UNICIENCIA 21
pp. 145-149
2004

# THE DIRECT EFFECT ON GEOID COMPUTATIONS

***Juan G. Serpas***
Escuela de Topografía, Catastro y Geodesia, Universidad Nacional
Heredia, Costa Rica
Email: jserpas@una.ac.cr

## ABSTRACT

The most popular technique for the reduction of gravity to the geoid is the Helmert's condensation method. Two different ways to apply this reduction are studied: the classical approach (Wang and Rapp, 1990, Heiskanen and Moritz, 1967), and the one by Vaniĉek and Kleusberg (1987), extended by Martinec *et al.* (1993). The classical approach (Wang and Rapp, 1990, Heiskanen and Moritz, 1967) argues that the effect of the condensed layer has to be evaluated at geoid level and not at the terrain level as stated by Vaniĉek and Kleusberg, 1987. Jekeli and Serpas (2003) conclude that both methods are correct from the theoretical point of view, and the difference is in the order of application of the remove, restore and downward continuation procedure. The computation of the geoid is analyzed under the two mentioned approaches and the results are compared to geoid undulations coming from GPS and orthometric heights in three different regions in the USA. Numerical assessment of the different approaches shows that both yield similar results in relatively flat areas, and that the classical approach provides better results in mountainous areas with rough topography.

**KEYWORDS**: Helmert's condensation method, geoid, direct effect.

## RESUMEN

La técnica más popular para la reducción de observaciones gravimétricas al geoide es el método de condensación de Helmert. Dos maneras diferentes de aplicar dicha reducción son estudiadas: el enfoque clásico (Wang y Rapp, 1990, Heiskanen y Moritz, 1967) y el enfoque por Vaniĉek and Kleusberg (1987), extendido por Martinec *et al.* (1993). El enfoque clásico (Wang y Rapp, 1990, Heiskanen y Moritz, 1967) argumenta que el efecto de la capa condensada tiene que ser evaluado en el geoide y no en el terreno como ha sido expuesto por Vaniĉek y Kleusberg, 1987. Jekeli y Serpas (2003) concluyen que ambos métodos son correctos desde el punto de vista teórico, siendo la diferencia el orden en la aplicación a la hora de remover y restaurar, y de la continuación descendente al geoide. El cálculo del geoide es analizado con los dos métodos mencionados y los resultados son comparados con ondulaciones del geoide obtenidas por la diferencia de alturas elipsoidales por GPS y alturas ortométricas en tres regiones diferentes en los Estados Unidos. Pruebas numéricas muestran que ambos enfoques producen resultados similares en áreas relativamente planas, y que el enfoque clásico provee mejores resultados en áreas montañosas con topografía variada.

**PALABRAS CLAVES**: método de condensación de Helmert, geoide, efecto directo.

## 1. INTRODUCTION

The knowledge of the geoid has gained importance nowadays due to the increasing interest of the use of GPS measurements for computing orthometric heights. The computation of a precise geoid model, at centimeter level of accuracy, is needed in order to take advantage of GPS measurements, which are more convenient than the classical geometric and trigonometric methods of leveling. The use of Helmert condensation method for the computation of geoid could provide the centimeter level accuracy needed. Many papers regarding the way to properly apply this method have been published during the last decade (Vanîcek and Kleusberg, 1987; Wang and Rapp, 1990; Martinec *et al.*, 1993; Heck, 1993; Vanîcek and Martinec, 1994, Najavandchi, 2001, Jekeli and Serpas, 2003). It is important therefore to assess the different solutions in order to establish the appropriate method to compute the geoid.

In this paper these solutions are studied and compared to the geoid coming from GPS observations on benchmarks with orthometric heights.

## 2. THE HELMERT'S CONDENSATION METHOD

The Helmert's condensation method for reducing gravity consists of the radial condensation of the topography above the geoid into a thin surface layer on the geoid (Heiskanen and Moritz, 1967). These masses are condensed along the local vertical with density (Heiskanen and Moritz, 1967; Vanîcek and Martinec, 1994):

$$k = \bar{\rho} h_p \qquad (2.1)$$

where $\bar{\rho}$ is the average density of the terrain along height $h_p$.

For the computation of the geoid by means of Stokes' integral we have:

$$N = \frac{R}{4\pi\gamma}\iint_{\sigma} \Delta g\, s(\psi) d\sigma + \delta N_I \qquad (2.2)$$

where in this case:

$$\Delta g = g_p - \gamma_Q + F - A_T + A_H \qquad (2.3)$$

with $g_p$ measured gravity

$\gamma_Q$ normal gravity at geoid

F the free air reduction

$A_H$ the attraction due to the condensed layer with density k

$A_T$ the attraction due to the topography

$A_T = 2\pi G\rho h_p - A_t$

$A_t$ the terrain correction

$\delta N_I$ the indirect effect

### 2.1 The classical approach

For the attraction of the condensed layer Wang and Rapp (1990), consider a point on the geoid and this attraction is given by:

$$A_H = -\left.\frac{\partial W}{\partial h_p}\right|_{h_p=0} = 2\pi G\rho h_p - \left[\frac{\partial}{\partial h_p}\iint \frac{G\rho h}{l_{P0}} dx\,dy\right]_{h_p=0}$$

$$= 2\pi G\rho h_p \qquad (2.1\text{-}1)$$

with $l_{P0}$ the distance from P on the terrain to a point on the geoid. Subtracting $A_T$ from equation (2.1-1), the direct effect can be written in planar approximation as the terrain correction (Wang and Rapp, 1990):

$$-A_T + A_H = \frac{1}{2}G\rho\iint \frac{(h-h_p)^2}{d^3} dx\,dy = A_t \qquad (2.1\text{-}2)$$

with $d^2=(x-x_P)^2+(y-y_p)^2$. Notice that this correction yields gravity anomalies that are exactly the Faye anomalies. We will refer to this approach as the W/R approach.

The term $\delta N_I$ in (2.1-1) is often neglected, but in this context should be considered since it can reach up to 5 cm for an elevation of 1000 m (Moritz, 1980). This term can be computed as (Heiskanen and Moritz, 1967; Wang and Rapp, 1990):

$$\delta N_I = \delta W_I / \gamma \qquad (2.1\text{-}3)$$

where $\delta W_I$ is the difference in potential between the mass of the actual topography and the mass of the condensed layer and is evaluated at geoid level. Equation (2.1-4) can be written in planar approximation as (Wang and Rapp, 1990):

$$\delta N_I = \frac{-\pi G\rho h^2}{\gamma} - \frac{1}{6}\frac{G\rho}{\gamma}\iint \frac{h^3 - h_p^3}{d^3} dx\,dy \qquad (2.1\text{-}4)$$

### 2.2 The Vanîcek and Kleusberg approach

Vanîcek and Kleusberg (1987) referred the attraction of the condensed layer to a point P on the topographic surface. The attraction of the condensed layer at a point P is given by (Wang and Rapp, 1990):

$$A_H^h = -\frac{\partial W}{\partial h_p}\bigg|_{h_p} = G\rho \iint \frac{h - h_p}{l_{P0}^3} dx\, dy$$

$$\approx 2\pi G\rho h_p - g_1 \qquad (2.2\text{-}1)$$

where:

$$g_1 = -G\rho h_p \iint \frac{h - h_p}{d^3} dx\, dy \qquad (2.2\text{-}2)$$

Equation (2.2-1) tells us that the direct effect can now be written as (Vaniĉek and Kleusberg, 1987):

$$- A_T + A_H^h = \frac{1}{2} G\rho \iint \frac{h^2 - h_p^2}{d^3} dx\, dy \qquad (2.2\text{-}3)$$

which is different from equation (2.1-2). Equation (2.1-2) is always positive, while equation (2.2-3) can take on positive and negative values according to the topography. We will call it the V/K approach.

Martinec *et al.* (1993) derived an expression where they argue that the effect of the condensed layer has to be evaluated at terrain level as stated by Vaniĉek and Kleusberg and not at the geoid level as argued by Wang and Rapp. They arrived at a different formulation for the evaluation of the gravity anomalies when using Helmert condensation method:

$$\Delta g = \Delta g^F - A_T + A_H^h + g_1 + \delta s \qquad (2.2\text{-}4)$$

with: $\Delta g^F = g_P - \gamma_Q + F$

$\delta s$ is the so-called secondary indirect effect and it is given by Heiskanen and Moritz (1967), eq. (3-51). In this formula also the term $g_1$ is included. This term represents the downward continuation of gravity anomalies from the topographical surface to the geoid, and it can be written with good approximation as:

$$g_1 \cong -\frac{R^2}{2\pi} h \int_\sigma \frac{\Delta g - \Delta g_P}{l_o^3} d\sigma \qquad (2.2\text{-}5)$$

with $l_o$ the distance on the geoid.

The evaluation of $g_1$ is usually carried out under the assumption of linear correlation between terrain and gravity anomalies:

$$\Delta g = a + 2\pi G\rho h \qquad (2.2\text{-}6)$$

Under this assumption the term $g_1$ can be written as:

$$g_1 \cong -G\rho R^2 h \int_\sigma \frac{h - h_P}{l_o^3} d\sigma \qquad (2.2\text{-}7)$$

If we now substitute this term in (2.2-4) the correction to be applied to the free air gravity anomalies can be written in planar approximation as:

$$A_T - A_H^h + g_1 = \frac{1}{2} G\rho \iint \frac{(h - h_p)^2}{d^3} dx\, dy \qquad (2.2\text{-}8)$$

It has to be noticed at this point that equation (2.2-8) is the same, considering the assumption on linear relationship between gravity anomaly and height, as equation (2.1-2) derived by Wang and Rapp (1990).

It has to be pointed out that Wang and Rapp (1990) do not assume any relationship between gravity anomalies and topography (Jekeli and Serpas, 2003). The equation derived by Wang and Rapp (1990) turns out to be the same as equation (2.2-8) under the approximations used.

From equation (2.2-4) we can see that the V/K approach neglects completely the term $g_1$ in their derivation and needs to be included. The inclusion of the downward continuation should be performed without considering any dependency between gravity anomalies and topography. With respect to the computation of the indirect effect, there is no difference between both approaches. Both approaches used equation (2.1-4) in their computations.

Vaniĉek and Martinec (1994) developed more thoroughly the theory on the principles by Vaniĉek and Kleusberg and identify the downward continuation as key factor in the reduction. Najavandchi (2001) came up with expressions developed from harmonic coefficient of the topography. Jekeli and Serpas (2003) developed equations for the direct effect on spherical approximation based on principles by Moritz (1980) and Pellinen (1962):

$$\delta g_{MP} = -\frac{4\pi G\rho}{R} h_p^2 + \frac{G\rho}{2} R^2 \iint_\sigma \left( \frac{(h-h_p)^2}{l_p^3} - \frac{3}{4}\frac{(h-h_p)^4}{l_p^5} + \cdots \right) d\sigma \tag{2.2-9}$$

and for the Vaniĉek and Martinec (V/M):

$$\delta g_{VM} = -\frac{4\pi G\rho}{R} h_p^2 + \frac{G\rho}{2} R^2 \iint_\sigma \left( \frac{h^2-h_p^2}{l_{p0}^3} + \frac{3}{4}\frac{(h-h_p)^2}{l_p^5}(h_p^2 - h^2 + 2h_p h) + \cdots \right) d\sigma \tag{2.2-10}$$

where: $l_p$ is the distance between two points on sphere of radius $R+h_p$

$l_{P0}$ is the distance between a point P and the sphere of radius R

Notice that the first integrand term in both equations is the same as the W/R and V/K approaches in spherical approximation. The constant term reaches sub milligal levels.

Finally, it is worth mentioning that the downward continuation is a key factor for the reduction of gravity anomalies to the geoid. The downward continuation is known to be an ill-posed problem and errors in the data are amplified by this procedure. For the case of the classical approach, the downward continuation of Bouguer anomalies is usually neglected under the assumption of smoothness of Bouguer anomalies, while the downward continuation of Helmert anomalies cannot be neglected and its contribution is necessary for the proper computation of the geoid when using the Vaniĉek and Kleusberg approach.

## 3. NUMERICAL RESULTS

In order to asses the accuracy of the different approaches, data of rough and flat topography terrain with different gravity signatures are considered in the computations. Three areas are evaluated. They will be referred as area 1, area 2 and area 3. See Table 3.1 for the description of these areas:

**Table 3.1. Areas used for the computations.**

| | Latitude | | Longitude | | Mean elev | Min elev | Max elev |
|---|---|---|---|---|---|---|---|
| Area | from | to | from | to | (m) | (m) | (m) |
| 1 | 42° | 48° | -101° | -91° | 433 | 183 | 1027 |
| 2 | 38° | 42° | -88° | -80° | 270 | 106 | 1456 |
| 3 | 37° | 44° | -110° | -103° | 2032 | 836 | 4334 |

Area 1 is known for its characteristic complicated gravity signature, which is not related linearly to the topography. For the case of area 3, this is located in the USA Rocky Mountains, which is important due to the high frequency content of gravity anomalies. Area 2 is located on a relative flat area with no special characteristics. The remove restore technique for the geoid computation by Stokes' integral was used. The low frequency part of gravity anomalies and geoid undulations from EGM96 were used. Edge effects are removed from the results and the comparisons are done in a reduced inner area. The geoid undulations are given at the same location as the gridded data of gravity anomalies. On the other hand, ellipsoidal heights in the same area are not at the same locations of gridded data. Therefore the geoid undulations were interpolated at the same locations where information of ellipsoidal heights are given. This was done by the use of cubic interpolation within the area of interest. With the information of geoid and ellipsoidal heights at the same locations, the computed geoid undulations coming from the M/P and V/M approaches, and a third one called V/M+DC, where the downward continuation is included for the V/M approach, are compared to geoid undulations coming from the GPS network. In the computations only the first term of the integral for the M/P and V/M approaches is used. Also differences for the G99SSS gravimetric geoid from NGS on the same locations are presented. The results are presented in Tables 3.2 to 3.4.

**Table 3.2. Differences geoid undulations area 1.**

| | Mean Diff [cm] | Std Dev [cm] | Min Diff [cm] | Max Diff [cm] |
|---|---|---|---|---|
| M/P | 88.0 | 6.2 | 74.9 | 113.9 |
| V/M | 86.9 | 6.2 | 73.7 | 112.3 |
| V/M+D/C | 87.7 | 6.2 | 46.2 | 113.0 |
| G99SSS | 77.8 | 6.1 | 66.9 | 102.5 |

**Table 3.3. Differences geoid undulations area 2.**

| | Mean Diff [cm] | Std Dev [cm] | Min Diff [cm] | Max Diff [cm] |
|---|---|---|---|---|
| M/P | 49.6 | 6.3 | 34.2 | 63.5 |
| V/M | 48.4 | 6.5 | 32.9 | 62.9 |
| V/M+D/C | 47.9 | 6.4 | 32.1 | 62.1 |
| G99SSS | 51.8 | 5.8 | 39.9 | 67.1 |

**Table 3.4. Differences geoid undulations area 3.**

| | Mean Diff [cm] | Std Dev [cm] | Min Diff [cm] | Max Diff [cm] |
|---|---|---|---|---|
| M/P | 40.9 | 8.5 | 20.2 | 63.3 |
| V/M | -26.2 | 23.4 | -90.5 | 22.1 |
| V/M+D/C | 12.6 | 14.6 | -27.0 | 47.1 |
| G99SSS | 63.0 | 8.6 | 32.6 | 86.3 |

## 4. CONCLUSIONS

We can observe that for the computation of the absolute geoid in areas 1 and 2 the three methods analyzed provide the same level of accuracy in terms of standard deviation being in the order of 6 cm. The picture is different for mountainous areas, where the M/P approach provides the best results with a standard deviation of 8.5 cm compared to the V/M and V/M+D/C approaches reaching 23 and 14.6 cm respectively.

The M/P approach provides similar results as those of G99SSS. This can be attributed to the fact that both use the same approach.

## REFERENCES


Blakely, R.J. 1995. *Potential Theory in Gravity & Magnetic Applications*, Cambridge Univ. Press, New York.

Forsberg, R. 1994. 4Terrain effects in geoid computations. In: *School for the determination and use of the geoid, Lecture notes*, Milan, Cotober 10-15.

Heck, B. (1993). A revision of the Helmert's second method of condensation in geoid and quasigeoid determination. In: *Proc. of IAG Symposium No.112:7th Symposium of Geodesy and Physics of the Earth*, Montag. H, Reigber C, Springer Verlag, Berlin, pp. 246-251.

Heiskanen, W. and H. Moritz. 1967. *Physical Geodesy*, Freeman and Co., San Francisco.

Jekeli, C. and J. Serpas. 2003. Review and Numerical assessment of the direct topographical reduction in geoid determination. *Journal of Geodesy*, 77:226-239.

Martinec, Z., C. Matyska, E.W. Grafared, and P. Vaniĉek. 1993. On Helmert's second condensation method. *Manuscripta Geodaetica*, 18: 417-421.

Moritz, H. (1980). *Advanced Physical Geodesy*. Herbert Wichmann, Verlag, Karlsruhe, Federal Republic of Germany.

Najavandchi, H. 2001. The direct topographical correction in gravimetric geoid determination by the Stokes-Helmert method. *Journal of Geodesy*, 74:488-496.

Pellinen, L.P. 1962. Accounting for topography in the calculation of quasi-geoidal heights and plumb-line defelections from gravity anomalies. *Bulletin Géodésique*, 63:57-65.

Smith, D.A. and D.G. Milbert. 1999. The GEOID96 high-resolution geoid model for the United States. *Journal of Geodesy*, 73:219-236.

Vaniĉek, P. and Kleusberg. 1987. The Canadian geoid – Stokesian approach. *Manuscripta Geodaetica*, 12:86-98.

Vaniĉek, P. and Z. Martinec. 1994. Stokes-Helmert scheme for the evaluation of a precise geoid. *Manuscripta Geodaetica*, 19:119-128.

Wang, Y.M. 1988. Downward continuation of the free air gravity anomalies to the ellipsoid using gradient solution, Poisson's integral and terrain correction. Numerical comparison and the computations. *Report 393 of the Department of Geodetic Science and Surveying*, Ohio State University. Columbus, Ohio.

Wang, Y.M. and R.H. Rapp. 1990. Terrain effects on geoid undulation computations. *Manuscripta Geodaetica*, 15:23-29.