Research paper

Nie-Tan Method and its Improved Version: A Counterexample

Método Nie-Tan y su Versión Mejorada: Un contraejemplo

***Juan D. Rojas*[1], *Omar Salazar*[1], *Humberto Serrano*[1]**

[1]Universidad Distrital Francisco José de Caldas. Correspondence email: osalazarm@correo.udistrital.edu.co

Received: 13-10-2016. Modified: 18-01-2016. Accepted: 30-03-2016

Abstract

Context: The bottleneck on interval type-2 fuzzy logic systems is the output processing when using Centroid Type-Reduction + Defuzzification (CTR+D method). Nie and Tan proposed an approximation to CTR+D (NT method). Recently, Mendel and Liu improved the NT method (INT method). Numerical examples (due to Mendel and Liu) exhibit the NT and INT methods as good approximations to CTR+D.
Method: Normalization to the unit interval of membership function domains (examples and counterexample) and variables involved in the calculations for the three methods. Examples (due to Mendel and Liu) taken from the literature. Counterexample with piecewise linear membership functions. Comparison by means of error and percentage relative error.
Results: NT vs. CTR+D: Our counterexample showed an error of 0.1014 and a percentage relative error of 30.53%. This is respectively 23 and 32 times higher than the worst case obtained in the examples. INT vs. CTR+D: Our counterexample showed an error of 0.0725 and a percentage relative error of 21.83%. This is respectively 363 and 546 times higher than the worst case obtained in the examples.
Conclusions: NT and INT methods are not necessarily good approximations to the CTR+D method.
Keywords: Type-2 fuzzy logic system, type-reduction, defuzzification, Nie-Tan method.
Language: English.

Resumen

Contexto: El cuello de botella en sistemas de lógica difusa tipo-2 de intervalo es el procesamiento de salida que usa reducción de tipo centroide + defusificación (método CTR+D). Nie y Tan propusieron una aproximación a CTR+D (método NT). Recientemente, Mendel y Liu mejoraron la propuesta (método INT). Ejemplos debidos a Mendel y Liu exhiben a NT e INT como buenas aproximaciones a CTR+D.
Método: Normalización al intervalo unitario de los dominios de las funciones de pertenencia (para ejemplos y contraejemplo) y de las variables que intervienen en los cálculos de los tres métodos. Ejemplos tomados de la literatura (debidos a Mendel y Liu). Contraejemplo con funciones de pertenencia lineales por tramos. Comparación por medio de métricas de error y porcentaje de error relativo.
Resultados: NT vs. CTR+D: El contraejemplo mostró un error de 0.1014 y error relativo porcentual de 30.53%. Esto es respectivamente 23 y 32 veces mayor que el peor caso obtenido en los ejemplos. INT vs. CTR+D: El contraejemplo mostró un error de 0.0725 y error relativo porcentual de 21.83%. Esto es respectivamente 363 y 546 veces mayor que el peor caso obtenido en los ejemplos.
Conclusiones: NT e INT no son necesariamente buenas aproximaciones al método CTR+D.
Palabras clave: Sistema de lógica difusa tipo-2, reducción de tipo, defusificación, método Nie-Tan.

 Cite this paper as: Rojas, J. D., Salazar, O., Serrano, H.: Nie-Tan Method and its Improved Version: A Counterexample. INGENIERÍA, Vol. 21, Num. 2, 2016 138:153.
En línea DOI: http://dx.doi.org/10.14483/udistrital.jour.reving.2016.2.a02

1. Introduction

Type-2 Fuzzy Logic Systems (T2FLS) (Figure 1) are used in several applications because Type-2 Fuzzy Sets (T2FS) provide greater flexibility than Type-1 Fuzzy Sets (T1FS) [1], [2]. In a T2FLS a crisp numerical input goes through three stages: fuzzification, inferencing, and the output processing. During the output processing a T2FS is converted into a crisp number. This last stage consists of two parts: type-reduction and defuzzification. Type-reduction is the procedure by which a T2FS is converted to a T1FS (called the type-reduced set). This set is then defuzzified to give a crisp number.

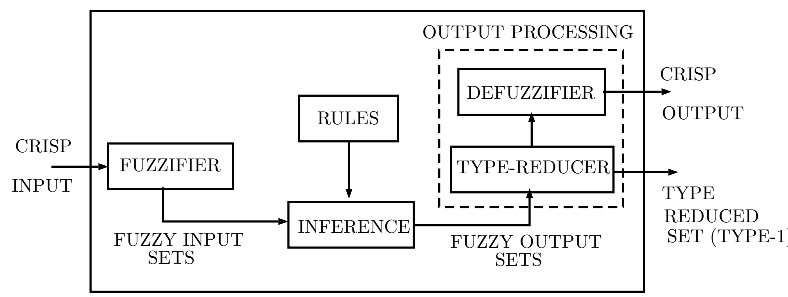

Figure 1. Type-2 Fuzzy Logic System. Taken from [3], [4].

In order to facilitate operations on T2FLSs, Interval Type-2 Fuzzy Sets (IT2FS) were introduced (Figure 2). IT2FSs are a simplified version of general T2FSs. IT2FSs are defined by two membership functions (MF): the Lower Membership Function (LMF) and the Upper Membership Function (UMF). Any MF between LMF and UMF is called an Embedded Membership Function (EMF). The region bounded by LMF and UMF is called Footprint of Uncertainty (FOU). The corresponding FLSs are called Interval Type-2 Fuzzy Logic Systems (IT2FLS) [4].

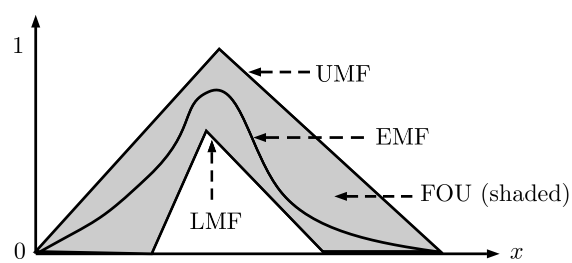

Figure 2. Interval Type-2 Fuzzy Set. LMF = Lower Membership Function. UMF = Upper Membership Function. EMF = Embedded Membership Function. FOU = Footprint of Uncertainty (shaded). Adapted from [4].

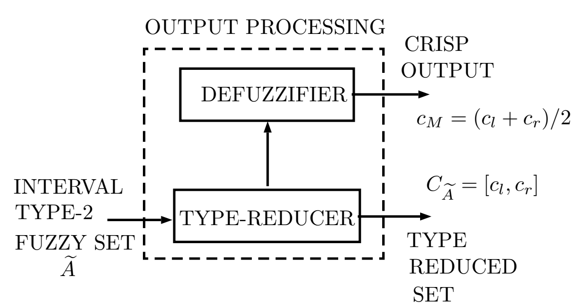

Figure 3. The CTR+D method. Adapted from [3], [4].

Since IT2FLSs were proposed, centroid type-reduction[1] (Figure 3) has been one of the main areas of study, mainly due to its high computational cost [3], [6]–[9]. If $\tilde{A}$ is an IT2FS, the main problem

[1]Centroid type-reduction is classified into two forms: discrete and continuous [5]. From a discretization of the MFs it is possible to switch from continuous to discrete. This paper discusses the continuous version, but several results are applied to the discrete case.

consists in finding the type-reduced set[2] $C_{\tilde{A}} = [c_l, c_r]$, where c_l and c_r are the endpoints of $C_{\tilde{A}}$. The interval $[c_l, c_r]$ contains the centroids of all EMFs in the FOU. Defuzzification, which consists in averaging c_l and c_r to get $c_M = (c_l + c_r)/2$, is a relatively simple step in IT2FLSs. Figure 3 is what we shall refer to as the Centroid Type-Reduction + Defuzzification (CTR+D) method[3].

Nie and Tan [13] proposed an approximation to the CTR+D method. It is known as the Nie-Tan (NT) method. It consists in averaging LMF and UMF to get an average MF (AMF). The defuzzified value c_{NT} is the centroid of this AMF (Figure 4(a)). Mendel and Liu [11], [12] improved the NT method. Their improvement is still an approximation. It is known as the Improved Nie-Tan (INT) method. It consists in adding to c_{NT} a correction factor δ, i.e., $c_{INT} = c_{NT} + \delta$ (Figure 4(b)).

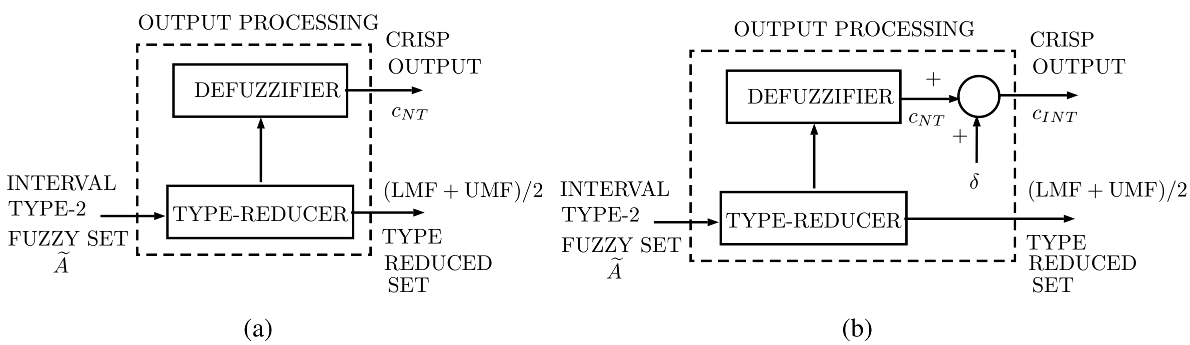

Figure 4. (a) The NT method. (b) The INT method.

Mendel and Liu showed four numerical examples in order to illustrate their theoretical results. These authors claimed that c_{NT} is a first-order approximation to $c_M = (c_l + c_r)/2$, and c_{INT} is a better third-order approximation to c_M. Their examples included IT2FSs defined over different domains, and they used the following metrics for comparison:

1. Absolute error: $E_{NT} = |c_{NT} - c_M|$ and $E_{INT} = |c_{INT} - c_M|$,
2. Percentage relative error: $RE_{NT} = \left(\dfrac{E_{NT}}{|c_M|}\right) \times 100\%$ and $RE_{INT} = \left(\dfrac{E_{INT}}{|c_M|}\right) \times 100\%$,
3. Difference of absolute errors: $E_{NT} - E_{INT}$, and
4. Absolute error ratio: $\dfrac{E_{NT}}{E_{INT}} = \dfrac{RE_{NT}}{RE_{INT}}$.

Their numerical results showed $0 \leq E_{NT} \leq 0.0844$, $0 \leq E_{INT} \leq 0.0014$, $0\% \leq RE_{NT} \leq 2.22\%$, and $0\% \leq RE_{INT} \leq 0.04\%$. In terms of error comparison, their results showed $0 \leq E_{NT} - E_{INT} \leq 0.0829$ and $4.29 \leq E_{NT}/E_{INT} \leq 58.93$. Although these results seem to exhibit the NT and INT methods as a good approximation to the CTR+D method, in this paper we will show this is not necessarily true.

The metrics shown above depend on two things: (1) the domain where LMF and UMF are defined and (2) the images of these two MFs. Let us explain this point in general terms. Let $\tilde{A}$ be an IT2FS

[2] An alternative notation as interval set is $C_{\tilde{A}} = 1/[c_l, c_r]$. In this paper we use standard mathematical notation [10].
[3] It is called the "KM + Defuzzification" method in [11], [12].

defined over a domain X. If $\mu_{A_e} : X \mapsto [0,1] : x \mapsto \mu_{A_e}(x)$ is an EMF then its centroid is given by

$$C_{A_e} = \frac{\displaystyle\int_X x\mu_{A_e}(x)dx}{\displaystyle\int_X \mu_{A_e}(x)dx}.$$

Therefore, C_{A_e} depends on two things for its calculation: (1) the domain X and (2) the image[4] of μ_{A_e}, denoted as $\mathrm{Im}(\mu_{A_e})$. As a consequence, if M is a metric calculated from $C_{A_{e1}}$ and $C_{A_{e2}}$ (centroids of two EMFs), M depends on X, $\mathrm{Im}(\mu_{A_{e1}})$ and $\mathrm{Im}(\mu_{A_{e2}})$. As we will see in next sections: c_M, c_{NT} and c_{INT} are calculated from X, LMF and UMF (two particular EMFs). Therefore, E_{NT}, E_{INT}, RE_{NT}, RE_{INT}, $E_{NT} - E_{INT}$, and $E_{NT}/E_{INT} = RE_{NT}/RE_{INT}$ depend on the domain where LMF and UMF are defined and the images of these two MFs.

The aim of this paper is to show a counterexample that exhibits higher errors than the corresponding errors in examples reported in the literature when comparing CTR+D method versus NT and INT methods[5]. We chose an IT2FS with piecewise linear MFs, mainly due to its simplicity. In order to reduce the effect of different domains on the metrics (as we explained above), all the domains (for examples and counterexample) were taken to a common domain: the unit interval $[0,1]$. This has a consequence: a change on a metric is due mainly to the change in the LMFs and UMFs (the shape of the FOU). Additionally, all the variables involved in the CTR+D, NT and INT methods were normalized to the unit interval.

After normalizing Mendel and Liu's results, their four numerical examples showed $0 \leq E^*_{NT} \leq 0.0044$, $0 \leq E^*_{INT} \leq 0.0002$, $0\% \leq RE^*_{NT} \leq 0.96\%$, and $0\% \leq RE^*_{INT} \leq 0.04\%$. In terms of error comparison, their examples showed $0 \leq E^*_{NT} - E^*_{INT} \leq 0.0044$ and $4.42 \leq E^*_{NT}/E^*_{INT} \leq 60.21$. Our counterexample showed $E^*_{NT} = 0.1014$ (23 times higher than $E^*_{NT} = 0.0044$), $E^*_{INT} = 0.0725$ (363 times higher than $E^*_{INT} = 0.0002$), $RE^*_{NT} = 30.53\%$ (32 times higher than $RE^*_{NT} = 0.96\%$), and $RE^*_{INT} = 21.83\%$ (546 times higher than $RE^*_{INT} = 0.04\%$). In terms of error comparison, our counterexample showed $E^*_{NT} - E^*_{INT} = 0.0289$ and $E^*_{NT}/E^*_{INT} = 1.3986$. We concluded, based on our results, that the NT and INT methods are not necessarily good approximations to the CTR+D method.

This paper is organized as follows: In Section 2 some preliminaries related to the CTR+D, NT and INT methods are presented. In Section 3, our normalization to the unit interval is described. In Section 4, the main results are shown. Finally, discussion and conclusions are presented in Section 5 and Section 6.

[4]The image of μ_{A_e} is $\mathrm{Im}(\mu_{A_e}) = \{\mu_{A_e}(x) \mid x \in X\}$.

[5]A comparative study (by means of statistical analysis) was carried out in [14] in order to compare accuracy and complexity for the Exhaustive Defuzzification method [15] versus the Karnik-Mendel iterative procedure [7] (EIASC algorithm [16, section III]), the Wu-Mendel approximation (WM algorithm [17, appendix III, pp. 635]), the Greenfield-Chiclana Collapsing Defuzzifier (collapsing algorithm [18]), and the NT method [13].

2. Preliminaries

2.1. The CTR+D method

Let $\tilde{A}$ be an IT2FS, which is determined by two MFs[6] $\underline{\mu} : X \mapsto [0,1]$ and $\overline{\mu} : X \mapsto [0,1]$, defined over a nonempty set $X \subset \mathbb{R}$, such that $\underline{\mu}(x) \leq \overline{\mu}(x)$ for all $x \in X$. $\underline{\mu}$ is called Lower Membership Function (LMF) and $\overline{\mu}$ is called Upper Membership Function (UMF). In many applications X is a closed interval[7], therefore from now on we will suppose $X = [a,b] \subset \mathbb{R}$, with $a < b$. The centroid (type-reduced set) of $\tilde{A}$, denoted by $C_{\tilde{A}}$, is $C_{\tilde{A}} = [c_l, c_r] \subseteq X$, where c_l and c_r are

$$c_l = \min_{\underline{\mu} \leq \theta \leq \overline{\mu}} \frac{\displaystyle\int_a^b x\theta(x)dx}{\displaystyle\int_a^b \theta(x)dx}, \quad \text{and} \quad c_r = \max_{\underline{\mu} \leq \theta \leq \overline{\mu}} \frac{\displaystyle\int_a^b x\theta(x)dx}{\displaystyle\int_a^b \theta(x)dx}, \tag{1}$$

and where $\theta : X \mapsto [0,1]$ is a MF such that

$$\underline{\mu}(x) \leq \theta(x) \leq \overline{\mu}(x) \tag{2}$$

for all $x \in X$. The defuzzified value of $\tilde{A}$ is

$$c_M = \frac{c_l + c_r}{2}. \tag{3}$$

It was shown [19], [20] that the θ functions to minimize and maximize (1) are respectively

$$\theta_l(x) = \begin{cases} \overline{\mu}(x), & x \leq x_l, \\ \underline{\mu}(x), & x > x_l, \end{cases} \quad \text{and} \quad \theta_r(x) = \begin{cases} \underline{\mu}(x), & x \leq x_r, \\ \overline{\mu}(x), & x > x_r, \end{cases} \tag{4}$$

where $x_l, x_r \in X$ are unknown (a priori) switch points between $\underline{\mu}$ and $\overline{\mu}$. The switch points x_l and x_r need to be found by means of iterative procedures in order to optimize (1).

Convex combination [21]–[23] was used to characterize all the θ functions that satisfy (2). It is known that for any θ which satisfies (2), there is at least one MF $\mu_\Lambda : X \mapsto [0,1]$ such that

$$\theta(x) = \underline{\mu}(x) + \mu_\Lambda(x)(\overline{\mu}(x) - \underline{\mu}(x)) \tag{5}$$

for all $x \in X$. If μ_Λ is taken in (5) as

$$\mu_{\Lambda_l}(x) = \begin{cases} 1, & x \leq x_l, \\ 0, & x > x_l, \end{cases} \quad \text{and} \quad \mu_{\Lambda_r}(x) = \begin{cases} 0, & x \leq x_r, \\ 1, & x > x_r, \end{cases} \tag{6}$$

then (4) is achieved. Therefore, $c_l = \min_{t\in X} \alpha(t)$ and $c_r = \max_{t \in X} \beta(t)$, where

$$\alpha(t) = \frac{\displaystyle\int_a^t x\overline{\mu}(x)dx + \int_t^b x\underline{\mu}(x)dx}{\displaystyle\int_a^t \overline{\mu}(x)dx + \int_t^b \underline{\mu}(x)dx}, \quad \text{and} \quad \beta(t) = \frac{\displaystyle\int_a^t x\underline{\mu}(x)dx + \int_t^b x\overline{\mu}(x)dx}{\displaystyle\int_a^t \underline{\mu}(x)dx + \int_t^b \overline{\mu}(x)dx}, \tag{7}$$

[6] In this paper all the MFs are supposed to be Riemann-Integrable.

[7] In several papers on the centroid, X is taken as $(-\infty, \infty)$ with the assumption that all integrals are convergent. However, over this domain the centroid of an IT2FS could not exist. See [5, sec. 2.1] for an example.

for all $t \in X$.

It was also shown [19], [20], [24] that $\alpha(c_l) = c_l$ and $\beta(c_r) = c_r$ (c_l and c_r are fixed points of α and β), i.e.,

$$c_l = \frac{\int_a^{c_l} x\overline{\mu}(x)dx + \int_{c_l}^b x\underline{\mu}(x)dx}{\int_a^{c_l} \overline{\mu}(x)dx + \int_{c_l}^b \underline{\mu}(x)dx}, \quad \text{and} \quad c_r = \frac{\int_a^{c_r} x\underline{\mu}(x)dx + \int_{c_r}^b x\overline{\mu}(x)dx}{\int_a^{c_r} \underline{\mu}(x)dx + \int_{c_r}^b \overline{\mu}(x)dx}. \tag{8}$$

From (8) it was shown [11], [12], [25] that the problem of finding c_l and c_r is equivalent to find the roots in $[a, b]$ of

$$\varphi(t) = \int_a^t (t-x)\overline{\mu}(x)dx + \int_t^b (t-x)\underline{\mu}(x)dx, \tag{9}$$

$$\omega(t) = \int_a^t (t-x)\underline{\mu}(x)dx + \int_t^b (t-x)\overline{\mu}(x)dx, \tag{10}$$

which are defined for all $t \in X$, and where $\varphi(c_l) = 0$ and $\omega(c_r) = 0$. It was also shown [25] that the Karnik-Mendel algorithm is equivalent to applying the Newton-Raphson method to find the roots of φ and ω.

2.2. The NT method

In the Nie and Tan's original method [13], the MF obtained after type-reducing is the average of $\overline{\mu}$ and $\underline{\mu}$, i.e., $\mu_{NT}(x) = (\overline{\mu}(x) + \underline{\mu}(x))/2$ for all $x \in X$. Therefore, its defuzzified value is

$$c_{NT} = \frac{\int_a^b x\mu_{NT}(x)dx}{\int_a^b \mu_{NT}(x)dx}. \tag{11}$$

Mendel and Liu [11], [12] claimed that c_{NT} is a first-order approximation to $c_M = (c_l + c_r)/2$.

2.3. The INT method

Mendel and Liu [11], [12] proposed the improved Nie-Tan method, which is

$$c_{INT} = c_{NT} + \delta, \tag{12}$$

where c_{NT} is given in Section 2.2, and δ is given by

$$\delta = \frac{2\left(\int_a^{c_{NT}} (c_{NT}-x)\overline{\mu}(x)dx + \int_{c_{NT}}^b (c_{NT}-x)\underline{\mu}(x)dx\right)}{\left(\int_a^b (\overline{\mu}(x) + \underline{\mu}(x))dx\right)^2} \times \left(\int_a^{c_{NT}} (\overline{\mu}(x) - \underline{\mu}(x))dx - \int_{c_{NT}}^b (\overline{\mu}(x) - \underline{\mu}(x))dx\right). \tag{13}$$

These authors claimed that c_{INT} is a better third-order approximation to $c_M = (c_l + c_r)/2$.

3. Normalization to the unit interval

Since the domain of $\underline{\mu}$ and $\overline{\mu}$ is $X = [a, b]$, with $a < b$, a bijective function is established $[a, b] \mapsto [0, 1] : x \mapsto y$ which is given by

$$y = \frac{x - a}{b - a}. \tag{14}$$

Its inverse function $[0, 1] \mapsto [a, b] : y \mapsto x$ (also a bijection) is given by

$$x = a + y(b - a). \tag{15}$$

By means of (15) we define MFs with normalized domain (to the unit interval) $\underline{\mu}^* : [0, 1] \mapsto [0, 1]$ and $\overline{\mu}^* : [0, 1] \mapsto [0, 1]$ given by

$$\overline{\mu}^*(y) = \overline{\mu}(a + y(b - a)), \tag{16}$$
$$\underline{\mu}^*(y) = \underline{\mu}(a + y(b - a)), \tag{17}$$

for all $y \in [0, 1]$, and where $\underline{\mu}^*(y) \leq \overline{\mu}^*(y)$ holds for all $y \in [0, 1]$.

3.1. Normalized CTR+D method

By means of (14)–(17), the normalized version of (7) is (after some algebra[8]):

$$\alpha^*(z) = \frac{\int_0^z y\overline{\mu}^*(y)dy + \int_z^1 y\underline{\mu}^*(y)dy}{\int_0^z \overline{\mu}^*(y)dy + \int_z^1 \underline{\mu}^*(y)dy}, \text{ and } \beta^*(z) = \frac{\int_0^z y\underline{\mu}^*(y)dy + \int_z^1 y\overline{\mu}^*(y)dy}{\int_0^z \underline{\mu}^*(y)dy + \int_z^1 \overline{\mu}^*(y)dy}, \tag{18}$$

where $z = (t - a)/(b - a)$, $\alpha^*(z) = (\alpha(t) - a)/(b - a)$, and $\beta^*(z) = (\beta(t) - a)/(b - a)$. Therefore $c_l^* = \min_{z\in[0,1]} \alpha^*(z)$, $c_r^* = \max_{z\in[0,1]} \beta^*(z)$ and $c_M^* = (c_l^* + c_r^*)/2$ where

$$c_l^* = \frac{c_l - a}{b - a}, \quad c_r^* = \frac{c_r - a}{b - a} \quad \text{and} \quad c_M^* = \frac{c_M - a}{b - a}. \tag{19}$$

It should be noted that from (14) we have $z, \alpha^*, \beta^*, c_l^*, c_r^*, c_M^* \in [0, 1]$. Similarly (9)–(10) are reduced to

$$\varphi^*(z) = \int_0^z (z - y)\overline{\mu}^*(y)dy + \int_z^1 (z - y)\underline{\mu}^*(y)dy, \tag{20}$$
$$\omega^*(z) = \int_0^z (z - y)\underline{\mu}^*(y)dy + \int_z^1 (z - y)\overline{\mu}^*(y)dy, \tag{21}$$

where $\varphi^*(z) = \varphi(t)/(b - a)^2$ and $\omega^*(z) = \omega(t)/(b - a)^2$. Since $\varphi(c_l) = 0$ and $\omega(c_r) = 0$ then $\varphi^*(c_l^*) = 0$ and $\omega^*(c_r^*) = 0$. Therefore c_l^* and c_r^* are roots in $[0, 1]$ of φ^* and ω^*. It is not difficult

[8] The substitution $x = a + y(b - a)$ yields $dx = (b - a)dy$, which is the required substitution for dx. The other variables are obtained by performing the corresponding substitutions.

to verify that $\varphi^*, \omega^* \in [-1/2, 1/2]$. Although φ^* and ω^* are not in the unit interval, we only need their roots in $[0,1]$. Additionally, there is a relation among φ^* and ω^*:

$$\varphi^*(z) + \omega^*(z) = z(B+D) - (A+C), \tag{22}$$

for all $z \in [0,1]$, where

$$A = \int_0^1 y\overline{\mu}^*(y)dy,\ B = \int_0^1 \overline{\mu}^*(y)dy,\ C = \int_0^1 y\underline{\mu}^*(y)dy,\ \text{and } D = \int_0^1 \underline{\mu}^*(y)dy. \tag{23}$$

3.2. Normalized NT method

By means of (14)–(17) the normalized version of (11) is (after some algebra):

$$c^*_{NT} = \frac{\displaystyle\int_0^1 y\mu^*_{NT}(y)dy}{\displaystyle\int_0^1 \mu^*_{NT}(y)dy}, \tag{24}$$

where $\mu^*_{NT}(y) = (\overline{\mu}^*(y) + \underline{\mu}^*(y))/2$ for all $y \in [0,1]$, and

$$c^*_{NT} = \frac{c_{NT} - a}{b - a}. \tag{25}$$

From (14) we have that $c^*_{NT} \in [0,1]$.

3.3. Normalized INT method

By means of (14)–(17), the normalized version of (12) is (after some algebra):

$$c^*_{INT} = c^*_{NT} + \delta^*, \tag{26}$$

where

$$c^*_{INT} = \frac{c_{INT} - a}{b - a}, \quad \text{and} \quad \delta^* = \frac{\delta}{b - a}. \tag{27}$$

From (14) we have that $c^*_{INT} \in [0,1]$. c^*_{NT} is given in Section 3.2, and δ^* is given by

$$\delta^* = \frac{2\left(\displaystyle\int_0^{c^*_{NT}} (c^*_{NT} - y)\overline{\mu}^*(y)dy + \int_{c^*_{NT}}^1 (c^*_{NT} - y)\underline{\mu}^*(y)dy\right)}{\left(\displaystyle\int_0^1 (\overline{\mu}^*(y) + \underline{\mu}^*(y))dy\right)^2} \times \left(\int_0^{c^*_{NT}} (\overline{\mu}^*(y) - \underline{\mu}^*(y))dy - \int_{c^*_{NT}}^1 (\overline{\mu}^*(y) - \underline{\mu}^*(y))dy\right). \tag{28}$$

4. Results

4.1. Original Mendel and Liu's numerical examples

Mendel and Liu [11], [12] showed the following four numerical examples.

1. Symmetric Gaussian MFs with uncertain deviation defined for all $x \in X = [0, 10]$:

$$\underline{\mu}_{\tilde{A}_1}(x) = \exp\left(-\frac{1}{2}\left(\frac{x-5}{0.25}\right)^2\right), \tag{29}$$

$$\overline{\mu}_{\tilde{A}_1}(x) = \exp\left(-\frac{1}{2}\left(\frac{x-5}{1.75}\right)^2\right). \tag{30}$$

2. Triangular LMF and Gaussian UMF defined for all $x \in X = [-5, 14]$:

$$\underline{\mu}_{\tilde{A}_2}(x) = \begin{cases} 0.6(x+5)/19, & x < 2.6, \\ 0.4(14-x)/19, & x \geq 2.6. \end{cases} \tag{31}$$

$$\overline{\mu}_{\tilde{A}_2}(x) = \begin{cases} \exp\left(-\frac{1}{2}\left(\frac{x-2}{5}\right)^2\right), & x < 7.185, \\ \exp\left(-\frac{1}{2}\left(\frac{x-9}{1.75}\right)^2\right), & x \geq 7.185. \end{cases} \tag{32}$$

3. Piecewise Gaussian MFs defined for all $x \in X = [0, 10]$:

$$\underline{\mu}_{\tilde{A}_3}(x) = \max\left\{0.5\exp\left(-\frac{(x-3)^2}{2}\right), 0.4\exp\left(-\frac{(x-6)^2}{2}\right)\right\}. \tag{33}$$

$$\overline{\mu}_{\tilde{A}_3}(x) = \max\left\{\exp\left(-\frac{(x-3)^2}{8}\right), 0.8\exp\left(-\frac{(x-6)^2}{8}\right)\right\}. \tag{34}$$

4. Piecewise Linear MFs defined for all $x \in X = [1, 8]$:

$$\underline{\mu}_{\tilde{A}_4}(x) = \max\left\{\begin{bmatrix}(x-1)/6, & 1 \leq x \leq 4, \\ (7-x)/6, & 4 \leq x \leq 7, \\ 0, & \text{otherwise.}\end{bmatrix}, \begin{bmatrix}(x-3)/6, & 3 \leq x \leq 5, \\ (8-x)/9, & 5 \leq x \leq 8, \\ 0, & \text{otherwise.}\end{bmatrix}\right\}. \tag{35}$$

$$\overline{\mu}_{\tilde{A}_4}(x) = \max\left\{\begin{bmatrix}(x-1)/2, & 1 \leq x \leq 3, \\ (7-x)/4, & 3 \leq x \leq 7, \\ 0, & \text{otherwise.}\end{bmatrix}, \begin{bmatrix}(x-2)/5, & 2 \leq x \leq 6, \\ (16-2x)/5, & 6 \leq x \leq 8, \\ 0, & \text{otherwise.}\end{bmatrix}\right\}. \tag{36}$$

Their results are summarized in Table I. These authors used the following metrics for comparison:

1. Absolute error: $E_{NT} = |c_{NT} - c_M|$ and $E_{INT} = |c_{INT} - c_M|$.
2. Percentage relative error: $RE_{NT} = \frac{E_{NT}}{|c_M|} \times 100\%$ and $RE_{INT} = \frac{E_{INT}}{|c_M|} \times 100\%$.

3. Difference of absolute errors: $E_{NT} - E_{INT}$.

4. Absolute error ratio: $\dfrac{E_{NT}}{E_{INT}} = \dfrac{RE_{NT}}{RE_{INT}}$.

Table I. Computation results in the continuous case. Taken and adapted from [11], [12].

	Domain		NT method			INT method			Error comparison	
IT2FS	$[a,b]$	$c_M = (c_l + c_r)/2$	c_{NT}	E_{NT}	RE_{NT}	c_{INT}	E_{INT}	RE_{INT}	$E_{NT} - E_{INT}$	E_{NT}/E_{INT}
$\tilde{A}_1$	$[0, 10]$	5.0	5.0	0	0	5.0	0	0	0	—
$\tilde{A}_2$	$[-5, 14]$	3.7984	3.7141	0.0844	2.22%	3.797	0.0014	0.04%	0.0829	58.93
$\tilde{A}_3$	$[0, 10]$	4.4152	4.3953	0.0200	0.45%	4.4158	0.0006	0.01%	0.0194	33.63
$\tilde{A}_4$	$[1, 8]$	4.3261	4.3208	0.0053	0.12%	4.3273	0.0012	0.03%	0.0041	4.29

4.2. Normalized results

The corresponding IT2FSs with a normalized domain (37)–(44) are found by means of (29)–(36) by substituting $x \in X$ by $a + y(b-a)$, where $y \in [0,1]$. The a and b values depend on the X-domain for each IT2FS. For example, for (35)–(36) we have $a = 1$ and $b = 8$.

1. Symmetric Gaussian MFs with uncertain deviation (Figure 5(a)) defined for all $y \in [0,1]$:

$$\underline{\mu}^*_{\tilde{A}_1}(y) = \exp\left(-\frac{1}{2}(40y - 20)^2\right), \tag{37}$$

$$\overline{\mu}^*_{\tilde{A}_1}(y) = \exp\left(-\frac{1}{2}\left(\frac{40y - 20}{7}\right)^2\right). \tag{38}$$

2. Triangular LMF and Gaussian UMF (Figure 5(b)) defined for all $y \in [0,1]$:

$$\underline{\mu}^*_{\tilde{A}_2}(y) = \begin{cases} 3y/5, & y < 2/5, \\ 2(1-y)/5, & y \geq 2/5. \end{cases} \tag{39}$$

$$\overline{\mu}^*_{\tilde{A}_2}(y) = \begin{cases} \exp\left(-\frac{1}{2}\left(\frac{19y - 7}{5}\right)^2\right), & y < 624/973, \\ \exp\left(-\frac{1}{2}\left(\frac{76y - 56}{7}\right)^2\right), & y \geq 624/973. \end{cases} \tag{40}$$

3. Piecewise Gaussian MFs (Figure 5(c)) defined for all $y \in [0,1]$:

$$\underline{\mu}^*_{\tilde{A}_3}(y) = \max\left\{0.5\exp\left(-\frac{(10y-3)^2}{2}\right), 0.4\exp\left(-\frac{(10y-6)^2}{2}\right)\right\}. \tag{41}$$

$$\overline{\mu}^*_{\tilde{A}_3}(y) = \max\left\{\exp\left(-\frac{(10y-3)^2}{8}\right), 0.8\exp\left(-\frac{(10y-6)^2}{8}\right)\right\}. \tag{42}$$

4. Piecewise Linear MFs (Figure 5(d)) defined for all $y \in [0,1]$:

$$\underline{\mu}^*_{\tilde{A}_4}(y) = \max\left\{\begin{bmatrix} 7y/6, & 0 \le y \le 3/7, \\ (6-7y)/6, & 3/7 \le y \le 6/7, \\ 0, & \text{otherwise.} \end{bmatrix}, \begin{bmatrix} (7y-2)/6, & 2/7 \le y \le 4/7, \\ 7(1-y)/9, & 4/7 \le y \le 1, \\ 0, & \text{otherwise.} \end{bmatrix}\right\}. \tag{43}$$

$$\overline{\mu}^*_{\tilde{A}_4}(y) = \max\left\{\begin{bmatrix} 7y/2, & 0 \le y \le 2/7, \\ (6-7y)/4, & 2/7 \le y \le 6/7, \\ 0, & \text{otherwise.} \end{bmatrix}, \begin{bmatrix} (7y-1)/5, & 1/7 \le y \le 5/7, \\ 14(1-y)/5, & 5/7 \le y \le 1, \\ 0, & \text{otherwise.} \end{bmatrix}\right\}. \tag{44}$$

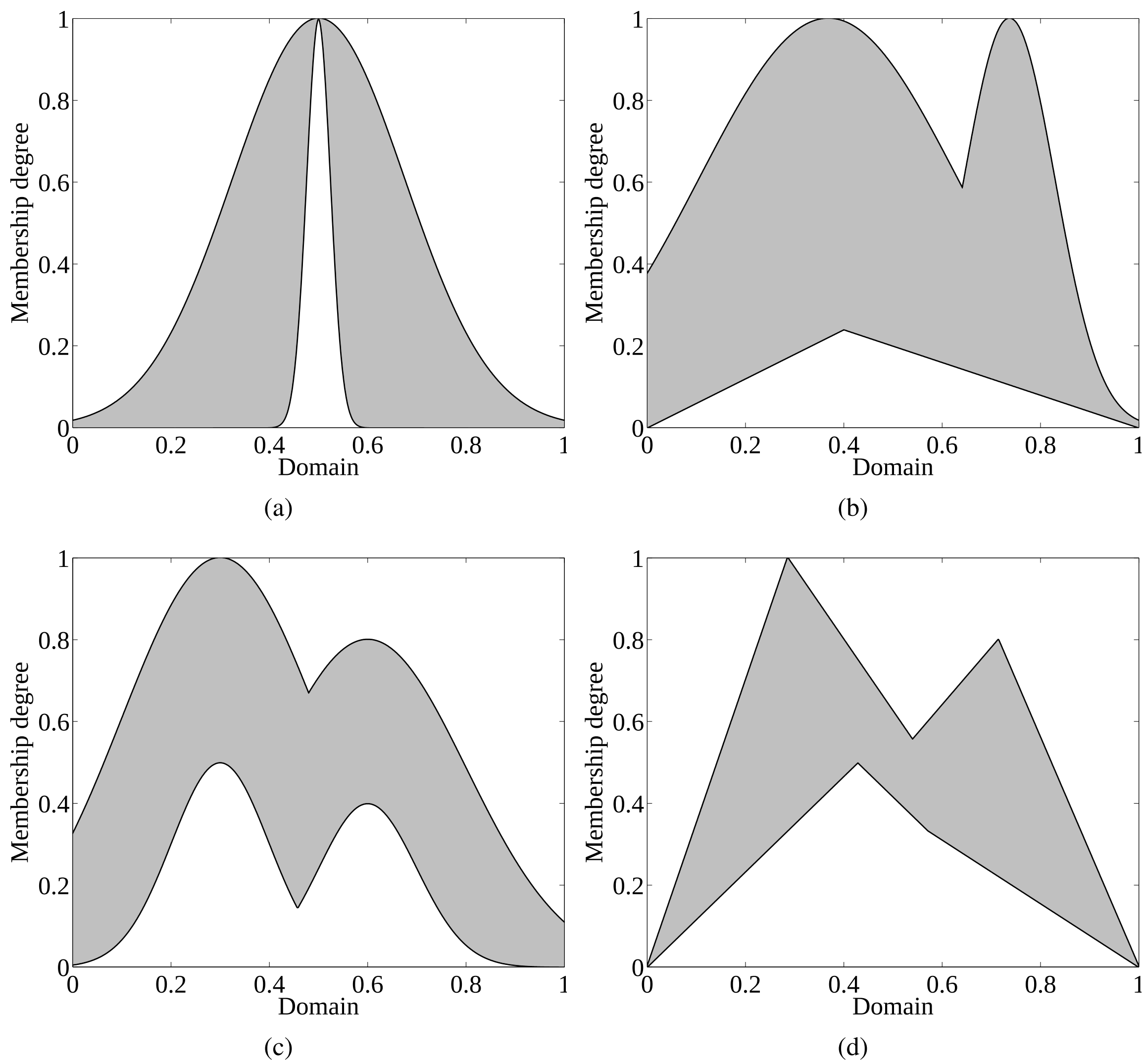

Figure 5. IT2FSs with normalized domain (37)–(44).

In Table II we show $c^*_M = (c_M - a)/(b-a)$, $c^*_{NT} = (c_{NT} - a)/(b-a)$ and $c^*_{INT} = (c_{INT} - a)/(b-a)$, which are the normalization to the unit interval of c_M, c_{NT} and c_{INT} in Table I. We recalculated the following metrics:

1. Absolute error: $E^*_{NT} = |c^*_{NT} - c^*_M|$ and $E^*_{INT} = |c^*_{INT} - c^*_M|$.
2. Percentage relative error: $RE^*_{NT} = \dfrac{E^*_{NT}}{c^*_M} \times 100\%$ and $RE^*_{INT} = \dfrac{E^*_{INT}}{c^*_M} \times 100\%$.
3. Difference of absolute errors: $E^*_{NT} - E^*_{INT}$.
4. Absolute error ratio: $\dfrac{E^*_{NT}}{E^*_{INT}} = \dfrac{RE^*_{NT}}{RE^*_{INT}}$.

Table II. Normalization to the unit interval of the results in Table I.

		Normalized NT method			Normalized INT method			Error comparison	
IT2FS†	$c^*_M = (c^*_l + c^*_r)/2$	c^*_{NT}	E^*_{NT}	RE^*_{NT}	c^*_{INT}	E^*_{INT}	RE^*_{INT}	$E^*_{NT} - E^*_{INT}$	E^*_{NT}/E^*_{INT}
$\tilde{A}^*_1$	0.5	0.5	0	0	0.5	0	0	0	—
$\tilde{A}^*_2$	0.4631	0.4586	0.0044	0.96%	0.4630	0.0001	0.02%	0.0044	60.21
$\tilde{A}^*_3$	0.4415	0.4395	0.0020	0.45%	0.4416	0.0001	0.01%	0.0019	33.17
$\tilde{A}^*_4$	0.4752	0.4744	0.0008	0.16%	0.4753	0.0002	0.04%	0.0006	4.42

† IT2FSs with normalized domain (37)–(44).

4.3. A counterexample

Let $\tilde{A}^*$ be an IT2FS defined over $[0, 1]$, and determined by piecewise linear MFs (Figure 6):

$$\underline{\mu}^*(y) = \begin{cases} (y + 3.24)/3.79, & y \leq 0.1, \\ 0, & \text{otherwise}, \end{cases} \tag{45}$$

$$\overline{\mu}^*(y) = \begin{cases} (y + 2.43)/2.84, & y \leq 0.41, \\ 1, & 0.41 < y < 0.68, \\ (1.24 - y)/0.56, & y \geq 0.68. \end{cases} \tag{46}$$

for all $y \in [0, 1]$.

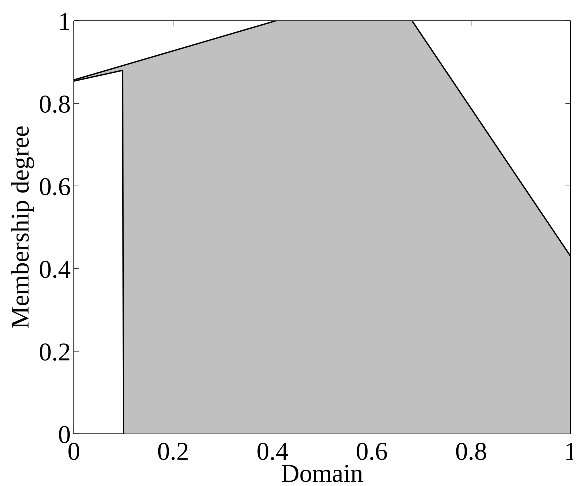

Figure 6. IT2FS with normalized domain (45)–(46).

After calculating[9] c^*_M, c^*_{NT}, c^*_{INT}, and the corresponding metrics for (45)–(46), we got the results in Table III.

Table III. Computation results for our counterexample.

		Normalized NT method			Normalized INT method			Error comparison	
IT2FS†	$c^*_M = (c^*_l + c^*_r)/2$	c^*_{NT}	E^*_{NT}	RE^*_{NT}	c^*_{INT}	E^*_{INT}	RE^*_{INT}	$E^*_{NT} - E^*_{INT}$	E^*_{NT}/E^*_{INT}
$\tilde{A}^*$	0.3321	0.4335	0.1014	30.53%	0.4046	0.0725	21.83%	0.0289	1.3986

† IT2FS with normalized domain (45)–(46).

5. Discussion

As we can see in Table III, our counterexample showed the following:

1. An absolute error $E^*_{NT} = 0.1014$. This is almost 23 times higher than $E^*_{NT} = 0.0044$ (worst case in Table II).
2. A percentage relative error $RE^*_{NT} = 30.53\%$. This is almost 32 times higher than $RE^*_{NT} = 0.96\%$ (worst case in Table II).
3. An absolute error $E^*_{INT} = 0.0725$. This is almost 363 times higher than $E^*_{INT} = 0.0002$ (worst case in Table II).
4. An percentage relative error $RE^*_{INT} = 21.83\%$. This is almost 546 times higher than $RE^*_{INT} = 0.04\%$ (worst case in Table II).

In terms of error comparison $E^*_{NT} - E^*_{INT} = 0.0289$ and $E^*_{NT}/E^*_{INT} = 1.3986$, our example showed that E^*_{NT} is comparable (in magnitude) with respect to E^*_{INT}, in contrast with the results in Table II.

6. Conclusions

This paper showed a counterexample that exhibits higher errors than the corresponding errors in examples reported in the literature when comparing the NT and INT methods versus the CTR+D method. We chose an IT2FS with piecewise linear MFs as our counterexample, mainly due to its simplicity. All the domains (for examples and counterexample) were taken to the unit interval $[0, 1]$ in order to reduce the effect of different domains on the metrics that we used for comparison. Additionally, all the variables involved in the three methods were normalized to the unit interval. We concluded, based on our results, that the NT and INT methods are not necessarily good approximations to the CTR+D method.

A. Source code for the counterexample in Section 4.3

The source code presented in this section was executed on MATLAB 7.14.0.739 (R2012a), on a laptop with Microsoft Windows XP Professional 32 bit, Intel(R) Atom(TM) CPU Z520 1.33 GHz,

[9] In Appendix A we present a source code for the numerical calculation

1014 MB of RAM. See the main text for a description of each variable in the following code.

```
%--------------------------------------------------------------------------
%      Definition of LMF and UMF.
%      LMF = Lower Membership Function
%      UMF = Upper Membership Function
%--------------------------------------------------------------------------
clear all
syms y z % Symbolic variables
LMF = ((y + 3.24)/3.79) * heaviside(0.1 - y);
UMF = ((y + 2.43)/2.84) * heaviside(0.41 - y) + ...
      (heaviside(y - 0.41) - heaviside(y - 0.68)) + ...
      ((1.24 - y)/0.56) * heaviside(y - 0.68);
%--------------------------------------------------------------------------
%      CTR+D method
%--------------------------------------------------------------------------
A = int(y * UMF, y, 0, 1);
B = int(UMF, y, 0, 1);
C = int(y * LMF, y, 0, 1);
D = int(LMF, y, 0, 1);
phi = int((z-y) * UMF, y, 0, z) + int((z-y) * LMF, y, z, 1);
omega = z * (B + D) - (A + C) - phi;
sol = solve(phi, 'Real', true); % Solve phi = 0. Find real values
index = find(sol >= 0 & sol <= 1);
c_l = vpa(sol(index), 4) % We choose the root of phi in [0,1] as c_l
sol = solve(omega, 'Real', true); % Solve omega = 0. Find real values
index = find(sol >= 0 & sol <= 1);
c_r = vpa(sol(index), 4) % We choose the root of omega in [0,1] as c_r
c_M = vpa((c_l + c_r)/2, 4) % CTR+D method
%--------------------------------------------------------------------------
%      NT method
%      AMF = Average Membership Function
%--------------------------------------------------------------------------
AMF = (LMF + UMF) / 2;
c_NT = int(y * AMF, y, 0, 1) / int(AMF, y, 0, 1);
c_NT = vpa(c_NT, 4) % Nie-Tan method
%--------------------------------------------------------------------------
%      INT method
%--------------------------------------------------------------------------
num1_delta = int((c_NT-y)*UMF, y, 0, c_NT) + int((c_NT-y)*LMF, y, c_NT, 1);
num2_delta = int(UMF-LMF, y, 0, c_NT) - int(UMF-LMF, y, c_NT, 1);
den_delta = (int(UMF, y, 0, 1) + int(LMF, y, 0, 1)) ^ 2;
delta = 2 * num1_delta * num2_delta / den_delta;
c_INT = c_NT + delta;
c_INT = vpa(c_INT, 4) % Improved Nie-Tan method
```

References

[1] J. M. Mendel and R. I. John, "Type-2 fuzzy sets made simple," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 117–127, 2002.

[2] O. Castillo and P. Melin, *Type-2 Fuzzy Logic: Theory and Applications*, ser. Studies in Fuzziness and Soft Computing. Springer-Verlag Berlin Heidelberg, 2008, vol. 223.

[3] N. N. Karnik and J. M. Mendel, "Type-2 fuzzy logic systems : Type-reduction," in *IEEE International Conference on Systems, Man, and Cybernetics*, vol. 2, San Diego, California, USA, Oct. 1998, pp. 2046–2051.

[4] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 6, pp. 808–821, Dec. 2006.

[5] O. Salazar, J. Soriano, and H. Serrano, "Centroid of an interval type-2 fuzzy set: Continuous vs. discrete," *Ingeniería*, vol. 16, no. 2, pp. 67–78, 2011, ISSN 0121-750X.

[6] N. N. Karnik, J. M. Mendel, and Q. Liang, "Type-2 fuzzy logic systems," *IEEE Transactions On Fuzzy Systems*, vol. 7, no. 6, pp. 643–658, 1999.

[7] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Information Sciences*, vol. 132, pp. 195–220, 2001.

[8] S. Coupland and R. John, "An investigation into alternative methods for the defuzzification of an interval type-2 fuzzy set," in *Proceedings of the 2006 IEEE International Conference on Fuzzy Systems*, Vancouver, Canada, Jul. 2006, pp. 1425–1432.

[9] J. M. Mendel, "Type-2 fuzzy sets and systems: An overview," *IEEE Computational Intelligence Magazine*, vol. 2, no. 1, pp. 20–29, Feb. 2007.

[10] J. Aisbett, J. T. Rickard, and D. G. Morgenthaler, "Type-2 fuzzy sets as functions on spaces," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 4, pp. 841–844, Aug. 2010.

[11] J. M. Mendel and X. Liu, "New closed-form solutions for karnik-mendel algorithm+defuzzification of an interval type-2 fuzzy set," in *Proceedings of the 2012 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, Brisbane, QLD, Jun. 2012, pp. 1–8.

[12] ——, "Simplified interval type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 6, pp. 1056–1069, 2013.

[13] M. Nie and W. W. Tan, "Towards an efficient type-reduction method for interval type-2 fuzzy logic systems," in *Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ 2008)*, 2008, pp. 1425–1432.

[14] S. Greenfield and F. Chiclana, "Accuracy and complexity evaluation of defuzzification strategies for the discretised interval type-2 fuzzy set," *International Journal of Approximate Reasoning*, vol. 54, pp. 1013–1033, 2013.

[15] J. M. Mendel, *Uncertainty Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Prentice-Hall PTR, 2001.

[16] D. Wu and M. Nie, "Comparison and practical implementation of type-reduction algorithms for type-2 fuzzy sets and systems," in *Proceedings of the IEEE International Conference on Fuzzy Systems (FUZZ 2011)*, 2011, pp. 2131–2138.

[17] J. M. Mendel and H. Wu, "Uncertainty bounds and their use in the design of interval type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 5, pp. 622–639, 2002.

[18] S. Greenfield, F. Chiclana, S. Coupland, and R. John, "The collapsing method of defuzzification for discretized interval type-2 fuzzy sets," *Information Sciences*, vol. 179, no. 13, pp. 2055–2069, 2009.

[19] J. M. Mendel and H. Wu, "Properties of the centroid of an interval type-2 fuzzy set, including the centroid of a fuzzy granule," in *Proceedings of the 2005 International Conference on Fuzzy Systems (FUZZ-IEEE 2005)*, 2005, pp. 341–346.

[20] ——, "New results about the centroid of an interval type-2 fuzzy set, including the centroid of a fuzzy granule," *Information Sciences*, vol. 177, pp. 360–377, 2007.

[21] O. Salazar and J. Soriano, "Generating embedded type-1 fuzzy sets by means of convex combination," in *Proceedings of the 2013 IFSA World Congress NAFIPS Annual Meeting*, Edmonton, Canada, Jun. 2013, pp. 51–56.

[22] ——, "Convex combination and its application to fuzzy sets and interval-valued fuzzy sets I," *Applied Mathematical Sciences*, vol. 9, no. 22, pp. 1061–1068, 2015, ISSN 1312-885X.

[23] ——, "Convex combination and its application to fuzzy sets and interval-valued fuzzy sets II," *Applied Mathematical Sciences*, vol. 9, no. 22, pp. 1069–1076, 2015, ISSN 1312-885X.

[24] J. M. Mendel and F. Liu, "Super-exponential convergence of the Karnik-Mendel algorithms for computing the centroid of an interval type-2 fuzzy set," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 2, pp. 309–320, Apr. 2007.

[25] X. Liu and J. M. Mendel, "Connect karnik-mendel algorithms to root-finding for computing the centroid of an interval type-2 fuzzy set," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 4, pp. 652–665, 2011.

Juan Diego Rojas Zambrano

Electronics Engineer, Universidad Distrital Francisco José de Caldas. His research interests include fuzzy systems, neural networks and mathematics. e-mail: jdrojasz@correo.udistrital.edu.co

Omar Salazar Morales

Electronics Engineer, Universidad Distrital Francisco José de Caldas. Master in Industrial Engineering, Universidad Distrital Francisco José de Caldas. Master in Mathematics, Universidad Nacional de Colombia. His research interests include fuzzy systems, control theory and mathematics. e-mail: osalazarm@correo.udistrital.edu.co

Jose Humberto Serrano Devia

Mathematician, Universidad Pedagógica Nacional. Master in Mathematics, Universidad Nacional de Colombia. His research interests include fuzzy systems and mathematics. e-mail: jhserrano@udistrital.edu.co