as the beginning of this century candidates for the Cambridge Tripos examination were examined on the contents of that book (Sections 1-3) and were expected to prove the propositions contained in it "by Newton's methods".

By that time, however, particularly in the specialist options of Schedule III, several new topics had been introduced -many stemming from the mathematics generated by the technological advances which gave rise to, and became associated with, the Industrial Revolution. Statistics and probability made a hesitant entry under the heading of "Theory of Chances, including Combination of Observations"; "Laplace's and allied funcions" was a new "applicable" option; there were also by 1884 specialist options on "Linear and

Planetary Theories", "Hydrodynamics", "Sound", "Vibrations of Strings and Bars", "Elastic solids", "Thermodynamics", "Conduction of Heat", "Electricity" and "Magnetism".

Perhaps the most striking feature of this list is the way that "physical" applications still dominated -indeed, entirely constituted the "applied mathematics" curriculum. ("Applied Mathematics" is still frequently used as a name to describe physical applications of mathematics). The manner in which applications from the natural and social sciences, from economics and management had to fight to gain a foothold in undergraduate courses certainly deserves serious study. The response of higher educational institutions was in general very slow and the reasons for this merit detailed examination. Certainly there was prejudice against upstart, "mathematically easy" subjects such as Operations Research, and, perhaps with a surer foundation, against those mathematical models in economics and elsewhere which lacked the precision of prediction traditionally supplied by models in the physical sciences. This, for example, led to considerable debate and acrimony when "catastrophe theory" bust upon the mathematical scene in the mid 1970s. Another obvious constraint was the lack of staff willing and competent to give such courses. Perhaps equally significant, but rarely made explicit, were inherent personal beliefs on what the aims of an undergraduate course in mathematics were and how these aims should be interpreted in terms of the course options offered. (Regrettably, few tried to emulate Whewell by making their beliefs explicit).

As I wrote earlier, these are matters demanding more detailed study. Here one can only comment that by the early 1980s the scope of the Cambridge Tripos had been extended to include Part I courses on "Probability and its applications", "Special relativity", "Introduction to Statistics", "Quantum mechanics", "Introduction to Optimization", "Markov chains" (including, *inter alia,* applications to genetics), and further Parts II and III options on statistics, probability, maximisation, control theory, quantum electronics, seismic waves, statistical physics, and mathematical economics. (Some of which can be clearly associated with the Second Industrial Revolution and the emergence of computers which could handle vast amounts of data drawn from social and other non-physical sources). Elsewhere courses on, for example, applications in medicine and biology were introduced.

This great widening in the range of applications is not echoed, but rather foreshadowed, in some of the school textbooks of the 1960s[13]. Thus, to take but a few examples, critical path analysis, linear programming, the Ford-Fulkerson theorem on maximal flow through networks, applications of Markov chains, can all be found in the School Mathematics Project books of that decade. Here we see how authors working at a lower level (or writing books which seek to popularise mathematics) do not feel impelled to have "twelve-lectures-worth of material" (i.e. sufficient for a full course) before they introduce particular ideas or topics. Again, this raises questions about the way in which traditional course structures and ideas about "standards" may work to inhibit change and the introduction of new applications.

Yet these Cambridge options can be viewed solely as updated versions of those lectures provided at Prague or elsewhere half a millenium ago. Students were given the latest information on how mathematics was being applied in certain fields, and, in addition, were now expected to work exercises and solve problems relating to these applications. Yet no attempt was made in them specifically to train (or perhaps, more realistically, to assist) students to become appliers of mathematics. Here, as became increasingly the case in the post-Second-World-War years, other higher educational institutions took the lead. Thus, for example, in the early 1970s Southampton University instituted a third-year option on "Mathematical Modelling" -a course whose major objectives lay in the "process" domain of learning how to apply mathematics rather than in the "product" domain of becoming acquainted with useful applications and models formulated by others. Similar courses were soon to be found in other institutions, particularly the polytechnics, and their development is charted through papers, books and congress proceedings[14]. The change was a significant one in that it demonstrated new beliefs concerning the purpose of education and also, perhaps less evidently, concerning the ability and motivation of students.

Again, university developments had counterparts at a school level. The theme of the lectures at the 1961 conference which gave rise to the School Mathematics Project was "Mathematics Models" and greater emphasis came to be placed on this aspect of mathematical education in the late 1970s and the 1980s. Yet admirable though the aim of turning all pupils into modellers might be, the extent to which it can be achieved is still an open question[15].

Bacon's prediction (see Section 5), then, was wholly accurate and the effects of this on mathematics education have now been considerable. Clearly, the promise that mathematics will be applied to new fields is still valid. What a cursory investigation suggests to be important is the study of how quickly mathematics educators, particularly in institutions of higher education, can make thoughtful and worthwhile responses to such innovations.

13. The innovators

Our outline study of the history of the way in which applications have been used in mathematics education has demonstrated a gradual evolution. Applications of mathematics to new fields, new pedagogical devices (e.g. exercises and examinations), and new technological developments (industrial changes and, for example, the availability of cheap squared paper[16] and, more recently, the micro) have slowly had an effect. Such is the way with education, for systems are now so large that changes usually occur by a slow process of diffusion.

There are always those, however, who wish to see changes made more rapidly, who wish to see existing pedagogical practices overturned completely. Often "innovations" occur which seemingly reflect failed initiatives of earlier times. It is worth looking briefly at one or two of these in order to see what lessons there are to be learned.

One innovator worthy of special mention is A.C. Clairaut, the 18th century French mathematician and physicist. His *Elements of Geometry,* (1741) was a revolutionary work (so much so that an English translation was deemed valuable as late as 1881).

> "It is true that to avoid the dryness naturally belonging to... Geometry, some authors have stated after every important proposition the practical use to which it can be applied, but while they thus prove the utility of Geometry they do not facilitate its study; for as every proposition comes before its application the mind does not return to concrete objects till after it has undergone the fatigue of conceiving abstract ideas (vii)".

In order to circumvent this pedagogical obstacle, Clairaut proposed to base the study of geometry on the study of certain practical problems, mainly concerned with land measurement: *Ought I not [then] to fear that these Elements may be confounded with the ordinary treatises of land surveying? This thought can only come to those who do not consider that the measurement of land is not the real object of this book, but that it serves as the means of discovering the chief truths of geometry (xi).*

In this attempt to forestall criticism, Clairaut effectively laid bare one of the problems with this approach -that it was all too easy for the utilitarian framework to disguise, and nullify, the pure mathematical intent. We shall look at other criticisms of Clairaut's book in the next section.

A significant British attempt to emulate Clairaut was made in 1907 by David Mair. *The material for mathematical education may be chosen for the* *value of the knowledge of the material itself, or for the value of the training got in the course of acquiring the knowledge. The knowledge-value is greater the more closely the thing studied is related to human life and interests (iii).* Accordingly, as in Clairaut, every chapter of the text began with a "real-life" problem situation and the mathematics (pre-determined by the author) drawn out from that. The work is a *tour-de-force*; the range of applications used in the exercises is impressive and the author displays interesting pedagogical insights. However, a textbook to be successful has to satisfy many demands, including that of a reference book to which one can turn for results, definitions, etc. Mair gives scant recognition to mathematics as a codification of knowledge and emphasises mathematics as an activity. Perhaps this is one reason why his book and others in a similar vein which have followed it have had little success in gaining general acceptance and adoption. Moreover, important questions arise concerning the ability of pupils to transfer knowledge gained through the study of a particular example to other situations.

The belief that a practical problem should provide the starting point for mathematical learning, can, of course, be readily extended to education as a whole. Why try to draw mathematics alone out of a particular situation? In a sense this was the driving force behind the "Activity Curriculum, or Project Method" which had some adherents at the beginning of this century: *wholehearted purposeful activity proceeding in a social environment* (KILPATRICK, 1918). *The part of the teacher in such a curriculum is to discover and utilise the children's 'centres of interest' making them the starting points of projects or activities that lead to a progressive organisation and enrichment of experience... As knowledge is needed for the furtherance of any project, it is acquired, and easily acquired, because of a compelling sense of its relevance and purpose* (SED, 1947 p. 24). Kilpatrick's work in the USA was to some extent mirrored in the immediately post-revolutionary Soviet Union. Krupskaya (Lenin's wife) and Lunacharsky (People's Commissar for Education) argued for the "complex method" in which *teachers were not to teach according to formal curricula for academic subjects. Instead, they were to take the problems of the children, of local production and of daily life as their starting point, and to examine them in the light of various disciplines simultaneously* (CASTLES and WÜSTENBERG, 1979, p. 50). The school curriculum, then, was not divided by subject, but into three main themes, "Nature", "Work" and "Society", and each year's programme was described under these heads. Thus in the second year Nature comprised "Air, water, soil, cultivated plants, domestic animals". Work was "The village at work, or the city area where the pupil lives"; and Society, "Social institutions of the village or the city" (see, e.g., CONNELL, 1980, Chapter 7). This "complex method" was adopted in various degrees by USSR schools in the 1920s, particularly with younger children. However, Stalin's rule was marked by a return to traditional methods and the familiar timetabled subjects.

Since the 1920s various types of experimental and progressive schools have attempted to employ "complex" or "project" methods, now normally described in "cross-curricular" terms. Some have raised objections to this practice on the grounds that each school subject has its own *traditions of practical, aesthetic, and intellectual activity* and that the timetable should be organised so that the pupils, contacts with these should not merely be *casual or episodical* (SPENS, 1938, p. 157). In particular, though, problems have been caused through trying to teach mathematics in such a way because the solution of a real-life problem provides no motivation to generalise or abstract (two central notions in mathematics) or to refine and practise techniques (still two essentials for mathematical progress).

Many arguments currently being advanced bear remarkable similarities to the two we have mentioned in this section. Their proponents would do well to see what there is to learn from earlier attempts to utilise particular pedagogical practices.

14. Some criticisms

In the preceding sections I have tried to give a brief overview of the history of teaching applications of mathematics. On occasion I have included some personal criticisms arising from a study of old textbooks and other materials and from accounts of the outcomes of innovations. Yet such important problems as the extent to which applications should be included in/be used to motivate/should threaten to dominate the teaching of mathematics have, of course, been the subject of specific discussion on many previous occasions.

Clairaut's geometry, for example, was criticised by the Oxford don, Williamson, in 1781: *Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy that you may pick and choose without ceremony. This is useful in fortification; you cannot play at billiards without this... And upon such terms, and with such inducements, who would not be a mathematician?* De Morgan, writing some fifty years later, observed that Williamson's remarks were not without force if directed against experimental geometry as "an ultimate course of study", but that they *lose their ironical character and become serious earnest when applied to the same as a preparatory method* (DE MORGAN, 1833, p. 36).

Here, the heart of the problem is somewhat concealed: how and when is the transition to be made from mathematic's as an "experimental science" designed to solve particular "real-life" problems, to mathematics as a structured, abstract, generalised, theoretical study? In the 150 years since De Morgan wrote, much attention has been paid to developing more Clairaut-type "starting-off" points, but very little attention has been devoted to the difficulties of the transition which must eventually be made.

De Morgan, himself, though was to point out other dangers associated with applications, for example, the untruthful way in which "whimsical" examples were often passed off as "real-life" ones. More particularly, though, he attacked the way in which the values of a mathematical edcuation were being claimed and measured purely in utilitarian terms: *There are places in abundance where bookkeeping is the great end of arithmetic, land-surveying and navigation of geometry and trigonometry. In some [institutions] a higher notion is cultivated; and in mechanics, astronomy, etc. is placed the ultimate use of such studies. These are all of the highest utility; and were they the sole end of mathematical learning, this last would well deserve to stand high among the branches of knowledge which had advanced civilization; but were this all, it must descend from the rank it holds in education.*

Perhaps, nowadays, we should ascribe other benefits to the study of mathematics than would De Morgan, but the sentiments of the passage quoted still demand our attention and appreciation.

As we have already mentioned (section 7), Fitch in 1881 was to question the use of mathematics as a medium for transmitting moral and social messages. A more telling criticism of applications made at that time[17] concerned the way in which real life situations shed their reality on entering the mathematics classroom. Mathematical models were rapidly made which completely ignored social and even practical constraints. The emphasis soon changed from solving a practical problem to establishing the "usefulness" of mathematics or practising some of its techniques. Although now the initial problem was "real", the subsequent treatment and discussion was as far removed from actual life as was the case with the whimsical problems to which De Morgan objected. It is not difficult to find present-day examples in which this occurs[18]. Indeed, it is almost impossible for it not to be so, because we are primarily teachers of mathematics and not primarily solvers of real-life problems. Honesty of presentation would seem the key here.

Other criticisms of "project" and "complex" methods were given in the last section. The message given there must, then, be remembered and can serve as a fitting end to this paper: those who propose innovations in this area would increase their opportunities of success by considering the historical lessons relating to the promulgation, acceptance, diffusion, and rejection of previous initiatives. History does have an important rôle to play in curriculum development. I hope this paper will encourage others to fill in details which I could only sketch and to develop and further substantiate arguments at which I could only hint.

NOTES

1 A brief description of Bede's work and a translation of Chapter 2 (whose authorship has been questioned) can be found in YELDHAM (1926). The curriculum of the eighth century school at York has been frequently reproduced (see, for example, SYLVESTER (1970)).

2 A man has to transport a wolf, goat and cabbage across a river. Only one object can accompany him in his boat. If the wolf cannot be left alone with the goat or the goat with the cabbage, how should he plan his crossings to ensure that he and all his belongings cross the river safely. See SINGMASTER (1988) for an account of sources and also generalizations of this problem.

3 According to SINGMASTER (1988), the horseshoe problem appeared for the first time in RUDOLFF (1526). See also SMITH (1970, p. 484) for a reproduction of an Italian manuscript of 1535.

4 Fifteen schoolgirls went for a daily walk arranged in five rows of three. Arrange it so that for seven consecutive days no girl will walk with any school-fellow in any triplet more than once. See, for example, ROUSE-BALL (1949, Chapter X).

5 See, for example, ZILSEL (1945) or LILLEY (1958).

6 *The art of teaching the art of applying mathematics* was the title of Lighthill's *Presidential Address* (1971) to the Mathematical Association.

7 Such "applications" of mathematics would now seem to have disappeared from "official" curricula and textbooks although one suspects that examples might still be found in industry and other forms of employment.

8 A 1988 GCSE Paper (set by the London and East Anglian Group) included *A gang of workers is digging a trench. When there are six workers they manage to dig a trench 18m long in one day... . Work out the length of trench that one worker could dig in one day. A group of workers digs 12m in one day. How many workers...?*

9 The following is a sample question displaying many social overtones:

A city barge, with chairs for the company and benches for the rowers, went a summer excursion, with two bargemen on every bench. The number of gentlemen on board was equal to the square of the number of bargemen, and the number of ladies was equal to the number of gentlemen, twice the number of bargemen, and one over. Among other provisions, there were a number of turtles equal to the square root of the number of ladies, and a number of bottles of wine less than the cube of the number of turtles by 361. The turtles in *dressing consumed a great quantity of wine, and the party having stayed out till the turtles were all eaten, and the wine all gone, it was computed, that supposing them all to have consumed an equal quantity, (viz, gentlemen, ladies, bargemen, and turtles) each individual would have consumed as many bottles as there were benches in the barge. Required the number of turtles.*

Ans. 19.

10 Interested readers of its history in England will wish to sample such key books as PRIESTLEY (1768), NEWMAN (1859), HUXLEY (1868), ARNOLD (1869) and, in the early years of this century, KENYON (1917).

11 We note that arguments for teaching mechanics in a practical manner are contained in EDGEWORTH (1801) who wished to base the teaching of concepts such as velocity, momentum, force and gravity on experiment, in particular, the construction of machines and models by the children themselves. (See SIMON, 1960).

12 Pepys (see HOWSON, 1982, p. 43) still considered music as part of mathematics and Stone's *Mathematical Dictionary* (1726) contains entries under, for example, "counter-tenor" and "harmony".

13 A most important reference on applications of mathematics at a school level is POLLAK (1979).

14 FREUDENTHAL (1968) still remains a useful reference, but the interested reader will find many others. In particular, attention is drawn to the Proceedings of the three International Conferences on the Teaching of Mathematical Modelling and Applications.

15 Few attempts have been made to evaluate large-scale projects aiming to produce "applied problem solvers". One such, directed at children in primary schools, is described in HOWSON *et al* (1981), Chapter 7. Regrettably, no significant gains were detected.

16 The effect which the introduction into schools of something so seemingly inconsequential as cheap square paper can have is illustrated in BROOK and PRICE (1980).

17 Regrettably I have lost the reference! I believe the author was Quick and that the example he analysed concerned the boiler of a steam engine.

18 For example: I recently heard one enthusiast for "real-life" problems suggest as a sensible question the location of a warehouse to service four given stores which had different levels of sales. He proceeded to "solve" the question by finding a point such that the sum of the four, weighted *straightline* distances from that point to the stores was a minimum. Thus, "real-life" transport networks were ignored as was, for instance, the problem of whether the "optimal" location was in a town centre, the centre of a national park, or the middle of a river estuary! A serious attempt to consider a similar problem was, in fact, made in the early 1970s. It was a package of data relating to the siting of an oil refinery. Many factors had to be weighed and mediated; financial, social and environmental. There was no simple "correct" answer; the question could not be "solved" within the confines of a single discipline, mathematics, geography or social studies. As a result, the kit was hardly used. The lessons of this for education should not be lost.

BIBLIOGRAPHY

ARNOLD, M. (1986): *Culture and Anarchy*, (ed. Dover Wilson 1932), Cambridge.

AUBREY, J. (1680?): *Brief Lives*, Folio Society (reprint 1975).

BALL, W.W.R. (1949): *Mathematical Recreations and Essays*, Macmillan, 11th edition.

BONNYCASTLE, J. (1808): *Arithmetic*, Johnson.

BRAYBROOKE, Lord (1828): *Memoirs and Correspondence of Samuel Pepys, Esq., FRS*, Vol. 5.

BROOK, W.H. and PRICE, M.H. (1980): "Squared paper in the nineteenth century: instrument of science and engineering and symbol of reform in mathematical education", *Educ. Studies in Maths, 11*, 365-381.

CASTLES, S. and WÜSTENBERG, W. (1979): *The Education of the Future*, Pluto Press.

CLAIRAUT, A.C. (1881): *Elements of Geometry* (trs.), Kegan Paul.

CLARENDON COMMISSION (1864): *Report of the Public Schools Commission*, HMSO.

CONNELL, W.F. (1980): *A History of Education in the Twentieth Century World*, Curriculum Development Centre, Canberra.

DE MORGAN, A. (1833): "On the Method of teaching the Elements of Geometry", *Q. Journal of Education, 6*, 35-49.

DE MORGAN, A. (1847): *Arithmetical Books* (reprinted in SMITH, 1970).

EASTON, J.B. (1967): "The early editions of Robert Recorde's *Ground of Artes*", *Isis, 58*, 515-32.

EDGEWORTH, R.L. and EDGEWORTH, M. (1801): *Practical Education*, (2nd Edn.)

FARRAR, F.W. (1867): *Essays on a Liberal Education*, London.

FITCH, J. (1881): *Lectures on Teaching*, Cambridge.

FREUDENTHAL, H. (ed.) (1968): "How to teach Mathematics so as to be useful", *Educ. Studies in Mathematics, 1* (special issue).

GREGORY, O. (1840): *Hints for Teachers*, Whittaker.

HERMELINK, H. (1978): "Arabic recreational mathematics as a mirror of age-old cultural relations between Eastern and Western civilizations" in Hassan, A.Y. *et al*, *Proc. First International Symposium for the History of Arab Science*, Aleppo, 44-52.

HOWSON, A.G. (1982): *A history of mathematics education in England*, Cambridge.

HOWSON, A.G. (1984): Euclid: 'a very English subject', *Sciences et Techniques en Perspective, V*, 60-102.

HOWSON, A.G., KEITEL, C. and KILPATRICK, J. (1981): *Curriculum Development in Mathematics*, Cambridge.

HOWSON, A.G. and MELLIN-OLSEN, S. (1986): "Social norms and external evaluation" in Christiansen B. *et al*, *Perspectives on Mathematical Education*, Reidel, 1-48.

HOYRUP, J. (1987): *The formation of "Islamic Mathematics": Sources and Conditions*, Roskilde University.

HUXLEY, T.H. (1868): "A Liberal Education and where to find it", in *Science and Education* (1895), Macmillan, Vol. 3.

KENYON, F.G. (ed.) (1917): *Education, Scientific and Humane*, Murray.

KILPATRICK, W.H. (1918): "The Project method", *Teachers College Record, 19*, 4 September.

KIRKMAN, T.P. (1850): "Problem", *Lady's and Gentleman's Diary*, 48.

LIGHTHILL, M.J. (1971): "The art of teaching the art of applying mathematics", *Math. Gazette, 55*, 249-270.

LILLEY, S. (1958): "Robert Recorde and the idea of progress", *Renaissance and Mod. Studies*, 2, 3-37.

McLACHLAN, H. (1931): *English Eduation under the Test Acts*, Manchester University.

MAIR, D.B. (1907): *A School Course of Mathematics*, Oxford.

MONTGOMERY, R.J. (1965): *Examinations*, Longman.

NEWMAN, J.H. (1859): *On University Education*, Everyman's Library (1915), Dent.

POLLAK, H. (1979): "The Interaction between mathematics and other school subjects" in *New Trends in Mathematics Teaching*, vol. 4. UNESCO, 232-248.

PRIESTLEY, J. (1768): *Essay on a Course of Liberal Education.*

ROACH, J. (1971): *Public examinations in England 1850-1900*,

RUDOLFF, C. (1526): *Kunstliche Rechnung.*

SAUNDERSON, N. (1756): *The Method of Fluxions.*

SCOTTISH EDUCATION DEPARTMENT, (1947): *Secondary Education*,

SIMON, B. (1960): *Studies in the History of Education 1780-1870*, Lawrence and Wishart.

SINGMASTER, D. (1988): *Sources in Recreational Mathematics*, South Bank Poly.

SMITH, D.E. (1970): *Rara Arithmetica* (reprint), Chelsea.

SPENS REPORT (1938): *Secondary Education*, HMSO.

SWETZ, F. (1978): *Socialist Mathematics Education*, Burgundy Press.

SYLVESTER, D.W. (1970): *Educational Documents 800-1816*, Methuen.

TAUNTON REPORT (1868): *Report of the Schools Inquiry Commission*,

WALLIS, J. (1697): *An Account of Some Passages in His Own Life.*

WHEWELL, W. (1835): *Thoughts on the Study of Mathematics*, Cambridge.

WHEWELL, W. (1845): *Of a Liberal Education*, Parker.

YELDHAM, F. (1926): *The Story of Reckoning in the Middle Ages*,

YELDHAM, F. (1936): *The Teaching of Arithmetic through Four Hundred Years (1535-1935)*, Harrap.

ZILSEL, E. (1945): "The genesis of the concept of scientific progress", *J. Hist., Ideas, 6*, 325-349.