Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 32, No. 1, 2014, pág. 71–90

Nemytskii operator on generalized bounded variation space

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Abstract. In this paper we show that if the Nemytskii operator maps the (ϕ,α)-bounded variation space into itself and satisfies some Lipschitz condition, then there are two functions g and h belonging to the (ϕ,α)-bounded variation space such that $f(t,y)=g(t)y+h(t)$ for all $t\in[a,b]$, $y\in\mathbb{R}$.
Keywords: Riesz p-variation, (ϕ,α)-bounded variation.
MSC2010: 26A45, 26B30, 26A16, 26A24.

El operador de Nemytskii en espacios de variación acotada generalizados

Resumen. En este artículo demostramos que si el operador de Nemytskii lleva el espacio de variación (ϕ,α)-acotada en sí mismo, y satisface cierta condición de Lipschitz, entonces existen dos funciones g y h perteneciendo al espacio de variación (ϕ,α)-acotada tal que $f(t,y)=g(t)y+h(t)$ para todo $t\in[a,b]$, $y\in\mathbb{R}$.

Palabras claves: p-variación de Riesz, variación (ϕ,α)-acotada.

1. Introduction

Two centuries ago, around 1880, C. Jordan (see [15]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones, when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such

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Received: 09 November 2013, Accepted: 17 March 2014.
To cite this article: R. E. Castillo, H. Rafeiro, E. Trousselot, Nemytskii operator on generalized bounded variation space, *Rev. Integr. Temas Mat.* 32 (2014), no. 1, 71–90.

Vol'pert, N. Wiener, among many others. It is noteworthy to mention that many of these generalizations where motivated by problems in such areas as calculus of variations, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation in mathematical physics see the monograph [13].

In his 1910 paper F. Riesz (see [27]) defined the concept of bounded p-variation $(1 \leq p < \infty)$ and proved that, for $1 < p < \infty$, this class coincides with the class of functions f, absolutely continuous with derivative $f^{'} \in L_p[a,b]$. Moreover the p-variation of a function f on $[a,b]$ is given by $\mathsf{V}_p(f,[a,b]) = \mathsf{V}_p(f) = \|f'\|^p_{L^p[a,b]}$.

In [3] the first and third named authors generalized the concept of bounded p-variation introducing a strictly increasing continuous function $\alpha : [a,b] \to \mathbb{R}$ and considering the bounded p-variation with respect to α. This new concept was called (p,α)-bounded variation and denoted by $\mathsf{BV}_{(p,\alpha)}[a,b]$. In [2] there was a further generalization, given rise to the concept of (ϕ,α)-bounded variation. In this paper we show that the Nemytskii operator is bounded in $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

2. Definitions and Notations

In this section, we gather definitions and notations that will be used throughout the paper. Let α be any strictly increasing, continuous function defined on $[a,b]$.

2.1. Lebesgue-Stieltjes measure and integral

Lebesgue-Stieltjes measure

Let α be a strictly increasing continuous function on $\mathbb{R}$ with finite value. Let $\gamma = \big\{[a,b) : a < b\big\}$ the set of all half-open intervals in $\mathbb{R}$. Define $\mu_\alpha : \gamma \to \mathbb{R}^+$ by

$$\mu_\alpha([a,b)) = \alpha(b) - \alpha(a).$$

It is not hard to show that μ_α is σ-additive on γ, that is, if $\big\{[a_n,b_n)\big\}_{n\in\mathbb{N}}$ is a disjoint sequence of members of γ, then

$$\mu_\alpha\left(\bigcup_{n=1}^{\infty}[a_n,b_n)\right) = \sum_{n=1}^{\infty}\mu_\alpha\left([a_n,b_n)\right).$$

Then there exists a unique extension of μ_α into the Borel sets of $\mathbb{R}$. The completeness of this extension is call the Lebesgue-Stieltjes measure induced by α, defined on the σ-algebra $\overline{\gamma}$, and will be denoted as μ_α too (see, e.g., [12]). In the case $\alpha(x) = x$ we get back the usual Lebesgue measure.

Example 2.1. *Let α be a strictly increasing absolutely continuous on $[a,b]$ with derivative α'. In this case the corresponding measure μ_α is defined on all subsets of $[a,b]$ which are Lebesgue measurable, and for each subset of this kind*

$$\mu_\alpha(A) = \int_A \alpha'(x)\,\mathrm{d}x.$$

Indeed, by virtue of the Lebesgue theorem, for each interval (c,d) *we have*

$$\mu_\alpha((c,d)) = \alpha(d) - \alpha(c) = \int_c^d \alpha'(x)\,\mathrm{d}x.$$

The Lebesgue-Stieltjes integral

Let μ_α be the Lebesgue-Stieltjes measure generated by a strictly increasing function α. For this measure we define in the usual way the class of integrable functions and we establish the concept of Lebesgue integral

$$\int_a^b f(x)\,\mathrm{d}\mu_\alpha(x).$$

One integral of this type taking with respect to a measure μ_α corresponding to a generated function α, is called Lebesgue-Stieltjes integral and it is denoted by

$$(LS)\int_a^b f(x)\,\mathrm{d}\alpha(x),$$

and we may write $f \in LS(\alpha)$.

Example 2.2. *If* α *is an absolutely continuous function, its Lebesgue-Stieltjes integral is*

$$\int_a^b f(x)\,\mathrm{d}\alpha(x) = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x.$$

Theorem 2.3. *If* f *is a continuous function on* $[a,b]$, *then* $f \in RS(\alpha)$ *(*$RS(\alpha)$ *stands for the set of all Riemann-Stieltjes integrable functions, see [12]). Moreover this integral coincides with the Lebesgue-Stieltjes integral*

$$(LS)\int_a^b f(x)\,\mathrm{d}\alpha(x) = (RS)\int_A^b f(x)\,\mathrm{d}\alpha(x).$$

Definition 2.4. A function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous with respect to α if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\{(a_j,b_j)\}_{j=1}^n$ are disjoint open subintervals of $[a,b]$, then

$$\sum_{j=1}^n (\alpha(b_j) - \alpha(a_j)) < \delta \quad \text{implies} \quad \sum_{j=1}^n |f(b_j) - f(a_j)| < \varepsilon.$$

Thus, the collection α-$\mathsf{AC}[a,b]$ of all α-absolutely continuous functions on $[a,b]$ is a function space and an algebra of functions.

Definition 2.5. Let $E \subset [a,b]$. The set E has measure zero (in the *Jordan sense*) with respect to μ_α if given $\varepsilon > 0$ there exists a numerable cover by open intervals of E $\{I_n = (a_n,b_n) : n \in \mathbb{N}\}$ such that $\sum_{j=1}^n (\alpha(b_n) - \alpha(a_n)) < \varepsilon$.

Definition 2.6. Suppose f and α are real-valued functions defined on the same open interval (bounded or unbounded). Suppose x_0 is a point in this interval. We say f is α-derivable at x_0 if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)} \quad \text{exists.}$$

If the limit exists we denote its value by $f'_\alpha(x_0)$, which we call the α-derivative of f at x_0.

Lemma 2.7. *Let $f \in \alpha\text{-}\mathsf{AC}[a,b]$; then f'_α exists and is finite on $[a,b]$ a.e.$[\mu_\alpha]$.*

For the proof of Lemma 2.7 see [4, 14, 27].

2.2. Nemytskii Operator

Suppose I, M and N are nonempty sets. Given a mapping $\varphi : I \times N \to M$, the operator $\varphi^\natural : N^I \to M^I$ defined by $(\varphi^\natural g)(x) = \varphi(x, g(x))$ for all $x \in I$ and $g \in N^I$ is called the Nemytskii operator.

The Nemytskii operator is, in Krasnosel'skii-Rutickii terminology [17], the "simplest" classical nonlinear operator acting between function spaces, and its study is very well documented (see, e.g., [1, 17]). This documentation is mainly done in many classical function spaces, such as Hölder, Lebesgue, Orlicz, Sobolev, among others. To the best of our knowledge, its properties on spaces of bounded variations and its generalizations (even for intervals) is less studied.

In his 1982 paper, J. Matkowski (see [19]) has shown that the operator F generated by $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ maps $Lip[a,b]$ into itself and it is globally Lipschitz, there exists a positive constant K such that

$$\|F(u) - F(v)\|_{Lip[a,b]} \leq K\|u - v\|_{Lip[a,b]}$$

where $u, v \in Lip[a,b]$ if and only if there exists $g, h \in Lip[a,b]$ such that

$$f(t,x) = g(t)x + h(t) \tag{1}$$

for $t \in [a,b]$, $x \in \mathbb{R}$.

Remark 2.8. Note that there are function spaces where the Matkowski result does not remain valid. For example, on the space $C[a,b]$ and $L_p[a,b]$ with $p \geq 1$, take $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \sin(x)$ and define

$$f(t,x) = g(x), \quad t \in [a,b],\ x \in \mathbb{R}.$$

The function g is Lipschitz on $\mathbb{R}$, but f does not satisfy the relation (1); however, the operator F generated by f maps each the above spaces into itself and

$$\begin{aligned}\|Fu - Fv\|_\infty &= \|\sin(u(\cdot)) - \sin(v(\cdot))\|_\infty \\ &\leq K\|u - v\|_\infty\end{aligned}$$

with $u, v \in C[a,b]$, and

$$\|Fu - Fv\|_{Lip[a,b]} = \left(\int_a^b |\sin(u(t)) - \sin(v(t))|^p dt \right)^{\frac{1}{p}}$$
$$\leq K\|u - v\|_{Lip[a,b]}$$

with $u, v \in L_p[a,b]$, where K is the Lipschitz constant of g.

Matkowski's result has been extended in the framework of various function spaces for single-valued as well as multivalued Lipschitzian Nemytskii operators, cf. [18, 28, 7, 8, 9, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31].

2.3. Functions of (ϕ, α)-bounded variation

Definition 2.9. Let $\phi : [0, \infty) \to [0, \infty)$ be a function such that

1. ϕ is continuous;
2. ϕ is strictly increasing;
3. $\phi(t) = 0$ if and only $t = 0$;
4. $\lim_{t\to\infty} \phi(t) = \infty$.

Then such a function is know as a ϕ-function.

Definition 2.10. Let f be a real-valued function on $[a,b]$ and ϕ be a ϕ-function. Let $\Pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$. We consider

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi) = \sum_{j=1}^{n} \phi\left(\frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})} \right) (\alpha(x_j) - \alpha(x_{j-1}))$$

and

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f;[a,b]) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = \sup_{\Pi} \sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi),$$

where the supremum is taken over all partitions Π of $[a,b]$. $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ is called the Riesz (ϕ,α)-variation of f on $[a,b]$. If $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty$, we say that f is a function of Riesz (ϕ,α)-bounded variation. The set of all this functions is denoted by

$$\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b] = \{f : [a,b] \to \mathbb{R} \mid \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty\}.$$

Note that if we set $\phi(t) = t^p (1 \leqslant p < \infty)$ we get back the concept of (p,α)-bounded variation defined in [3].

Definition 2.11. Let ϕ be a convex ϕ-function. Then

$$\{f : [a,b] \to \mathbb{R} | \exists \lambda > 0 \text{ such that } \lambda f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\}$$
$$= \{f : [a,b] \to \mathbb{R} | \exists \lambda > 0 \text{ such that } \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) < +\infty\}$$

is called the vector space of (ϕ,α)-bounded variation function in the sense of Riesz, and we denote it by $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

Conclusion:

$$\mathsf{RBV}_{(\phi,\alpha)}[a,b] = \left[\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\right] \subset \mathsf{B}[a,b],$$

where $\mathsf{B}[a,b]$ is the set of bounded functions.

Definition 2.12. Let ϕ be a convex ϕ-function . We introduce the norm

$$\|\cdot\|^{\mathsf{R}}_{(\phi,\alpha)} : \mathsf{RBV}_{(\phi,\alpha)}[a,b] \to \mathbb{R},$$

with $f \mapsto |f(a)| + |f - f(a)|^{\mathsf{R}}_{(\phi,\alpha)} = |f(a)| + \inf\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\}.$

In their 1987 paper, L. Maligranda and W. Orlicz gave a lemma which supplies a test to check that some function spaces are Banach algebras (see [3]), specifically they stated the following

Lemma 2.13. Let $(\mathsf{X}, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under multiplication of functions. Let us assume that $f \cdot g \in \mathsf{X}$ and

$$\|fg\| \leqslant \|f\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty$$

for any $f, g \in \mathsf{X}$. Then the space X equipped with the norm

$$\|f\|_1 = \|f\|_\infty + \|f\|$$

is a normed Banach algebra. Also, if $\mathsf{X} \hookrightarrow \mathsf{B}[a,b]$, then the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. Moreover, if $\|f\|_\infty \leqslant M\|f\|$ for $f \in \mathsf{X}$, then $(\mathsf{X}, \|\cdot\|_2)$ is a normed Banach algebra with $\|f\|_2 = 2M\|f\|$, $f \in \mathsf{X}$ and the norms $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent.

We have the following results, proved in [2]:

Theorem 2.14. *Let ϕ be a convex ϕ-function. Then*

1. $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *with the norm* $\|f\|^{\mathsf{R}}_1 = \|f\|_\infty + \|f\|_{(\phi,\alpha)}$ $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *is a Banach algebra;*
2. $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *with the norm* $\|f\|^{\mathsf{R}}_2 = 2\max\{1, M\}\|f\|^{\mathsf{R}}_{(\phi,\alpha)}$ $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *is a Banach algebra, where*

$$M = \max\left\{\frac{1}{(\alpha(b)-\alpha(a))\phi\left(\frac{1}{\alpha(b)-\alpha(a)}\right)}, (\alpha(b) - \alpha(a)\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)\right\}.$$

3. *The norms* $\|\cdot\|^{\mathsf{R}}_{(\phi,\alpha)}$, $\|\cdot\|^{\mathsf{R}}_1$ *and* $\|\cdot\|^{\mathsf{R}}_2$ *are equivalent.*

Lemma 2.15. *Let ϕ be a convex ϕ-function defined on $[0,\infty)$ with $\phi(0) = 0$. Then the function $\psi : (0,\infty) \to \mathbb{R}$, with $x \mapsto \phi(x)/x$, is increasing.*

2.4. Medved'ev's theorem

In what follows, we need to justify why we need to introduce another condition on ϕ ((∞_1) condition) to avoid the theory became trivial.

Theorem 2.16. *Let ϕ be a convex ϕ-function. Then $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \mathsf{B}[a,b]$, i.e., all function of (ϕ,α)-bounded variation in the sense of Riesz is a function of bounded variation. Moreover*

$$\mathsf{V}(f,[a,b]) \leqslant (\alpha(b)-\alpha(a)) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

Proof. Let $\Pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$. Note that

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| = \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})}(\alpha(x_j) - \alpha(x_{j-1})).$$

Let

$$E = \left\{ j \in \{1,2,\ldots,n\} : \frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})} \leqslant 1 \right\}.$$

If $j \in E$, then $|f(x_j) - f(x_{j-1})| \leqslant (\alpha(x_j) - \alpha(x_{j-1}))$. If $j \notin E$, then $\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})} \geqslant 1$ and by Lemma 2.15 we obtain

$$\frac{\phi(1)}{1} \leqslant \frac{\phi\left(\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\right)}{\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}},$$

and thus

$$\frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})} \leqslant \frac{1}{\phi(1)}\phi\left(\frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})}\right)$$

for $j \notin E$.

Then

$$\begin{aligned}
&\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&= \sum_{j\in E}^{n} |f(x_j) - f(x_{j-1})| + \sum_{j\notin E}^{n} \frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})}(\alpha(x_j) - \alpha(x_{j-1})) \\
&\leqslant \sum_{j\in E}^{n} |f(x_j) - f(x_{j-1})| + \frac{1}{\phi(1)}\sum_{j\notin E}^{n} \phi\left(\frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})}\right)(\alpha(x_j) - \alpha(x_{j-1})) \\
&\leqslant \alpha(b) - \alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < +\infty
\end{aligned}$$

for all partitions Π of $[a,b]$; therefore,

$$\mathsf{V}(f,[a,b]) \leqslant \alpha(b) - \alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$ ☑

We will need the following:

Definition 2.17. Let ϕ be a convex ϕ-function. If $\lim\limits_{n\to\infty} \dfrac{\phi(x)}{x} = +\infty$, then we say that ϕ satisfies the (∞_1) condition.

We might observe that this limit exists since ϕ is convex. If the ϕ convex ϕ-function does not satisfy the (∞_1) condition, then there exists $r > 0$ such that $\lim_{n\to\infty} \frac{\phi(x)}{x} = r < +\infty$, that is, there exists $M > 0$ such that $\phi(x) \leqslant rx$ for $x \geqslant M$. Since $\frac{\phi(x)}{x}$ is increasing (Lemma 2.15), we have

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = \sup_{x\in(0,\infty)} \frac{\phi(x)}{x}.$$

Theorem 2.18. *Let ϕ be a convex ϕ-function which does not satisfy the (∞_1) condition, that is, if there exists $r > 0$ such that*

$$\lim_{x\to\infty)} \frac{\phi(x)}{x} = \sup_{x\in(0,\infty)} \frac{\phi(x)}{x} < +\infty,$$

then

$$\mathsf{BV}[a,b] \subset \mathsf{RBV}_{(\phi,\alpha)}[a,b].$$

Moreover,

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leqslant r\mathsf{V}\left(f,[a,b]\right).$$

Proof. Let $f \in \mathsf{BV}[a,b]$ and $\Pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be a partition of $[a,b]$. Let us consider

$$\frac{\phi\left(\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\right)}{\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}} \leqslant r, \quad j = 1,2,\ldots,n.$$

Then

$$\phi\left(\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\right)(\alpha(x_j)-\alpha(x_{j-1})) \leqslant r|f(x_j)-f(x_{j-1})|, \quad j = 1,2,\ldots,n,$$

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f,\Pi) \leqslant r\sum_{j=1}^{n} |f(x_j)-f(x_{j-1})|$$

for all partition Π of $[a,b]$,

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f,\Pi) \leqslant r\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$$

and

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leqslant r\mathsf{V}(f,[a,b]).$$

Therefore $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. ☑

From Theorem 2.16 and 2.18 we deduce

Corollary 2.19. *Let ϕ be a convex ϕ-function such that $\lim_{x\to\infty}\frac{\phi(x)}{x} = r < +\infty$. Then* $\mathsf{RBV}_{(\phi,\alpha)}[a,b] = \mathsf{BV}[a,b]$ *and*

$$\frac{1}{r}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leqslant \mathsf{V}(f,[a,b]) \leqslant \alpha(b)-\alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

To avoid this case (Corollary 2.19) we will assume that ϕ satisfy the (∞_1) condition.

Theorem 2.20. *Let ϕ be a convex ϕ-function which satisfy the (∞_1) condition, and let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Then f is absolutely continuous with respect to α on $[a,b]$, that is,*

$$\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \alpha\text{-}\mathsf{AC}[a,b].$$

Proof. Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Given $\varepsilon > 0$, let us consider (a_j,b_j), $j=1,2,\ldots,n$, a finite collection of disjoint subintervals contained in $[a,b]$. Let $m>0$ such that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \frac{m\varepsilon}{2}$. Since ϕ satisfy the (∞_1) condition, there exists $x_0 \in (0,\infty)$ such that $\phi(x) \geqslant mx$ for $x \geqslant x_0$. Next, let us consider the following set:

$$E = \left\{ j \in \{1,2,\ldots,n\} : \frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)} \geqslant x_0 \right\}.$$

If $j \in E$, then

$$x_0 \leqslant \frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}.$$

Since ϕ satisfy the (∞_1) condition, we have

$$m\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)} \leqslant \phi\left(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\right),$$

and thus

$$|f(b_j)-f(a_j)| \leqslant \frac{1}{m}\phi\left(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\right)(\alpha(b_j)-\alpha(a_j)).$$

From this last inequality we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} |f(b_j)-f(a_j)| \\
&\quad= \sum_{j\in E} |f(b_j)-f(a_j)| + \sum_{j\notin E} |f(b_j)-f(a_j)| \\
&\quad\leqslant \frac{1}{m}\sum_{j\in E}\phi\left(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\right)(\alpha(b_j)-\alpha(a_j)) + x_0\sum_{j\notin E}(\alpha(b_j)-\alpha(a_j)) \\
&\quad\leqslant \frac{1}{m}\sum_{j=1}^{n}\phi\left(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\right)(\alpha(b_j)-\alpha(a_j)) + x_0\sum_{j=1}^{n}(\alpha(b_j)-\alpha(a_j)) \\
&\quad< \frac{1}{m}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) + x_0\sum_{j=1}^{n}(\alpha(b_j)-\alpha(a_j)).
\end{aligned}$$

Now, choose $0<\delta<\varepsilon/(2x_0)$. Thus, if $\sum_{j=1}^{n}(\alpha(b_j)-\alpha(a_j))<\delta$, then

$$\sum_{j=1}^{n}|f(b_j)-f(a_j)|<\frac{\varepsilon}{2}+x_0\delta<\varepsilon.$$

Finally, collecting all of this information we conclude that, given $\varepsilon>0$, there exists $\delta>0$ such that for all finite family of disjoint subintervals $\{(a_j,b_j): j=1,2,\dots,n\}$ of $[a,b]$ such that $\sum_{j=1}^{n}(\alpha(b_j)-\alpha(a_j))<\delta$, then $\sum_{j=1}^{n}|f(b_j)-f(a_j)|<\varepsilon$, which means that $f\in\alpha\text{-}\mathsf{AC}[a,b]$. ☑

The coming result is a generalization of the result due to Medved'ev (see [16, 11]) and also provide us with a characterization to find out the (ϕ,α)-bounded variation of a function f in the sense of Riesz.

Theorem 2.21. *Let ϕ be a convex ϕ-function which satisfy the (∞_1) condition, and let $f:[a,b]\to\mathbb{R}$. Then:*

1. *If f is α-absolutely continuous function on $[a,b]$ and $\int_a^b\phi\left(|f'_\alpha(x)|\right)\mathrm{d}\mu_\alpha(x)<\infty$, then $f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and*

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)\leqslant\int_a^b\phi\left(|f'_\alpha(x)|\right)\mathrm{d}\mu_\alpha(x).$$

2. *If $f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$, that is $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)<\infty$, then f is α-absolutely continuous on $[a,b]$ and*

$$\int_a^b\phi\left(|f'_\alpha(x)|\right)\mathrm{d}\mu_\alpha(x)\leqslant\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

Proof. Since $f\in\alpha\text{-}\mathsf{AC}[a,b]$, by Lemma 2.7 f'_α there exists a.e. $[\mu_\alpha]$ on $[a,b]$. Let $x_1,x_2\in[a,b]$ with $x_1<x_2$; then

$$\begin{aligned}\phi\left(\frac{|f(x_2)-f(x_1)|}{\alpha(x_2)-\alpha(x_1)}\right)(\alpha(x_2)-\alpha(x_1))=&\phi\left(\frac{\int_{x_1}^{x_2}f'_\alpha(x)\,\mathrm{d}\alpha(x)}{\alpha(x_2)-\alpha(x_1)}\right)(\alpha(x_2)-\alpha(x_1))\\ \leqslant&\phi\left(\frac{\int_{x_1}^{x_2}|f'_\alpha(x)|\,\mathrm{d}\alpha(x)}{\alpha(x_2)-\alpha(x_1)}\right)(\alpha(x_2)-\alpha(x_1))\\ =&\phi\left(\frac{\int_{x_1}^{x_2}|f'_\alpha(x)|\,\mathrm{d}\alpha(x)}{\int_{x_1}^{x_2}\mathrm{d}\alpha(x)}\right)(\alpha(x_2)-\alpha(x_1))\\ \leqslant&\frac{\int_{x_1}^{x_2}\phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x)}{\int_{x_1}^{x_2}\mathrm{d}\alpha(x)}(\alpha(x_2)-\alpha(x_1))\\ =&\int_{x_1}^{x_2}\phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x).\end{aligned}$$

Now, let us consider $\Pi = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is a partition of $[a, b]$. Then

$$\begin{aligned}\sum_{j=1}^{n} \phi\left(\frac{|f(x_j) - f(x_{j-1})|}{\alpha(x_j) - \alpha(x_{j-1})}\right)(\alpha(x_j) - \alpha(x_{j-1})) &\leqslant \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x)\\ &= \int_a^b \phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x) < \infty,\end{aligned}$$

and hence for all partitions Π of $[a, b] <$ we have

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leqslant \int_a^b \phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x).$$

Therefore $f \in \mathsf{RBV}_{(\phi,\alpha)}[a, b]$.

ii) Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a, b]$; then, by Theorem 2.20 f is absolutely continuous with respect to α on $[a, b]$ and thus f'_α there exists a.e. $[\mu_\alpha]$ on $[a, b]$. Let $n \in \mathbb{N}$ and $\Pi_n = \{a = x_{0,n} < x_{1,n} < \ldots < x_{n,n} = b\}$ be a partition of $[a, b]$ given by

$$x_{j,n} = a + \frac{j(b-a)}{n}, \quad j = 1, \ldots, n.$$

Next, let us consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : [a, b] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} \dfrac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} & \text{if} \quad x_{j,n} \leqslant x < x_{j+1,n},\\ \text{any other reasonable value} & \text{if} \quad x = b.\end{cases}$$

We claim that $\{f_n\}_{n\in\mathbb{N}}$ converge to f'_α on those points where f is α-differentiable and different of $x_{i,n}$, $i = 0, 1, \ldots, n$, that is, on

$$A = \Big\{x \in [a, b] : \{f'_\alpha \text{ exists}\} \backslash \{x_{i,n} : n \in \mathbb{N}, i = 0, 1, \ldots, n\}\Big\}.$$

Let $x \in A$; then, for each $n \in \mathbb{N}$ there exists $k \in \{0, \ldots, n\}$ such that $x_{k,n} \leqslant x < x_{k+1,n}$, so that

$$\begin{aligned}f_n(x) =& \frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})}\\ =& \frac{f(x_{k+1,n}) - f(x) + f(x) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})}\\ =& \frac{\alpha(x_{k+1,n}) - \alpha(x)}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})}\,\frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x)} + \frac{\alpha(x) - \alpha(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})}\,\frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}.\end{aligned}$$

Therefore $f_n(x)$ is a convex combination of the points

$$\frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x)} \quad \text{and} \quad \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}.$$

Now, when $n \to \infty$, then $x_{n,k} \to x$ and $x_{k+1,n} \to x$. Since f is differentiable in x, the expressions

$$\frac{f(x_{k+1,n}) - f(x)}{\alpha(x_{k+1,n}) - \alpha(x)} \quad \text{and} \quad \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}$$

go to the α-derivative of f in x, that is, f'_α; from this we have

$$\lim_{n\to\infty} f_n(x) = f'_\alpha(x), \quad x \in A$$

a.e $[\mu_\alpha]$ on $[a,b]$). Since ϕ is continuous, then

$$\lim_{n\to\infty} \phi(|f_n(x)|) = \phi\left(\lim_{n\to\infty} |f_n(x)|\right) = \phi(|f'_\alpha(x)|), \quad x \in A.$$

Using Fatou's lemma, we obtain

$$\begin{aligned}
\int_a^b \phi(|f'_\alpha|)\,\mathrm{d}\alpha(x) &= \int_a^b \lim_{n\to\infty} \phi(|f_n(x)|)\,\mathrm{d}\alpha(x) \\
&\leqslant \liminf_{n\to\infty} \int_a^b \phi(|f_n(x)|)\,\mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \int_{x_{i,n}}^{x_{i+1,n}} \phi(|f_n(x)|)\,\mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \phi\left(\frac{|f(x_{i+1,n}) - f(x_{i,n})|}{\alpha(x_{i+1,n}) - \alpha(x_{i,n})}\right) \int_{x_{i,n}}^{x_{i+1,n}} \mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \phi\left(\frac{|f(x_{i+1,n}) - f(x_{i,n})|}{\alpha(x_{i+1,n}) - \alpha(x_{i,n})}\right) (\alpha(x_{i+1,n}) - \alpha(x_{i,h})) \\
&\leqslant \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty.
\end{aligned}$$

☑

Corollary 2.22. *Let ϕ be a convex ϕ-function that satisfies the (∞_1) condition. If $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, then f is α-absolutely continuous on $[a,b]$ and*

$$\int_a^b \phi(|f'_\alpha(x)|)\,\mathrm{d}\alpha(x) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

Corollary 2.23. *Let ϕ be a convex ϕ-function that satisfies the (∞_1) condition. Then $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ if and only if f is α-absolutely continuous on $[a,b]$ and $\int_a^b \phi(|f'_\alpha|)\,\mathrm{d}\alpha(x) < \infty$. Moreover, $\int_a^b \phi(|f'_\alpha|)\,\mathrm{d}\alpha(x) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$.*

Corollary 2.24. *Let ϕ be a convex ϕ-function that satisfies the (∞_1) condition. Let $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$; then,*

$$|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\left\{\varepsilon > 0 : \int_a^b \phi\left(\frac{|f'_\alpha(x)|}{\varepsilon}\right)\mathrm{d}\alpha(x) \leqslant 1S(\Lambda)\right\}.$$

Proof. From Corollary 2.22 we have

$$|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\}$$
$$= \inf\left\{\varepsilon > 0 : \int_a^b \phi\left(\frac{|f'_\alpha(x)|}{\varepsilon}\right) \mathrm{d}\alpha(x) \leqslant 1\right\}, \quad f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]. \qquad ☑$$

3. Nemytskii operator on $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$

We will use the Medved'ev generalized Lemma and the equivalent norms obtained by the Maligranda-Orlicz Lemma to show the following

Theorem 3.1. *Let ϕ be a convex ϕ-function which satisfies the (∞_1) condition. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$. The Nemytskii operator associated to f defined by*

$$F : \mathsf{RBV}_{(\phi,\alpha)}[a,b] \to \mathsf{RBV}_{(\phi,\alpha)}[a,b]$$
$$u \mapsto F(u)$$

with $F(u) = f(t, u(t))$, $t \in [a,b]$ act on $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and is globally Lipschitz, that is, there exists $K > 0$ such that

$$\|F(u_1) - F(u_2)\|^{\mathsf{R}}_{(\phi,\alpha)} \leqslant K\|u_1 - u_2\|^{\mathsf{R}}_{(\phi,\alpha)},$$

$u_1, u_2 \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, if and only if there exist $g, h \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ such that

$$f(t,y) = g(t)y + h(t), \quad t \in [a,b], \quad y \in \mathbb{R}.$$

Proof. *Sufficiency.* Let $y \in \mathbb{R}$. Define

$$u_0 : [a,b] \to \mathbb{R}$$
$$t \mapsto u_0(t) = y \text{ (constant function)}$$

and

$$F : \mathsf{RBV}_{(\phi,\alpha)}[a,b] \to \mathsf{RBV}_{(\phi,\alpha)}[a,b]$$

with $F(u_0)(t) = f(t, u_0(t)) = f(t,y)$. Then $f(t,y) \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, for all $y \in \mathbb{R}$.

By hypothesis $f(\cdot, y)$ is α-absolutely continuous.

Let $t, t' \in [a,b]$, $t < t'$, $y_1, y_2, y'_1, y'_2 \in \mathbb{R}$. Let us define u_1, u_2 by

$$u_i : [a,b] \to \mathbb{R}$$
$$s \mapsto u_i(s) = \begin{cases} y_i, & a \leqslant s \leqslant t; \\ \dfrac{y'_i - y_i}{\alpha(t') - \alpha(t)}(\alpha(s) - \alpha(t)) + y_i, & t \leqslant s \leqslant t'; \\ y'_i, & t' < s \leqslant b. \end{cases}$$

Then we may observe that $u_i \in \alpha\text{-}\mathsf{AC}[a,b]$, $i = 1, 2$.

We have

$$
\begin{aligned}
&u_1-u_2:[a,b]\to\mathbb{R}\\
&\qquad\qquad s\mapsto\begin{cases} y_1-y_2, & a\leqslant s\leqslant t;\\ \dfrac{y_1'-y_1-(y_2'-y_2)}{\alpha(t')-\alpha(t)}(\alpha(s)-\alpha(t))+y_1-y_2, & t\leqslant s\leqslant t';\\ y_1'-y_2', & t'<s\leqslant b.\end{cases}
\end{aligned}
$$

Since $u_1-u_2\in\alpha\text{-}\mathsf{AC}[a,b]$ its α-derivative exists and is given by

$$
(u_1-u_2)'_\alpha(s)=\begin{cases} 0, & a\leqslant s\leqslant t;\\ \dfrac{y_1'-y_1-(y_2'-y_2)}{\alpha(t')-\alpha(t)}, & t\leqslant s\leqslant t';\\ 0, & t'<s\leqslant b.\end{cases}
$$

To calculate $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{u_1-u_2}{\varepsilon}\right)$ we use the theorem of Medved'ev:

$$
\begin{aligned}
\int_a^b\phi\left(\left|\left(\frac{u_1-u_2}{\varepsilon}\right)'_\alpha(s)\right|\right)\mathrm{d}\alpha(s)&=\int_t^{t'}\phi\left(\left|\frac{y_1'-y_1-y_2'+y_2}{\varepsilon(\alpha(t')-\alpha(t))}\right|\right)\mathrm{d}\alpha(s)\\
&=\phi\left(\left|\frac{y_1'-y_1-y_2'+y_2}{\varepsilon(\alpha(t')-\alpha(t))}\right|\right)(\alpha(t')-\alpha(t))<+\infty.
\end{aligned}
$$

Since $u_1-u_2\in\alpha\text{-}\mathsf{AC}[a,b]$ we conclude that $u_1-u_2\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and

$$
\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{u_1-u_2}{\varepsilon}\right)=\phi\left(\left|\frac{y_1'-y_1-y_2'+y_2}{\varepsilon(\alpha(t')-\alpha(t))}\right|\right)(\alpha(t')-\alpha(t)).
$$

To calculate $\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)|}$ we make

$$
\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{u_1-u_2}{\varepsilon}\right)\leqslant 1
$$

if and only if

$$
\phi\left(\left|\frac{y_1'-y_1-y_2'+y_2}{\varepsilon(\alpha(t')-\alpha(t))}\right|\right)(\alpha(t')-\alpha(t))\leqslant 1.
$$

Next, applying ϕ^{-1} we have

$$
\left|\frac{y_1'-y_1-y_2'+y_2}{\varepsilon(\alpha(t')-\alpha(t))}\right|\leqslant\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right),
$$

and this is equivalent to

$$
\left|\frac{y_1'-y_1-y_2'+y_2}{(\alpha(t')-\alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)}\right|\leqslant\varepsilon
$$

and thus

$$\inf\left\{\varepsilon>0:\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{u_1-u_2}{\varepsilon}\right)\leqslant 1\right\}=\frac{|y_1'-y_1-y_2'+y_2|}{(\alpha(t')-\alpha(t))\phi^{-1}\left(\frac{1}{|\alpha(t')-\alpha(t)|}\right)}.$$

Then

$$\begin{aligned}\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)}&=|(u_1-u_2)(a)|+\inf\left\{\varepsilon>0:\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{u_1-u_2}{\varepsilon}\right)\leqslant 1\right\}\\&=|y_1-y_2|+\left|\frac{y_1'-y_1-y_2'+y_2}{(\alpha(t')-\alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)}\right|.\end{aligned}$$

By hypothesis $F(u_1)$ and $F(u_2)$ belong to $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and thus also $F(u_1)-F(u_2)\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ with

$$\begin{aligned}F(u_i):[a,b]&\to\mathbb{R}\\ s&\mapsto F(u_i)(s)=f(s,u_i(s))\end{aligned}$$

for $i=1,2$, with

$$f(s,u_i(s))=\begin{cases}f(s,y_i), & a\leqslant s<t;\\ f\left(s,\frac{y_i'-y_i}{\alpha(t')-\alpha(t)}(\alpha(s)-\alpha(t))+y_i\right), & t\leqslant s\leqslant t';\\ f(s,y_i'), & t'<s\leqslant b.\end{cases}$$

Let us consider the partition $\Pi=\{a<t<t<t'<b\}$; then,

$$\begin{aligned}\phi\left(\left|\frac{(F(u_1)-F(u_2))(t')-(F(u_1)-F(u_2))(t)}{\varepsilon(\alpha(t')-\alpha(t))}\right|\right)&=\sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{F(u_1)-F(u_2)}{\varepsilon},\Pi\right)\\&\leqslant\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{F(u_1)-F(u_2)}{\varepsilon},\Pi\right)\\&\leqslant 1;\end{aligned}$$

applying ϕ^{-1} we have

$$\overbrace{\left|\frac{(F(u_1)-F(u_2))(t')-(F(u_1)-F(u_2))(t)}{(\alpha(t')-\alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)}\right|}^{I}\leqslant\varepsilon,$$

and hence

$$I\leqslant\inf\left\{\varepsilon>0:\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{F(u_1)-F(u_2)}{\varepsilon}\right)\leqslant 1\right\}.$$

Finally,

$$\begin{aligned}
I &\leqslant |F(u_1) - F(u_2)(a)| + \inf\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{F(u_1) - F(u_2)}{\varepsilon}\right) \leqslant 1\right\} \\
&\leqslant \|F(u_1) - F(u_2)\|^{\mathsf{R}}_{(\phi,\alpha)} \\
&\leqslant K\|u_1 - u_2\|^{\mathsf{R}}_{(\phi,\alpha)} \\
&= K\left(|y_1 - y_2| + \left|\frac{y_1' - y_1 - y_2' + y_2}{(\alpha(t') - \alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)}\right|\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\left|\frac{f(t', y_1') - f(t', y_2') - f(t, y_1) + f(t, y_2)}{\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)(\alpha(t') - \alpha(t))}\right| \\
&\qquad\qquad \leqslant K\left(|y_1 - y_2| + \left|\frac{y_1' - y_1 - y_2' + y_2}{(\alpha(t') - \alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)}\right|\right).
\end{aligned}$$

Multiplying both sides by $(\alpha(t') - \alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t')-\alpha(t)}\right)$ we have

$$\begin{aligned}
&|f(t', y_1') - f(t', y_2') - f(t, y_1) + f(t, y_2)| \\
&\qquad \leqslant K\left((\alpha(t') - \alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t') - \alpha(t)}\right)|y_1 - y_2| + |y_1' - y_1 - y_2' + y_2|\right).
\end{aligned}$$

Since ϕ satisfies the (∞_1) condition, we obtain

$$\lim_{t' \to t}(\alpha(t') - \alpha(t))\phi^{-1}\left(\frac{1}{\alpha(t') - \alpha(t)}\right);$$

then,

$$|f(t, y_1') - f(t, y_2') - f(t, y_1) + f(t, y_2)| \leqslant k|y_1' - y_1 - y_2' + y_2|. \tag{2}$$

Next, putting

$$y_1' = w + z; \quad y_2' = w; \quad y_1 = z; \quad y_2 = 0$$

into (2), we have

$$|f(t, w + z) - f(t, w) + f(t, 0) - f(t, z)| \leqslant K|w + z - w + 0 - z| = 0,$$

and thus

$$f(t,w+z)-f(t,w)+f(t,0)-f(t,z)=0;$$

then

$$f(t,w+z)-f(t,0)=(f(t,w)-f(t,0))+(f(t,z)-f(t,0)).$$

Put

$$P_t(\cdot)=f(t,\cdot)-f(t,0);$$

hence

$$P_t(w+z)=P_t(w)+P_t(z).$$

Note that $P_t(\cdot)=f(t,\cdot)-f(t,0)$ is continuous, so that the Cauchy functional equation is satisfied and its unique solution is given by $P_t(y)=g(t)y$ with $g:[a,b]\to\mathbb{R}$, $y\in\mathbb{R}$. Let be

$$\begin{aligned} h:[a,b]&\to\mathbb{R}\\ t&\mapsto h(t)=f(t,0). \end{aligned}$$

Then $h\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$, and we can write $P_t(y)=f(t,y)-f(t,0)$ as $g(t)y=f(t,y)-h(t)$; therefore,

$$f(t,y)=g(t)y+h(t).$$

Since

$$f(t,1)-f(t,0)=(P_t(1)+f(t,0))-f(t,0)=g(t),\quad t\in[a,b],$$

we conclude that $g\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Conversely, let $g,h\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ such that $f(t,y)=g(t)y+h(t)$, that is

$$\begin{aligned} f:[a,b]\times\mathbb{R}&\to\mathbb{R}\\ (t,y)&\mapsto f(t,g)=g(t)y+h(t). \end{aligned}$$

The Nemytskii operator generated by f is given by

$$F(u)(t)=f(t,u(t))=g(t)u(t)+h(t),\quad f\in[a,b].$$

Since $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ is an algebra, we conclude that

$$F:\mathsf{RBV}_{(\phi,\alpha)}[a,b]\to\mathsf{RBV}_{(\phi,\alpha)}[a,b].$$

Let us show that F satisfy the global Lipschitz condition. Let the functions $u_1,u_2\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$; then

$$\begin{aligned} \|F(u_1)-F(u_2)\|^{\mathsf{R}}_{(\phi,\alpha)}&=\|f(\cdot,u_1(\cdot))-f(\cdot,u_2(\cdot))\|^{\mathsf{R}}_{(\phi,\alpha)}\\ &=\|g(\cdot)u_1(\cdot)-h(\cdot)-g(\cdot)u_2(\cdot)+h(\cdot)\|^{\mathsf{R}}_{(\phi,\alpha)}\\ &=\|g(\cdot)(u_1(\cdot)-u_2(\cdot))\|^{\mathsf{R}}_{(\phi,\alpha)}\\ &=\|g(u_1-u_2)\|^{\mathsf{R}}_{(\phi,\alpha)}. \end{aligned}$$

By the Maligranda-Orlicz lemma we have

$$\begin{aligned}\|F(u_1)-F(u_2)\|^{\mathsf{R}}_{(\phi,\alpha)} &\leqslant \|g\|_\infty\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)}+\|u_1-u_2\|_\infty\|g\|^{\mathsf{R}}_{(\phi,\alpha)}\\ &\leqslant \|g\|_\infty\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)}+\max\{1,M\}\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)}\|g\|^{\mathsf{R}}_{(\phi,\alpha)}\\ &=\left(\|g\|_\infty+\max\{1,M\}\|g\|^{\mathsf{R}}_{(\phi,\alpha)}\right)\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)}.\end{aligned}$$

Hence,

$$\|F(u_1)-F(u_2)\|^{\mathsf{R}}_{(\phi,\alpha)}\leqslant K\|u_1-u_2\|^{\mathsf{R}}_{(\phi,\alpha)},$$

with

$$K=\|g\|_\infty+\max\{1,M\}\|g\|^{\mathsf{R}}_{(\phi,\alpha)}$$

and

$$M=\max\left\{\frac{1}{(\alpha(b)-\alpha(a))\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)},(\alpha(b)-\alpha(a))\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)\right\}. \qquad \boxtimes$$

Acknowledgements

We want to thank the referee for his comments and suggestions. H. Rafeiro was partially supported by Pontificia Universidad Javeriana under the research project "Study of variable bounded variation spaces", Id Pry: 5863.

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