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Some remarks on a generalized vector product

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Abstract. In this paper we use a generalized vector product to construct an exterior form $\wedge : (\mathbb{R}^n)^k \to \mathbb{R}^{\binom{n}{k}}$, where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, $k \leq n$. Finally, for $n = k-1$ we introduce the reversing operation to study this generalized vector product over palindromic and antipalindromic vectors.
Keywords: alternating multilinear function, antipalindromic vector, exterior product, palindromic vector, reversing, vector product.
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Algunas observaciones sobre un producto vectorial generalizado

Resumen. En este artículo usamos un producto vectorial generalizado para construir una forma exterior $\wedge : (\mathbb{R}^n)^k \to \mathbb{R}^{\binom{n}{k}}$, en donde como es natural, $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, $k \leq n$. Finalmente, para $n = k-1$ introducimos la operación reversar para estudiar este producto vectorial generalizado sobre vectores palindrómicos y antipalindrómicos.
Palabras claves: función multilineal alternante, producto exterior, producto vectorial, reversar, vector palindrómico, vector antipalindrómico.

Introduction

It is well known that the vector product over $\mathbb{R}^3$ is an alternating 2-linear function from $\mathbb{R}^3 \times \mathbb{R}^3$ onto $\mathbb{R}^3$. Although this vector product is a natural topic to be studied in any course of basic linear algebra, there is a plenty of textbooks on this subject in where it is not considered over $\mathbb{R}^n$. The following definition, with interesting remarks, can be found also in [3, 7, 8]. Let

$$A_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ldots, A_{n-1} = \left(a_{(n-1)1}, a_{(n-1)2}, \ldots, a_{(n-1)n}\right)$$

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be $n-1$ vectors in $\mathbb{R}^n$. The vector product over $\mathbb{R}^n$ is a function $\times : (\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$ such that

$$\times (A_1,\ A_2,\ \ldots,\ A_{n-1}) = A_1 \times A_2 \times \cdots \times A_{n-1} = \sum_{k=1}^{n} (-1)^{1+k} \det (X_k)\, e_k, \qquad (1)$$

where e_k is the $k-$th unity vector of the standard basis of $\mathbb{R}^n$ and X_k is the square matrix obtained through the deleting of the $k-$th column of the $(a_{ij})_{(n-1)\times n}$. Notice that in this case the function is not binary and sends a matrix M of size $(n-1)\times n$ to a vector of its $\binom{n}{n-1}$ maximal minors.

One aim of this paper is to give an algorithm to construct, using elementary techniques, a function with domain in $(\mathbb{R}^n)^k$ and codomain $\mathbb{R}^{\binom{n}{k}}$ which will be an alternating k-linear function that obviously generalizes the previous vector product defined over $\mathbb{R}^n$.

Using techniques and methods of algebraic geometry we can see that the vector product obtained here, without signs, corresponds to the *Plücker coordinates* of the matrix M (see [4, 5]). Although this vector product is known and useful to define the concept of *Grassmanian variety* (see [4]), we present an alternative construction, avoiding algebraic geometry, which lead us to known results that can be found as for example in [6].

Another aim of this work, following [1, 2], is the presentation of some original results concerning the vector product for $n = k-1$ in palindromic and antipalindromic vectors by means of *reversing operation.*

The way this paper is presented can allow students and teachers of basic linear algebra the implementation of these results on their courses. This is our final aim.

1. A generalized vector product

In this section we set some preliminaries, properties and the Cramer's rule as application of the generalized vector product.

1.1. Preliminaries

Following [3, 7] we define the generalized vector product over $\mathbb{R}^n$ as the function

$$\times : (\mathbb{R}^n)^{n-1} \to \mathbb{R}^n$$

such that for $A_1 = (a_{11}, a_{12}, \ldots, a_{1n}), \ldots, A_{n-1} = \left(a_{(n-1)1}, a_{(n-1)2}, \ldots, a_{(n-1)n}\right)$, $n-1$ vectors of $\mathbb{R}^n$, their vector product is given by

$$\times (A_1,\ A_2,\ \ldots,\ A_{n-1}) = A_1 \times A_2 \times \cdots \times A_{n-1} = \sum_{k=1}^{n} (-1)^{1+k} \det (X_k)\, e_k, \qquad (2)$$

where e_k is the $k-$th element of the canonical basis for $\mathbb{R}^n$ and X_k is the square matrix obtained after the elimination of the $k-$th column of the matrix $(a_{ij})_{(n-1)\times n}$. The definition presented in expression (2) corresponds to a natural generalization of the vector product of two vectors belonging to $\mathbb{R}^3$.

1.2. Some properties

Let $A_1, A_2, \ldots, A_n$ be vectors of $\mathbb{R}^n$. The following statements hold.

1) $\times(A_1,\ A_2,\ \ldots,\ A_{n-1})$ is an orthogonal vector for the given vectors.
2) Assume $\alpha, \beta \in \mathbb{R}$, $B_i \in \mathbb{R}^n$; then

$$A_1 \times A_2 \times \cdots \times (\alpha A_i + \beta B_i) \times \cdots \times A_{n-1} = \\ A_1 \times A_2 \times \cdots \times \alpha A_i \times \cdots \times A_{n-1} + A_1 \times A_2 \times \cdots \times \beta B_i \times \cdots \times A_{n-1}.$$

3) Let A be the matrix given by $A = (A_1, A_2, \ldots, A_n)$. Then

$$\det A = A_1 \cdot (A_2 \times \cdots \times A_n) = (-1)^{1+j} A_j \cdot (A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times A_n).$$

4) The vectors $A_1, A_2, \ldots, A_{n-1}$ are $n-1$ linearly dependent vectors for $\mathbb{R}^n$ if and only if $A_1 \times A_2 \times \cdots \times A_{n-1} = 0$.

It is well known that these properties can be proven using the properties of the determinant (see, for example, [3, 7]).

1.3. Cramer's rule

Consider the following system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

that can be expressed in vectorial way as

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = B, \tag{3}$$

being $A_i = (a_{1i}, a_{2i}, \ldots, a_{ni})$ with $i = 1, 2, \ldots, n$ and $B = (b_1, b_2, \ldots, b_n)$. Suppose that $\det(A_1, A_2, \ldots, A_n) \neq 0$. Therefore the system has a unique solution that can be obtained applying the scalar product between the equation (3) and $A_2 \times A_3 \times \cdots \times A_n$; so we obtain

$$\begin{aligned} (x_1A_1 + x_2A_2 + \cdots + x_nA_n) \cdot A_2 \times A_3 \times \cdots \times A_n &= B \cdot A_2 \times A_3 \times \cdots \times A_n, \\ x_1A_1 \cdot A_2 \times A_3 \times \cdots \times A_n &= B \cdot A_2 \times A_3 \times \cdots \times A_n, \end{aligned}$$

since $A_j \cdot A_2 \times A_3 \times \cdots \times A_n = 0$ for $j = 2, 3, \ldots, n$. Therefore,

$$x_1 = \frac{B \cdot A_2 \times A_3 \times \cdots \times A_n}{A_1 \cdot A_2 \times A_3 \times \cdots \times A_n} = \frac{\det(B, A_2, A_3, \cdots, A_n)}{\det(A_1, A_2, A_3, \cdots, A_n)}. \tag{4}$$

In a general way, we can obtain

$$\begin{aligned} x_i &= \frac{B\cdot A_1\times A_2\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_n}{A_i\cdot A_1\times A_2\times\cdots A_{i-1}\times A_{i+1}\times\cdots\times A_n} \\ &= \frac{(-1)^{i+1}\det\left(A_1,A_2,\ldots,A_{i-1},B,A_{i+1},\cdots,A_n\right)}{(-1)^{i+1}\det(A_1,A_2,A_3,\cdots,A_n)} \\ &= \frac{\det\left(A_1,A_2,\ldots,A_{i-1},B,A_{i+1},\cdots,A_n\right)}{\det(A_1,A_2,A_3,\cdots,A_n)}, \end{aligned}$$

that is, the well-known *Cramer's rule.*

2. Didactic way to define $\wedge$: algorithm and properties

In this section we propose a didactic way to define the exterior product $\wedge$. To do this, we set an algorithm to the construction of $\wedge$ and as consequence of this construction arise some properties.

2.4. Algorithm to the construction of $\wedge$

Here we present an algorithm and some simple examples to illustrate it.

Step 1

Consider $n\in\mathbb{N}$ and $1\leq k\leq n$, being k an integer. We define

$$I=\{i_1 i_2\cdots i_k : 1\leq i_1<i_2<\cdots<i_k\leq n\};$$

this means that the elements belonging to I are chains of numbers conformed in agreement with the lexicographic order.

Example 2.1. *For $n=5$ and $k=3$ we have*

$$I=\{123,\ 124,\ 125,\ 134,\ 135,145,234,235,245,345\}.$$

As we can see, $\#I=\binom{n}{k}=\binom{5}{3}=10$.

Example 2.2. *For $n=5$ and $k=2$, we obtain $\binom{5}{2}=10$. For instance, I is given by*

$$I=\{12,13,14,15,23,24,25,34,35,45\}.$$

Step 2

We set that I should be ordered lexicographically:

$$I_{(1)}<I_{(2)}<\cdots<I_{\left(\binom{n}{k}\right)}.$$

In this way, if $I_s\in I$, then there exists p (only one) such that $I_s=I_{(p)}$. Thus, we can define p as the *rank* of I_s and will be denoted by $r\left(I_s\right)=p$. That is, p is the place of I_s in I as set of ordered elements lexicographically.

In Example 2.1 we can see that $r\left(234\right)=7$, $r\left(345\right)=10$. In Example 2.2 we have $r\left(25\right)=7$, $r\left(35\right)=9$.

Step 3

Let $u_1 = (u_{11}, u_{12}, \ldots, u_{1n}), \ldots, u_k = (u_{k1}, u_{k2}, \ldots, u_{kn})$, be k vectors of $\mathbb{R}^n$, with $k \leq n$. Consider the matrix $U = (u_{ij})$ of order $k \times n$ conformed by these vectors. Assume $i_1 i_2 \cdots i_k \in I$ and let $U_{i_1 i_2 \cdots i_k}$ be the matrix of order k, conformed by the k columns $i_1, i_2, \cdots, i_k$ of U. From now on, U always will be a matrix of this kind.

Example 2.3. *Consider*

$$U = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix};$$

in this case, $U_{123} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ *and* $U_{245} = \begin{pmatrix} a_2 & a_4 & a_5 \\ b_2 & b_4 & b_5 \\ c_2 & c_4 & c_5 \end{pmatrix}$.

Notice that when we choose a particular number of columns of such matrix U, which exactly corresponds to delete in U the non-selected columns.

Step 4

Consider

$$(\mathbb{R}^n)^k := \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k-\text{times}}.$$

Now we define the function *exterior product* $\wedge : (\mathbb{R}^n)^k \to \mathbb{R}^{\binom{n}{k}}$ as follows:

$$\wedge (U) = \sum_{i \in I} (-1)^{\binom{n}{k} - r(i)} \det (U_i)\, e_{\binom{n}{k} - r(i) + 1},$$

where $e_{\binom{n}{k} - r(i)+1}$ corresponds to the $\left(\binom{n}{k} - r(i) + 1\right)-$th unity vector of the standard basis of $\mathbb{R}^{\binom{n}{k}}$.

For convenience, we can write

$$\wedge (U) = \wedge (u_1, u_2, \ldots, u_k) = u_1 \wedge u_2 \wedge \ldots \wedge u_k.$$

Example 2.4. *Consider the vectors* $(2, 3, -1, 5), (4, 7, 2, 0) \in \mathbb{R}^4$. *The vector* $(2, 3, -1, 5) \wedge (4, 7, 2, 0)$ *belongs to* $\mathbb{R}^{\binom{4}{2}} = \mathbb{R}^6$. *In this case*

$$\begin{aligned} I &= \{12, 13, 14, 23, 24, 34\}, \\ U &= \begin{pmatrix} 2 & 3 & -1 & 5 \\ 4 & 7 & 2 & 0 \end{pmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \wedge (U) &= -\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} e_6 + \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} e_5 - \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} e_4 + \begin{vmatrix} 3 & -1 \\ 7 & 2 \end{vmatrix} e_3 - \begin{vmatrix} 3 & 5 \\ 7 & 0 \end{vmatrix} e_2 \\ &\quad + \begin{vmatrix} -1 & 5 \\ 2 & 0 \end{vmatrix} e_1 \\ &= -2e_6 + 8e_5 + 20e_4 + 13e_3 + 35e_2 - 10e_1 \\ &= (-10, 35, 13, 20, 8, -2). \end{aligned}$$

Example 2.5. *Consider the canonical basis for* $\mathbb{R}^4$*, that is,* $e_1 = (1,0,0,0)$*,* $e_2 = (0,1,0,0)$*,* $e_3 = (0,0,1,0)$ *and* $e_1 = (0,0,0,1)$*. Thus, the exterior product* $e_i \wedge e_j$ *for* $i < j$ *is given by*

$$\begin{aligned}
e_1 \wedge e_2 &= -(0,0,0,0,0,1) = -e_6 \in \mathbb{R}^6, \\
e_1 \wedge e_3 &= (0,0,0,0,1,0) = e_5 \in \mathbb{R}^6, \\
e_1 \wedge e_4 &= -(0,0,0,1,0,0) = -e_4 \in \mathbb{R}^6, \\
e_2 \wedge e_3 &= (0,0,1,0,0,0) = e_3 \in \mathbb{R}^6, \\
e_2 \wedge e_4 &= -(0,1,0,0,0,0) = -e_2 \in \mathbb{R}^6, \\
e_3 \wedge e_4 &= (1,0,0,0,0,0) = e_1 \in \mathbb{R}^6.
\end{aligned}$$

As we can see, the set $B = \{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\} \subset \mathbb{R}^6$ *is a basis for* $\mathbb{R}^6$*.*

Notice that in a given basis B for $\mathbb{R}^n$, the exterior product of them taken in sets of k-elements without repetition constitutes a basis B' for $\mathbb{R}^{\binom{n}{k}}$.

2.5. Some properties of $\wedge$

The following properties are satisfied by $\wedge$:

1) If $k = n$, then $\wedge(U) = \det(U)$.
2) If $k = n-1$, then $\wedge$ is the generalized vector product.
3) If n is even and $k = 1$, then U is orthogonal to $\wedge(U)$.
4) $\wedge$ is $k-$linear:

$$\wedge(u_1, \ldots, u_i + b, \ldots, u_k) = \wedge(u_1, \ldots, u_i, \ldots, u_k) + \wedge(u_1, \ldots, b, \ldots, u_k).$$

5) If M_p is a permutation of two rows (being fixed the other ones) of M, then $\wedge(M_p) = -\wedge(M)$.
6) If $u_1, \ldots, u_k$ are k $(\leq n)$ linear dependent vectors of $\mathbb{R}^n$, then $\wedge(u_1, \ldots, u_k) = 0 \in \mathbb{R}^{\binom{n}{k}}$.

Proof. We proceed according to each item.

1) Assuming $k = n$ we have $\binom{n}{k} = \binom{n}{n} = 1$ and $r(i) = 1$ (due to I has only one element). So,

$$\begin{aligned}
\wedge(U) &= \sum_{i \in I} (-1)^{\binom{n}{k} - r(i)} \det(U_i)\, e_{\binom{n}{k} - r(i) + 1} \\
&= \det(U_i).
\end{aligned}$$

Trivially we can see that for $\mathbb{R}$, $e_1 = 1$.

2) Assuming $k = n-1$, we have $\binom{n}{k} = \binom{n}{n-1} = n$; in this way, I has n elements. Owing to the symmetry of $\binom{n}{k}$, the election of $n-1$ columns of the matrix U corresponds to the elimination of one column of U (precisely the avoided column in the election). In other words, we can see that

$$U_i = X_{n-r(i)+1},$$

where $X_{n-r(i)+1}$ corresponds to the matrix that has been obtained throughout U deleting the $(n-r(i)+1)$-th column, so that

$$\begin{aligned} \wedge(U) &= \sum_{i\in I} (-1)^{n-r(i)} \det(U_i)\, e_{n-r(i)+1} \\ &= \sum_{i\in I} (-1)^{(n-r(i)+1)+1} \det(U_i)\, e_{n-r(i)+1} \\ &= \sum_{j=1}^{n} (-1)^{j+1} \det(X_j)\, e_j \\ &= u_1 \times \ldots \times u_k. \end{aligned}$$

3) For $n = 2p$ and $k = 1$, we have $\binom{2p}{1} = 2p$; thus, the cardinality of I is even and

$$I = \{1, 2, \ldots, p, p+1, \ldots, 2p\}.$$

Furthermore, $r(i) = 1$. In this way, $\wedge(U) \in \mathbb{R}^{2p}$. On the other hand, considering $U = (u_1, u_2, \ldots, u_{2p})$ and $\wedge(U) = (v_1, v_2, \ldots, v_{2p})$, we obtain

$$\begin{aligned} \wedge(U) &= \sum_{i\in I} (-1)^{2p-i} \det(U_i)\, e_{2p-i+1} \\ &= \sum_{i=1}^{2p} (-1)^{i} u_i e_{2p-i+1} \\ &= (u_{2p}, -u_{2p-1}, \ldots, u_2, -u_1), \end{aligned}$$

where it follows that $v_j = (-1)^{j+1} u_{2p-j+1}$ for $j = 1, 2, \ldots, 2p$. Therefore,

$$\begin{aligned} U \cdot \wedge(U) &= (u_1, u_2, \ldots, u_{2p-1}, u_{2p}) \cdot (u_{2p}, -u_{2p-1}, \ldots, u_2, -u_1) \\ &= u_1 u_{2p} - u_2 u_{2p-1} + \ldots + u_{2p-1} u_2 - u_{2p} u_1 \\ &= (u_1 u_{2p} - u_{2p} u_1) + \ldots + (-1)^{p+1} (u_p u_{p+1} - u_{p+1} u_p) \\ &= 0. \end{aligned}$$

Items 4), 5) and 6) can be proven using the properties of the determinant in similar way as the previous ones. ☑

3. Reversing operation over $\wedge$

The *reversing operation* has been applied successfully over rings and vector spaces (see [1, 2]). In this section we apply the reversing operation to obtain some results that involve the exterior product with the *palindromic* and *antipalindromic* vectors. The following results correspond to a generalization of some results presented in [2]. Consider the matrix $M = (m_{i,j})$ of size $m \times n$. The reversing of M, denoted by $\overleftarrow{M}$, is given by $\overleftarrow{M} = (\overleftarrow{m}_{i,j})$, where $\overleftarrow{m}_{i,j} = m_{i,n-j+1}$. We can see that the size of $\overleftarrow{M}$ is $m \times n$ too. We denote by $J_n = \overleftarrow{I_n}$ the reversing of the identity matrix I_n of size n. Thus, the following properties can be proven (see [2]).

1. The double reversing:

$$\overleftarrow{\overleftarrow{M}} = (\overleftarrow{\overleftarrow{m}}_{i,j}) = (\overleftarrow{m}_{i,n-j+1}) = \left(m_{i,n-(n-j+1)+1}\right) = (m_{i,j}) = M,$$

2. $\overleftarrow{M} = MJ_n$,
3. $J_nJ_n = I_n$.

The following definitions were introduced in [2]. A matrix M is called palindromic whether it satisfies $\overleftarrow{M} = M$. In the same way, a matrix M is called antipalindromic whether it satisfies $\overleftarrow{M} = -M$. In particular, for $m = 1$ we get palindromic and antipalindromic vectors, respectively.

As we can see, the palindromic matrix M satisfies $m_{i,j} = m_{i,n-j+1}$, and therefore M has at least $\dfrac{n}{2}$ pair of equal columns if n is even (as well $\dfrac{n}{2} - 1$ when n is odd). This fact lead us to the following result.

Proposition 3.1. $\det(J_n) = \begin{cases} (-1)^{n/2}, & n = 2k, \ k \in \mathbb{Z}^+ \\ (-1)^{\frac{n+3}{2}}, & n = 2k-1, \ k \in \mathbb{Z}^+. \end{cases}$

Proof. We proceed by induction over n. Assuming $n = 1$, we have that $I_n = 1$ and $J_n = 1$, thus $\det(J_n) = 1 = (-1)^{\frac{1+3}{2}}$. Let the proposition be true for n; thus we will prove that it is also true for $n+1$. We start considering that n is even, so we get

$$\begin{aligned} \det(J_{n+1}) &= 1\,(-1)^{1+(n+1)} \det(J_n) \\ &= (-1)^{n+2}(-1)^{\frac{n}{2}} \\ &= (-1)^{\frac{n}{2}} = (-1)^{\frac{(n+1)+3}{2}}. \end{aligned}$$

Now, considering n as an positive odd integer, we have

$$\begin{aligned} \det(J_{n+1}) &= 1\,(-1)^{1+(n+1)} \det(J_n) \\ &= (-1)^{n+2}(-1)^{\frac{n+3}{2}} \\ &= (-1)(-1)^{\frac{n+3}{2}} \\ &= (-1)^{\frac{n+5}{2}} = (-1)^{\frac{n+1}{2}}. \end{aligned}$$ ☑

Now, we study the relationship between the exterior product $\wedge$ and the reversing operation. We start considering $k = n-1$, that is, the generalized vector product over $\mathbb{R}^n$. Consider $M_1 = (m_{11}, m_{12}, \ldots, m_{1n}), \ldots, M_{n-1} = \left(m_{(n-1),1}, a_{(n-1),2}, \ldots, m_{(n-1),n}\right)$, $n-1$ vectors in $\mathbb{R}^n$. The generalized vector product is given by the equation (1), therefore we obtain

$$\times (M_1,\ M_2,\ \ldots,\ M_{n-1}) = \sum_{k=1}^{n} (-1)^{1+k} \det\left(M^{(k)}\right) e_k; \tag{5}$$

here e_k is the k-th element of the canonical basis for $\mathbb{R}^n$ and $M^{(k)}$ is the square matrix obtained after the deleting of the k-th column of the matrix $M = (m_{ij})_{(n-1)\times n}$. The matrix $M^{(k)}$ is a square matrix of size $(n-1)\times(n-1)$ and is given by

$$M^{(k)} = \left(m_{i,j}^{(k)}\right) = \begin{cases} m_{i,j}, \text{ if } j < k \\ m_{i,j+1}, \text{ if } j \geq k. \end{cases} \tag{6}$$

Proposition 3.2. *If we consider $M = (m_{ij})_{(n-1)\times n}$, then $\overleftarrow{M}^{(k)} = M^{(n-k+1)} J_{n-1}$, for $1 \leq k \leq n$.*

Proof. We know that $\overleftarrow{M} = M J_n$, that is, $(\overleftarrow{m}_{i,j}) = (m_{i,n-j+1})$, $1 \leq j \leq n$. Therefore

$$\begin{aligned} \overleftarrow{M}^{(k)} &= \left(\overleftarrow{m}_{i,j}^{(k)}\right) = \begin{cases} \overleftarrow{m}_{i,j}, \text{ if } j < k \\ \overleftarrow{m}_{i,j+1}, \text{ if } j \geq k \end{cases} \\ &= \begin{cases} m_{i,n-j+1}, \text{ if } j < k \\ m_{i,n-(j+1)+1}, \text{ if } j \geq k. \end{cases} \end{aligned}$$

On the other hand,

$$M^{(n-k+1)} = \left(m_{i,j}^{(n-k+1)}\right) = \begin{cases} m_{i,j}, \text{ if } j < n-k+1 \\ m_{i,j+1}, \text{ if } j \geq n-k+1. \end{cases} \tag{7}$$

Now, we obtain

$$\begin{aligned} M^{(n-k+1)} J_{n-1} &= \left(m_{i,(n-1)-j+1}^{(n-k+1)}\right) = \left(m_{i,n-j}^{(n-k+1)}\right) \\ &= \begin{cases} m_{i,(n-j)}, \text{ if } n-j < n-k+1 \\ m_{i,(n-j)+1}, \text{ if } n-j \geq n-k+1 \end{cases} \\ &= \begin{cases} m_{i,(n-j)}, \text{ if } j > k-1 \\ m_{i,(n-j)+1}, \text{ if } j \leq k-1 \end{cases} \\ &= \begin{cases} m_{i,n-j}, \text{ if } j \geq k \\ m_{i,n-j+1}, \text{ if } j < k \end{cases} = \overleftarrow{M}^{(k)}. \end{aligned}$$

☑

The following proposition is a generalization of a result presented in [2], where it was analyzed the reversing of the vector product in $\mathbb{R}^3$.

From now on, for suitability we denote $M = (M_1, M_2, \ldots, M_{n-1})$, i.e., M is the matrix that has as rows the vectors $M_1, M_2, \ldots, M_{n-1}$; thus, we obtain

$$\overleftarrow{M} = \left(\overleftarrow{M}_1,\ \overleftarrow{M}_2,\ \ldots,\ \overleftarrow{M}_{n-1}\right).$$

In the same way, for suitability we write

$$\mathfrak{M} = \times\left(\overleftarrow{M}_1,\ \overleftarrow{M}_2,\ \ldots,\ \overleftarrow{M}_{n-1}\right).$$

Proposition 3.3. *The generalized vector product of* $\overleftarrow{M}_i$, $1 \leq i \leq n-1$, *satisfies*

$$\mathfrak{M} = \begin{cases} (-1)^{\frac{3n}{2}}\left(\overleftarrow{\times\left(M_1,\ M_2,\ \ldots,\ M_{n-1}\right)}\right), & n = 2k, \\ (-1)^{\frac{3n+1}{2}}\left(\overleftarrow{\times\left(M_1,\ M_2,\ \ldots,\ M_{n-1}\right)}\right), & n = 2k-1, \end{cases}$$

where $k \in \mathbb{Z}^+$.

Proof. For suitability we denote $M = (M_1, M_2, \ldots, M_{n-1})$, i.e., M is the matrix that has as rows the vectors M_1, M_2, $\ldots$, M_{n-1}; thus, we obtain

$$\overleftarrow{M} = \left(\overleftarrow{M}_1,\ \overleftarrow{M}_2,\ \ldots,\ \overleftarrow{M}_{n-1}\right).$$

In the same way, for suitability we write

$$\mathfrak{M} = \times\left(\overleftarrow{M}_1,\ \overleftarrow{M}_2,\ \ldots,\ \overleftarrow{M}_{n-1}\right).$$

Now, applying the generalized vector product we obtain

$$\begin{aligned} \mathfrak{M} &= \sum_{k=1}^{n} (-1)^{k+1} \det\left(\overleftarrow{M}^{(k)}\right) e_k \\ &= \sum_{k=1}^{n} (-1)^{k+1} \det\left(M^{(n-k+1)} J_{n-1}\right) e_k \\ &= \sum_{k=1}^{n} (-1)^{k+1} \det\left(M^{(n-k+1)} J_{n-1}\right) e_k \\ &= \sum_{k=1}^{n} (-1)^{k+1} \det\left(M^{(n-k+1)}\right) \det\left(J_{n-1}\right) e_k \\ &= \det\left(J_{n-1}\right) \sum_{k=1}^{n} (-1)^{n-k} \det\left(M^{(k)}\right) e_{n-k+1} \\ &= (-1)^{n+1} \det\left(J_{n-1}\right) \sum_{k=1}^{n} (-1)^{k+1} \det\left(M^{(k)}\right) e_{n-k+1} \\ &= (-1)^{n+1} \det\left(J_{n-1}\right) \left(\sum_{k=1}^{n} (-1)^{k+1} \det\left(M^{(k)}\right) e_k\right) J_n \\ &= (-1)^{n+1} \det\left(J_{n-1}\right) \left(\overleftarrow{\times\left(M_1,\ M_2,\ \ldots,\ M_{n-1}\right)}\right), \end{aligned}$$

and therefore

$$\begin{aligned}\mathfrak{M} &= \begin{cases} (-1)^{n+1}(-1)^{\frac{(n-1)+3}{2}}\left(\overleftarrow{\times(M_1,\ M_2,\ \ldots,\ M_{n-1})}\right), & n=2k\\ (-1)^{n+1}(-1)^{\frac{n-1}{2}}\left(\overleftarrow{\times(M_1,\ M_2,\ \ldots,\ M_{n-1})}\right), & n=2k-1\end{cases}\\ &= \begin{cases} (-1)^{\frac{3n}{2}}\left(\overleftarrow{\times(M_1,\ M_2,\ \ldots,\ M_{n-1})}\right), & n=2k\\ (-1)^{\frac{3n+1}{2}}\left(\overleftarrow{\times(M_1,\ M_2,\ \ldots,\ M_{n-1})}\right), & n=2k-1.\end{cases}\end{aligned}$$ ☑

If M is a palindromic matrix, then the minors $M^{(k)}$ have at least $\dfrac{n}{2}-1$ pair of equal columns when n is even, and respectively $\dfrac{n-1}{2}-1$ when n is odd. This implies that for $n \geq 4$, the minors have at least one pair of equal columns and therefore $\det\left(M^{(k)}\right)=0$ for all $1 \leq k \leq n$, so that

$$\times(M_1,\ M_2,\ \ldots,\ M_{n-1}) = \mathbf{0} \in \mathbb{R}^n. \tag{8}$$

This means that the generalized vector product of $(n-1)$ palindromic vectors in $\mathbb{R}^n$ is interesting when $1 \leq n \leq 3$. The same result is obtained when we assume M as an antipalindromic matrix.

Final Remarks

When we consider the exterior product for $k \neq n-1$, the previous results cannot be applied due to the fact that, in general, they are not true. To illustrate it, we present the following example.

Example 3.4. *Consider the vectors* $(2,3,-1,5)$ *and* $(4,7,2,0)$ *in* $\mathbb{R}^4$. *In this case,*

$$M=\begin{pmatrix} 2 & 3 & -1 & 5\\ 4 & 7 & 2 & 0\end{pmatrix} \quad \text{and} \quad \overleftarrow{M}=\begin{pmatrix} 5 & -1 & 3 & 2\\ 0 & 2 & 7 & 4\end{pmatrix}$$

As we have seen before,

$$(2,3,-1,5)\wedge(4,7,2,0)=(-10,35,13,20,8,-2).$$

Therefore,

$$\begin{aligned}(5,-1,3,2)\wedge(0,2,7,4) &= \\ & -\begin{vmatrix} 5 & -1\\ 0 & 2\end{vmatrix} e_6 + \begin{vmatrix} 5 & 3\\ 0 & 7\end{vmatrix} e_5 - \begin{vmatrix} 5 & 2\\ 0 & 4\end{vmatrix} e_4 + \\ & + \begin{vmatrix} -1 & 3\\ 2 & 7\end{vmatrix} e_3 - \begin{vmatrix} -1 & 2\\ 2 & 4\end{vmatrix} e_2 + \begin{vmatrix} 3 & 2\\ 7 & 4\end{vmatrix} e_1\\ &= -(10)e_6+(35)e_5-(20)e_4+(-7-6)e_3-\\ & \quad (-4-4)e_2+(12-14)e_1\\ &= (-2,8,-13,-20,35,-10).\end{aligned}$$

Thus, in general, the exterior product does not satisfies

$$\overleftarrow{\bigwedge U} = (-1)^p \bigwedge \overleftarrow{U}, \quad \text{for some } \ p \in \mathbb{Z}.$$

Finally, although this paper is presented in a didactic way, there are original results corresponding to the relations between the reversing operation and the generalized vector product.

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