THE DISTRIBUTED NATURE OF PATTERN GENERALIZATION

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Drawing on a review of recent work conducted in the area of pattern generalization (PG), this paper makes a case for a distributed view of PG, which basically situates processing ability in terms of convergences among several different factors that influence PG. Consequently, the distributed nature leads to different types of PG that depend on the nature of a given PG task and a host of cognitive, sociocultural, classroom-related, and unexplored factors. Individual learners draw on a complex net of parallel choices, where every choice depends on the strength of ongoing training and connections among factors, with some factors appearing to be more predictable than others.

Keywords: Algebraic thinking; Distributed view of pattern generalization processing; Mathematical structures; Mathematical thinking; Pattern generalization

La naturaleza distribuida de la generalización de patrones

Sobre la base de una revisión de trabajos recientes en el área de generalización de patrones (PG), este artículo aboga por una visión distribuida de PG, que básicamente sitúa la capacidad de procesamiento en términos de convergencias entre diferentes factores que influyen en PG. En consecuencia, la naturaleza distribuida conduce a diferentes tipos de PG que dependen de la naturaleza de una tarea PG dada y una serie de factores cognitivos, socioculturales, inexplorados y relacionadas con el aula. Alumnos individuales se basan en una compleja red de opciones paralelas, donde cada elección depende de la fortaleza de la formación continua y las conexiones entre los factores, con algunos factores más predecibles que otros.

Términos clave: Estructuras matemáticas; Generalización de patrones; Pensamiento algebraico; Pensamiento matemático; Vista distribuida del procesamiento de la generalización de patrones

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Pattern generalization ability involves the proficiency to construct and justify an interpreted well-defined structure from a constrained set of initial cues (Rivera, 2013). Such structure is mathematical in which case it refers to "a mental construct that satisfies a collection of explicit formal rules on which mathematical reasoning can be carried out" (National Research Council, 2013, p. 29). Hence, pattern generalization (PG) and mathematical structure are intimately and conceptually intertwined, meaning to say that PG ability is interpretive and rule-driven in nature and enables learners to employ predictive and inferential reasoning despite the initial constraint of having only an incomplete knowledge of the target objects for generalization (e.g., stages in a pattern such as those shown in Figure 1 or a set of particular instances, situations, or cases).

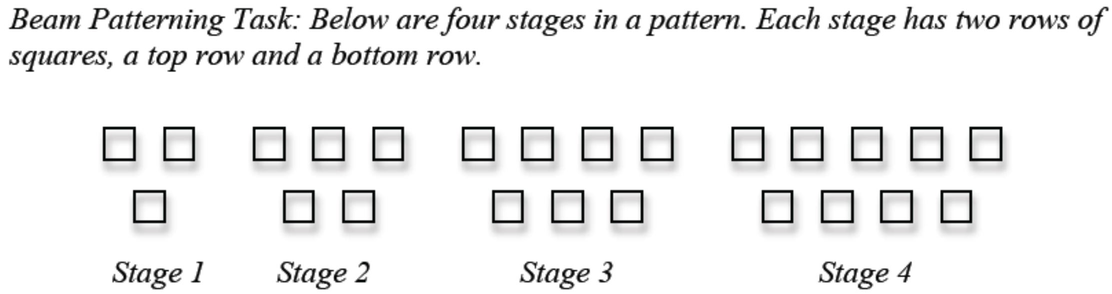


Figure 1. Beam pattern

One unintended consequence of recent results on PG processing is the translation of evidence to interpretive norms of practice (e.g., how teachers expect students to model PG processing). While certainly at this stage we can more or less claim that cognitive shifts from the additive to the multiplicative, from the factual to the symbolic, from the arithmetical to the algebraic, and so on provide successful indications of generalized (especially functional) thinking among learners in various grade levels and tasks, the truth of the matter is that a complex of factors will always influence PG performance. In this article, we explore a *distributed view of PG*, which basically situates processing ability in terms of convergences among several different cognitive and non-cognitive factors that influence PG. The distributed nature of PG inevitably leads to a natural occurrence of different types of PG. That is, depending on the nature of a given PG task and a host of cognitive, sociocultural, classroom-related, and other unexplored (e.g., neural) factors, PG processing draws on a complex net of parallel choices, where every choice depends on the strength of ongoing training and connections among factors, with some factors appearing to be more predictable than others. The primary mode of evidence for a distributed view of PG processing nestles on a review of recent work on generalization in both pattern and non-pattern contexts.

Consider the following interview episode below with a US second-grade student named Skype (S; age 7 years) who was asked to obtain a PG for the Beam pattern task shown in Figure 1 in a clinical interview setting that took place after a one-week teaching experiment on growing figural patterns (Rivera, 2013). During the experiment, the students in Skype's class explored linear patterns to begin to develop the habit of noticing and paying attention to parts in figural stages that appeared to them as being common and shared across the given stages. Once those parts have been identified, they were then instructed to color those parts using the same color from one stage to the next and to use a different color to shade the remaining parts. Next, they were asked to extend the given stages to two more near stages (oftentimes Stages 5 and 6) before dealing with the task of either drawing or explaining to a friend how some far stages in their pattern (e.g., Stages 10, 25, and 100) might behave on the basis of their initial inferences.

Interviewer (I): Your job is to figure out what comes next, Stage 5.

Student (S): Okay. [He draws Stage 5 on paper. See Figure 2.]

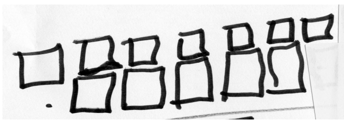


Figure 2. Skype's constructed Stage 5

I: Okay, so how did you know what to do?

S: Because here [referring to the bottom row of Stage 4] it has 4 and that I added 1 more. And right here [referring to the top row of Stage 4] I added 1 more. And this [top row][in Stage 4] one has more [in comparison with Stage 3]. And the top row has more [pointing to the top rows in Stages 1, 2, and 3] and the bottom row [pointing to the bottom rows in Stages 1, 2, and 3] has less.

I: Okay, so can you show me what comes next in Stage 6. [He draws Stage 6 on paper. See Figure 3.]

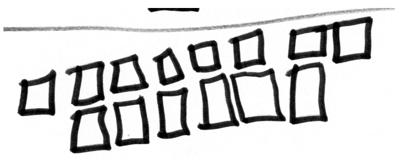


Figure 3. Skype's constructed Stage 6

I: Okay, and how do you know how many to put?

S: I just added 1 more on each side [referring to Stage 6] and then right here [referring to the top row of Stage 5] has less than this [referring to the top row of Stage 6].

I: Okay so when you said you added 1 extra [square] on each side, could you explain that a little more on this drawing [referring to Stage 5]?

S: Hmm. Like you added another 1 right here [on the right corner of the top row in Stage 5] and another here [on the left corner of the top row in Stage 5]. Then you just add one more right here [on each corner of the bottom row of Stage 5].

I: Okay, and how did you know how many to put here [referring to the entire Stage 5]?

S: Because over here [Stage 5] it gets bigger and bigger [points to Stages 1, 2, 3, and 4].

I: Okay, so what about Stage 10. Could you explain to me how to make Stage 10, give me instructions so that I could draw it, how would you explain it?

S: Hmm, you need 10 on the bottom and 11 on the top.

I: Okay, and what about Stage 25?

S: 25 on the bottom and 26 on the top.

I: And a hundred? What about Stage 100?

S: 100 on the bottom and 101 on the top.

When the interviewer asked Skype to check whether his constructed Stages 5 and 6 (Figures 2 and 3) were consistent with the verbal description he offered for Stages 25 and 100, he redrew them and produced the figural stages shown in Figures 4.

Figures 4. Skype's corrected Stages 5 and 6

In a follow up interview that occurred the next day, Skype was presented with the modified beam pattern task shown in Figure 5.

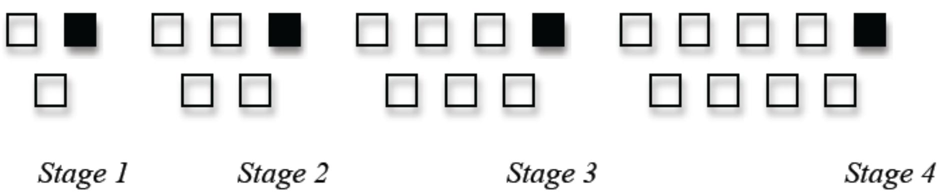


Figure 5. Modified beam pattern

Following the same protocol as before, the interviewer asked Skype to first extend the figural pattern to Stages 5, 6, and 7. Figure 6 shows his drawn Stages 5, 6, and 7. When the interviewer asked him to explain his pattern, he reasoned as follows.

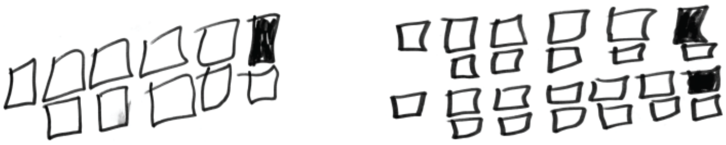


Figure 6. Skype's extensions of the Figure 5 pattern Stage 5 and Stage 6 and 7

S: Right here [referring to Stage 4] it has 4. You add 1 more [referring to the top row in Stage 3]. And then you add 1 more right here [referring to the bottom row of Stage 4 in comparison with the bottom row of Stage 3]…. Then in Stage 5, there's 5 [squares] here and 1 black square [referring to the top row], and then 5 [squares] right here [referring to the bottom row].

I: Okay, great! So what would Stage 10 look like if you just explain it to me?

S: 10 on this side [using Stage 5 as a point of reference and pointing to the top row] and10 on this side [the bottom row].

I: And what about Stage 25?

S: 25 on the top and 1 black one and 25 on the bottom.

I: And what about Stage 100?

S: 100 on the top and a black one and 100 on the bottom.

In two separate clinical interview sessions, Skype established two structural generalizations for the same pattern. While both generalizations were expressed in functional form, which enabled him to predict the structures of the stipulated far generalizing cases, various task constraints influenced the content of his two incipient generalizations. For example, the black corner square on each stage in Figure 5 played an important role in the generalization that he used to describe Stages 10, 25, and 100 of his pattern, which he did not observe when he first dealt with the same pattern in Figure 1. The two separate but related interviews with Skype underscore how certain factors come into play when students engage in PG processing.

Different Factors that Shape Pattern Generalization Processing

Figure 7 is a conceptual framework that takes into account various cognitive and noncognitive factors that shape PG processing. Fundamental differences in individual learners' PG processing can be due to differences in and the simultaneous layering of or complex connections among such factors, which influence various aspects of constructing, expressing, and justifying interpreted structures.

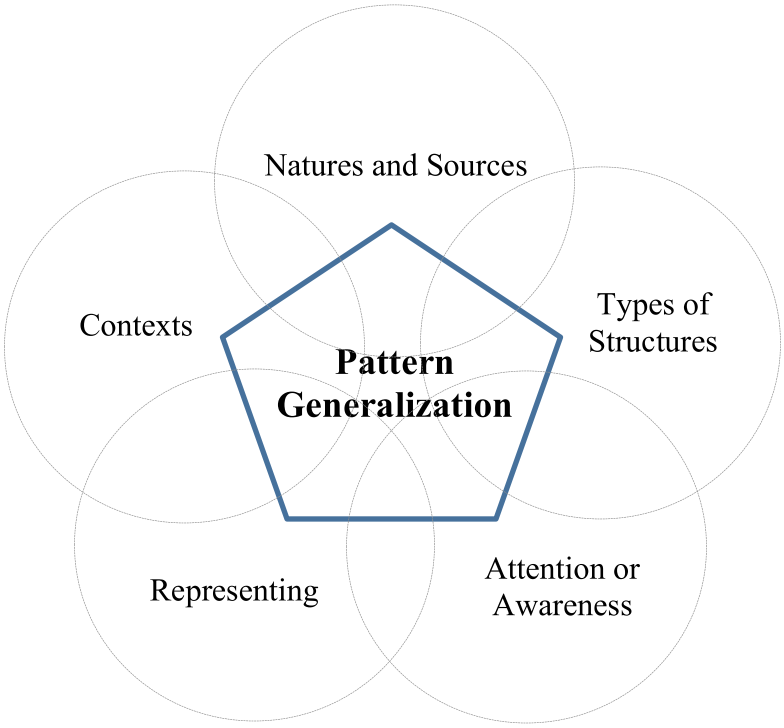


Figure 7. Factors that shape pattern generalization processing

Natures and Sources of Generalization

Several research results converge on the view that individual learners in some cases tend to generalize invariant relationships by paying attention on the given instances or stages in a pattern, while in other cases they tend to dwell on the generality of their inferred ideas, methods, or processes than on the objects themselves. One implication of such findings seems to suggest the need for students to be provided with every opportunity to reflect on their generalizing activity—that is, whether they are working with particular objects or working with general ideas and methods but with a clear and sustained focus on the need for justification.

In one of his studies on older students' mathematical induction, Harel (2001) carefully distinguishes between process pattern generalization (PPG) and result pattern generalization (RPG). In PPG, students establish regularities in an emerging process. In RPG, they dwell on perceived regularities that they infer on the available results. An example of a RPG is shown in Figure 8, which demonstrates a student's empirical-based conviction that his or her generalization about the product rule for logarithms is correct on the basis of several randomly drawn instances that enabled him or her to verify its validity.

$$\log(3 \cdot 4 \cdot 7) = \log 84 = 1.924.$$
$$\log 4 + \log 3 + \log 7 = 1.924.$$
$$\log(4 \cdot 3 \cdot 6) = \log 72 = 1.857;$$
$$\log 4 + \log 3 + \log 6 = 1.857.$$
$$\therefore \log(a_1 \cdot a_2 \cdot ... \cdot a_n) = \log a_1 + \log a_2 + ... + \log a_n$$

Figure 8. An empirical-driven RPG example involving the product rule for logarithms (Harel, 2001, p. 180)

In contrast, Figure 9 illustrates another student's PPG on the same proposition noted in Figure 8, which reflects an indirect use of mathematical induction.

(1) $\log(a_1a_2) = \log a_1 + \log a_2$ $\log(a_1a_2a_3) = \log a_1 + \log a_2 + \log a_3$ *by definition.*

(2) $\log(a_1a_2a_3) = \log a_1 + \log a_2 + \log a_3$. *Similar to* $\log(ax)$ *as in step (1), where this time* $ax = a_1a_2$.

Then $\log(a_1a_2a_3) = \log a_1 + \log a_2 + \log a_3$.

(3) We can see from step (2) any $\log(a_1a_2a_3...a_n) = \log a_1 + \log a_2 + \log a_3$ *can be repeatedly broken down to* $\log a_1 + \log a_2 + ... + \log a_n$.

Figure 9. A transformational-driven PPG example involving the product rule for logarithms (Harel, 2001, p. 180)

Harel notes that while a RPG might initially motivate a PPG, the latter has a more transformational content than empirical, that is, there is

> *(a) consideration of the generality aspects of the conjecture, (b) [an] application of mental operations that are goal oriented and anticipatory—an attempt to predict outcomes on the basis of general principles—and (c) [a sequence of] transformations of images that govern the deduction in the evidencing process.* (Harel, 2001, p. 191)

In other words, there are two different levels of evidence, and PPG goes beyond RPG since it focuses on aspects that involve general principles in order to anticipate and predict subsequent outcomes in a pattern.

In a study conducted with a group of Grade 8 Japanese students (mean age of 14 years), Iwasaki and Yamaguchi (1997) carefully illustrate differences between generalization of objects (GO) and generalization of method (GM). The students participated in a two-hour problem-solving session involving the two tasks shown in Figures 10 and 11. In part (1) of the Figure 10 algebraic task, they performed calculations on the numbers using a trial-and-error strategy that assisted some of them to infer the mathematical relation "the sum of three numbers on the vertical direction in the frame is equal to that on the horizontal direction". In part (2) of the same task, they moved the location of the frame several times to other parts on the calendar that also enabled them to verify the invariance of the inferred relationship. When they were asked to generalize, they pointed out the invariance of adding and subtracting by 1 and 7 to the central number in the frame with the central number seen as a variant since the frame was allowed to move freely on the calendar. They also saw the invariance of 1 and 7 in the context of how each triad of numbers in one direction appeared to them as being arranged in a particular way. In part (3) of the same task, they constructed new mathematical relationships on the basis of changing either the shape of the frame or the arrangement of the numbers (e.g., rotating the frame by 45^o).

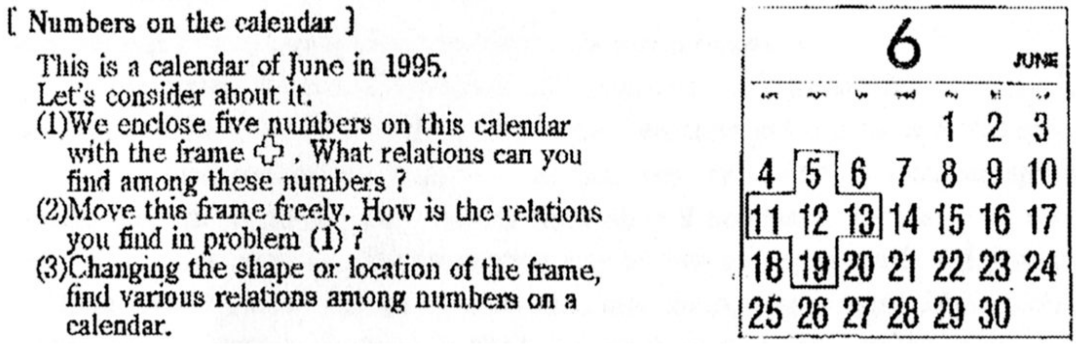

[Numbers on the calendar]
This is a calendar of June in 1995.
Let's consider about it.
(1)We enclose five numbers on this calendar with the frame ✣ . What relations can you find among these numbers ?
(2)Move this frame freely. How is the relations you find in problem (1) ?
(3)Changing the shape or location of the frame, find various relations among numbers on a calendar.

Figure 10. Numbers on the calendar task (Iwasaki & Yamaguchi, 1997, p. 108)

In Figure 11, parts (2) and (3) of the geometric task encouraged some students to draw several pentagrams and to measure the five vertex angles with a protractor that helped them to infer that the sum of the angles measured 180^o across the constructed examples. Several other students "realized the limitations" of the measurement approach, which then encouraged them to prove the statement in part (3) deductively. In this particular situation, they used a single drawn pentagram as a general icon or prototype that enabled them to transition in reasoning from the inductive to the deductive.

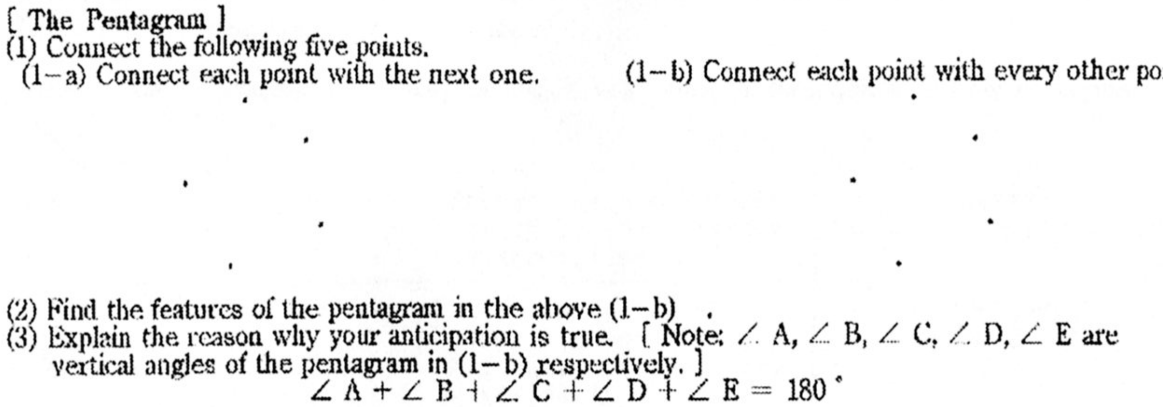

[The Pentagram]
(1) Connect the following five points.
(1−a) Connect each point with the next one. (1−b) Connect each point with every other po
(2) Find the features of the pentagram in the above (1−b) .
(3) Explain the reason why your anticipation is true. [Note: ∠ A, ∠ B, ∠ C, ∠ D, ∠ E are vertical angles of the pentagram in (1−b) respectively.]
$\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

Figure 11. The pentagram task (Iwasaki & Yamaguchi, 1997, p. 110)

Based on the students' thinking on the two tasks shown in Figures 10 and 11, Iwasaki and Yamaguchi note that

> *there are two types of generalization: one is the generalization of object in the algebraic situation, the other is that of method in the geometrical situation. In other words, in [the Figure 10 task], the object of one's thinking such as the concrete number is generalized by the use of letter n. On the other hand, in [the Figure 11 task], the way of viewing itself is generalized. It realizes the change of inference form, that is, from inductive to deductive.* (Iwasaki & Yamaguchi, 1997, pp. 111-112)

Several other research results articulate similar findings. Yerushalmy (1993) distinguishes between generalization of ideas (GI) and generalization from examples (GE). GI and GE represent two situations in which generalization processing is focused on constructing a more general statement from several specific ideas and developing a generalization drawn from observing particular cases or examples in a given set, respectively. However, in a GI, which "is assumed to be the most complex type of generalization" (Yerushalmy, 1993, pp. 68-69), it is not crucial to draw on examples since what matters more are the relevant ideas that can be dropped, ignored, relaxed, and/or combined in order to achieve a greater generality (cf. Holland, Holyoak, Nisbett, & Thagard, 1986; Yevdokimov, 2008). Drawing on their work with young and older students and teachers, Zazkis, Liljedahl, and Chernoff (2008) note that while a GE enables students to notice commonalities among examples, employing big numbers as a purposeful strategy can "serve as a stepping stone toward expressing generality with algebraic symbols" (p. 13) and, thus, can help students to infer possible underlying structures.

Types of Structures

Recent studies that have documented ways in which learners engaged in figural PG processing underscore the following two core features of structures involving the given stages in a figural pattern: there is a clear unit of repeat; and the unit of repeat is the basis for multiplicative thinking, which involves iterating the unit of repeat, leading to the construction of a function-based generalization. Possible complications happen due to the interpretive nature of structural discernment, formation, and construction in exact terms that students tend to express in several different ways.

Drawing on their patterning investigations with 2 to 7-year old children involving repeating sequences of objects and informed by the available research evidence in the field, Clements and Sarama (2009) suggest the following PG learning trajectory actions among young children: (a) pre-explicit patterning, (b) pattern recognizing, (c) pattern fixing, (d) pattern extending, (e) pattern unit recognition, and (f) numeric patterning (pp. 195-198). In the pre-explicit patterning phase, two-year-old children have an implicit and approximate sense of what constitutes a pattern. In the pattern recognition phase, which takes place at around age 3, they begin to recognize a pattern. At age 4, the pattern fixing phase, they are able to fill in the unknown object in a repeating pattern in at least three different ways. One way involves constructing their own repeating pattern in another location with a close eye on a given pattern involving two objects (i.e., duplicating). A second way involves adding elements at the end of a given row of repeating pattern involving two objects (i.e., extending). A third way involves duplicating far more complex repeating patterns beyond two objects. At age 5, the pattern extending phase, they can extend simple repeating patterns. In the pattern unit recognition phase, which occurs around age 6, they begin to interpret, recognize, and construct a core unit of repeat for a given pattern, which also enable them to translate the same pattern in different media and in some cases create patterns of their own choice. At age 7, the numeric patterning phase, they are able to describe figural growth patterns numerically and translate between their figural and numerical representations.

In several studies, Mulligan and her colleagues (e.g., Mulligan, Prescott, & Mitchelmore, 2004; Papic, Mulligan, & Mitchelmore, 2011, 2009) also note changes in structural representations among their samples of young children who dealt with repeating and nonrepeating patterning tasks in various contexts. Based on their empirical studies conducted with several different cohorts of Australian preschool and Grade 1 children (with ages ranging from 3.75 to 6.7 years), young children's structural representations transition from the prestructural stage, followed by emergent and then partial, before finally achieving the full stage of structural development. Of course, appropriate habituation lessons that focus on the notions of a common unit (of repeat) and relevant spatial concepts tend to support the development of, and growth in, multiplicative and functional thinking. Students in the prestructural phase often produce idiosyncratic responses that have little to no semblance of any kind of structure in both aspects of numerical count and spatial arrangement. Students in the emergent phase produce invented or approximate structures in either numerical count or spatial arrangement and are often influenced by what they find meaningful and relevant. In this phase, shape and not count appears to be a factor in the children's structural discernment of a given task. Students in the partial phase produce at least one consistent and organized structural feature with some missing or incomplete necessary features. In the case of patterns, either shape or count is correctly accounted for but not both. Shape consistency without numerical consistency might indicate a valid recognition of a correct and replicable unit of repeat, which is central to any structural arrangement involving objects and sets of objects. Finally, students in the full stage of structural development exhibit responses that represent an organized, exact (versus approximate), and integrated interplay of all the relevant structural features (i.e., numerical count and spatial arrangement), which are deemed to be consistent and valid within their respective contexts.

In several articles that have been drawn from their longitudinal studies with middle school students, Rivera and Becker (2011) provide an account of the existence of several different types of fully algebraically-useful structures relative to figural PG tasks (e.g., Figure 1). Figural patterns involve shapes as the primary objects of generalization. As with all shapes in mathematics, they are analyzed in terms of subconfigurations or parts or components that operate or make sense within some interpreted structures. The term fully algebraically-useful structure fulfills all the requirement of a full stage of structural development and the additional constraint that an interpreted structure is conveyed through an explicit formula in function or closed form.

Attending to, or Awareness of, Structure

One very interesting strand of studies on generalization addresses the fundamental issue of the nature of learner-constructed structures in PG contexts. Should structures emerge over the course of their domains by comparing the most reasonable number of particular cases? Or, should they be imposed first by relying on single generic cases or prototype models that can then be the basis for further construction of related instances or cases? Küchemann (2010) captures this fundamental issue in terms of the sequential versus generic approach to PG activity.

Noss, Healy, and Hoyles (1997) and Küchemann (2010) emphasize the view that any kind of structural awareness would need to occur first in situations or "environments in which the only way to manipulate and reconstruct [the relevant] objects is to express explicitly the relationships between them" (Noss et al., 1997, p. 207). Küchemann (2010) recommends the use of PG tasks that are not always presented "in the form of sequential elements" because they tend to engender empirical generalizations that are "divorced" from structures that produce them in the first place. Ainley, Wilson, and Bills (2003) point out as well that "term-by-term" patterns might "obscure the need for algebraic generalization" (p. 15). Mason, Stephens, and Watson (2009) also distinguish between empirical counting and structural generalization. Küchemann (2010) proposes the use of PG tasks that can be "tackled generically" say, by "inspecting a single generic case," where the primary focus of students' attention is "directly on the search for structure" (Küchemann, 2010, p. 242; e.g., Figure 13) or equivalent structures that emerge as a consequence of the "semantically ambiguous nature" of the relevant symbolic expressions of generality (Samson, 2011; Samson & Schäfer, 2011).

Drawing on their case studies with two 12 to 13-year-old UK students who engaged in PG in the context of a computer microworld, Noss et al. (1997) express the view in which "language as the mechanism for controlling objects" (in the dynamic microworld) can, in fact, "make the algebra of relationships between things semi-formal, concrete, [and] meaningful" (p. 207). Hence, instead of presenting the Figure 1 pattern in terms of stages, individual students can be presented first with a generic stage that can help them infer a plausible general structure or "write a general procedure (i.e., a program)" (p. 211) that can "draw any term in the sequence…, [determine]how many matches would be needed…, [and] predict the number of matches for any terms of the sequence" (p. 214).

Based on their study, the authors note that the two students did not focus on the relevant numerical data. Instead, a "microworld's activity structure supported—perhaps even encouraged—a theoretical rather than a pragmatic approach, in which an elaboration of structure was a means to an (empirical) end, rather than an end in itself" (p. 230).

Steele and Johanning's (2004) PG research with eight US Grade 7 students centered around PG tasks that fit Küchemann's (2010) recommended generic approach. However, while their findings underscore the need for students to build structural schemes as a way of establishing a PG, their data indicate a *sequential (inductive) processing* route in the construction of such structures. For example, 7th grade student Cathy's generalizing scheme, which enabled her to establish her formula $(N \cdot 4) - 4 = \#$ *of squares* for the pattern in Figure 12, started with the 3×3 grid, in which case she counted 8 shaded outside squares. Next she drew a 4×4 grid, a 5×5 grid, followed by a 6×6 grid, and counted the totals in each case. She then made the following claims below based on the diagrams she drew.

In the diagram a 3 x 3 grid of squares is colored so that only outside squares are shaded. This leaves one square on the inside that is not shaded and 8 squares that are shaded. If you had a 25 x 25 grid of squares and only the outside edge of squares are shaded, how many squares would be shaded? If n represents the number of squares on a side and you have all the outside squares of a n x n grid shaded, write an expression representing the total number of shaded squares in the figure.

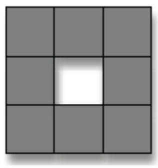


Figure 12. $N \times N$ square grid patterning task (Steele & Johanning, 2004, p. 74)

In a 25×25 grid, 96 squares would be shaded.

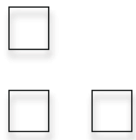
> *I got this because there are 25 squares to a side. But 4 squares, the corner ones, are shared so you don't count them twice. I will use a smaller example to tell what I mean.*
>
> *If you count the corners twice, you get 12. If you don't, you get 8. .... So knowing this information, the formula is* $(N \cdot 4) - 4 =$ *# of squares. You times n by 4 because a square has 4 sides that are all equal. You subtract 4 because the corners are shared.* (Steele & Johanning, 2004, p. 74)

Another possible complication with a generic approach to PG involves a situation called *selective processing*. A few studies conducted with elementary school children show a predilection towards either "impos[ing] a pattern by modifying or ignoring some elements in a given configuration" (Lee & Freiman, 2004, p.

249) or engaging in a narrow and, thus, invalid, specializing on a single case. For example, among the 35 kindergarten Canadian children (ages 5 to 6 years) that Lee and Freiman (2004) interviewed in their PG study, 9 to 12 of them interpreted some growing patterns with given initial stages as repeating sequences. Rivera (2010) also investigated 21 US second grade students' PG ability involving the two related open-ended patterning tasks shown in Figures 13 and 14.

Let us begin with a square and call it step 1.

Now suppose step 2 looks like as shown. How many squares do you see?



A. How might step 3 appear to you? Show me with the blocks.

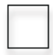
B. Show me steps 4 and 5. How many squares do you see?

C. Pretend we don't have any more blocks and suppose we skip steps. If someone asks you how step 8 looks like, how might you respond? Can you describe or draw for me?

D. If someone asks you how step 10 looks like, how might you respond? Can you describe or draw for me?

Figure 13. Open-ended patterning task for second grade students (Rivera, 2013, p. 132)

The students' pattern extensions in the case of the Figure 14 task specialized on the second step without connecting all the pattern stages to the given first stage, which led them to answer 4, 5, and 6 squares for Stages 3, 4, and 5. The answers had no structure other than the fact that the students' answers conveyed the use of the successor property ("after 3 squares comes 4, then 5, and then 6"). Even when Rivera (2010) modified the Figure 13 task so that instead of two stages they were given three stages in a growing L-shaped pattern shown in Figure 14, the results indicate that most of them still consistently ignored the first two stages and narrowly specialized on the third stage with only 2 out of 21 students producing a reasonable structure for their emerging patterns.

Let us begin with a square and call it step 1. 

Now suppose steps 2 and 3 looks like as shown. How many squares do you see?

Step 2 Step 3

A. How might step 4 appear to you? Show me with the blocks.

B. Show me step 5. How many squares do you see?

C. If someone asks you how step 10 looks like, how might you respond? Can you describe or draw for me?

Figure 14. Modified Figure 13 patterning task for second grade students (Rivera, 2013, p. 132)

Regardless of the initial context in which a PG task is framed, what seems to matter more is how individual learners attend to an emerging structure. The following theoretical framework below, taken from Mason and colleagues (Mason et al., 2009; Watson, 2009), emphasizes at least five different awareness levels or attentional states relative to the discernment, appreciation, and formation of structural generalizations that apply to both sequential and generic PG activity.

- Holding wholes (gazing).
- Discerning details (making distinctions).
- Recognizing relationships (among specific discerned elements).
- Perceiving properties (as generalities which maybe instantiated in specific situations).
- Reasoning on the basis of identified properties.

(Mason, Stephens, & Watson, 2009, p. 11)

Holding wholes involves a certain manner of gazing, producing personally-constructed images (i.e., propositions or diagrams) that need to be analyzed in more detail. At this awareness level, the focus is basically on "foregrounding and backgrounding structures inherent in the object of attention" (Watson, 2009, p. 219). Graphs of functions, for example, can be noticed primarily at the level of their notational forms and overall shape and direction. *Discerning details* shifts the attention towards further analysis and deeper description. Parts are constructed and described in detail depending on meaningfulness. Also, content may focus on parts that either change or remain the same. Referring to the graphs of functions, they may be investigated for symmetry, number and nature of the calculated intercepts, changes in the graphs when coefficients are modified by domain, etc.

Recognizing relationships involves delineating and analyzing varying and invariant relationships more critically than before. Further, constructed (conceptual) relationships may be covarying, correlational, and/or causal along some dimensions. For example, graphs of linear functions can be further categorized by the nature of their slopes, graphs of polynomial functions can be explained in terms of the parity nature of their corresponding degrees, and behaviors of algebraic and transcendental functions can also be assessed for similarities and differences.

Perceiving properties marks the phase of explicating generalities. This phase also enables a further categorization of different objects and "general classes of related objects" (Watson, 2009, p. 220).

Reasoning on the basis of the identified properties is the important phase in which empirical-based reasoning about specific objects and related objects is situated within a deductive reasoning context, which then enables the investigation of what, how, and/or whether other instances may belong to a well-established structure. Following a deductive model, structural understanding proceeds from the experiential to the formal, from generalizing within classes to abstracting that can include the most number of classes, and from the available objects to the generation and construction of "new objects and relations that might never be perceived except in the imagination" (Watson, 2009, p. 220).

Representing Generalizations

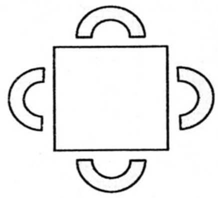
Representing—that is, explicitly communicating, expressing, and conveying—PG in some recognizable medium (or media) range in form and content, from approximate, idiosyncratic, and unstructured expressions to exact, sophisticated, and structured systems that give meaning to constructed expressions. Bastable and Schifter (2008) note that some children tend to "use English, a natural language, to describe relationships that are more frequently expressed with algebraic formalisms" (p. 175). For example, a Grade 4 student in their study articulated the following verbal observation regarding square numbers: "If you take two consecutive numbers, add the lower number and its square to the higher number, you get the higher number's square" (p. 173). While the fourth-grade class inductively verified the student's claim on two near cases (2 and 3; 7 and 8), they also expressed their inference verbally. The authors note a finding drawn from a US second-grade class in which case students expressed their general understanding of square numbers in a verbal format. When the students in that particular class paid particular attention on their construction process, they all expressed their written conclusions verbally (e.g.: "square numbers go odd, even, odd, even;" "if you times a square number by a square number, you get a square number;"

"when you add a row at the bottom and a row to the side and make a corner, you get another square number").

Drawing on their findings with years 7-10 students (ages 12-15 years) in Australia, Stacey and MacGregor (2001) also articulate the importance and necessity of the "verbal description phase" in the "process of recognizing a function and expressing it algebraically" (p. 150). They underscore how some students experienced difficulties in "transitioning from a verbal expression to an algebraic rule," especially those "students with poor English skills" who were either unable to "construct a coherent verbal description" or produce "verbal description[s that] cannot be [conveniently and logically] translated directly to algebra" (MacGregor and Stacey, 1992, pp. 369-370). Bastable and Schifter (2008) point out, however, the "ambiguities of natural language [that] may, at times, cause concern" (p. 175). For example, the following written verbal generalization of a US second grade student regarding square numbers, "take any square number, add two zeros to it, and you will get another square number," actually used the word "add" to convey a sense of "concatenating" rather than adding in an operational sense (p. 175).

Another documented way of representing in PG contexts involves the use of tables. Carraher, Martinez, and Schliemann (2008), for example, initially asked a group of US Grade 3 students (ages 8-9 years) to use the data table of values shown in Figure 15 to help them make sense of the Separated Dinner Tables task and eventually obtain a direct expression for the pattern. The constructed generalizations that the students developed ranged from additive-arithmetical to multiplicative-arithmetical generalizations. Vale and Pimentel (2010) also used tables in assisting their Grade 3 participants from Portugal to obtain PGs. However, they note that tables from patterns that only show totals and primarily encourage students to recursively establish outputs by a process of differencing "do not allow an understanding of the structure of the patterns" (p. 245). Instead, they propose a multi-representational approach that employs words, mathematical language, and a structural generalizing table of values in order to help students focus on relationships that can be translated in variable form as direct expressions.

Handout: Detached Tables



Name: ______________________________

In your restaurant, a maximum of four people can sit at each <u>dinner table</u>.

Fill in the following <u>data table</u>.

If you know the number of tables, figure out the <u>maximum number of people</u> you can seat.

If you already know the number of people, figure out the <u>minimum number of tables</u> you need.

Number of Dinner Tables	Show How	Number of People
1	1 × 4 →	
2	2 × 4 →	
3	→	
4	→	
	←	24
	←	20
	←	11

How many people can you seat at **t** tables? [hint: More than **t** people? Less than t people? Exactly **t** people?]

How many tables do you need to seat **n people**? [hint: More than **n** tables? Less than **n** tables? Exactly **n** tables?]

Figure 15. Separated dinner tables patterning task (Carraher et al., 2008, p. 9)

Recent PG studies by Cooper and Warren (2011) and Britt and Irwin (2011) emphasize the significance of having students either engage in quasi-generalizing or use quasi-variables in which case expressing generalizations involves using "specific numbers and even to an example of any number before they can provide a generalization in language or symbols" (Cooper & Warren, 2011, p. 193) or "thinking of numbers themselves as variables" (Britt & Irwin, 2011, p. 152), respectively. For example, Cooper and Warren's (2011) work with Australian Years 2 to 6 students (ages 6-10 years) on figural growth patterns indicates students' success in establishing quasi-generalizations from tables. They also note that figural-driven quasi-generalizations tend to produce "more equivalent solutions" and "better process generalizations than quasi-generalizations that relied on data tables alone (p. 198). In Britt and Irwin's (2011) study, they note that a significant number of New Zealand Year 8 prealgebra students (mean age of 12 years) who participated in their numeracy project that emphasized algebraic-driven operational strategies seems to be far more successful than their counterparts in a control group on the basis of a test on basic arithmetical tasks that involves transferring relevant knowledge in a variety of situations and examples. The authors' numeracy project primarily focused on "the development of awareness of generality" (Britt & Irwin, 2011, p. 147) by emphasizing both the construction of arithmetical relationships and the use of quasi-variables. Instruction, especially, focused on the many different ways in which arithmetical problems (e.g., find the sum of two whole numbers) can be dealt with by reflectively drawing on their concrete experiences (e.g., with a ten-frame relative to an addition-fact task). Eventually, the reflective experience enabled the students to acquire an understanding of some underlying structure or generalization such as the additive compensation strategy $a+b=(a+c)+(b-c)$ on the basis of their manipulations with quasi-variables. The reported empirical studies of Cooper and Warren (2011) and Britt and Irwin (2011) demonstrate representational changes that tend to occur when PG activity is supported by students' experiences with concrete models and numbers that are employed as quasi-variables, followed by words, and by the use of literal symbols and variables in algebra.

In a series of papers, Radford (1999, 2000, 2001a, 2001b, 2003, 2006) empirically demonstrates the semiotic emergence of direct or closed variable-based expressions in the context of culturally mediated activity. Semiotic emergence refers to ways in which expressions of generalizations come about "in processes of sign use" (Radford, 1999, p. 90) at least initially with others (teachers, students) in joint activity using readily available tools and processes (e.g. shared language, notations, and practices). Results of his longitudinal study with a cohort of 120 Grade 8 students and their six teachers in Canada over the course of three years of classroom research indicate the existence of the following three types of direct expressions in the context of figural PG tasks: factual; contextual; and symbolic. The patterns in Figure 16 show three of several figural PG tasks that Radford used in his studies. The protocol involves asking them to construct near generalizations (e.g., Stages 9 and below) and far generalizations (e.g., Stages 10 and up) and then to convey a generalization for any stage or figure number in the pattern.

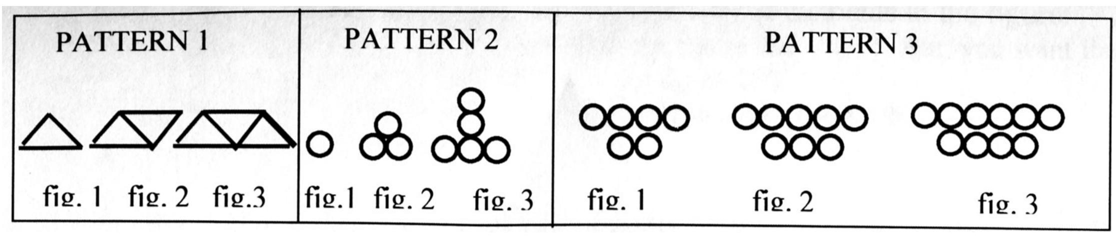


Figure 16. Figural patterns in studies on generalization (Radford, 2000, p. 83)

Radford distinguishes between arithmetical generalization and algebraic generalization. For example, one way of obtaining the total number of toothpicks needed to construct Stage 25 of pattern 1 in Figure 16 involves the painstaking recursive process of counting toothpicks figure after figure up to Stage 25, which exemplifies an arithmetical counting strategy. However, algebraic strategies tend to model more efficient nonrecursive modes of counting because the primary source is often rooted in an interpreted multiplicative structure of a pattern and its parts (i.e., parts are themselves seen as multiplicative units in an emerging structure of the pattern). Having such a structure has the same effect as initially drawing on axioms in order to make sense of resulting propositions relevant to some (sets of) objects and their relationships.

In the context of figural PG, Radford's initial layer of structural generalization is factual, that is, it is "a generalization of numerical actions in the form of an operational scheme that remains bound to the numerical level, nevertheless allowing the students to virtually tackle any particular case successfully" (Radford, 2001b, pp. 82-83). So, for example, one group of Grade 8 students in his study noticed that since the first two stages in pattern 1 of Figure 16 seem to follow the sense "it's always the next… 1 + 2, 2 + 3," that allowed them to impose the factual structure of "25 plus 26" in the case of Stage 25 of the pattern. Here the multiplicative dimension pertains to the two growing composite parts corresponding to the top and bottom rows of circles (versus the additive strategy of counting-all in which case circles are counted one by one and from stage to stage). Factual generalizations are often accompanied by the use of adverbs such as "the next" or "always," including the effects of rhythm of an utterance and movement (e.g., a pointing gesture). While perhaps necessary in the beginning stage of generalizing, unfortunately, factual generalizations remain context-bound and numerical and often draw on shared "implicit agreements and mutual comprehension" (Radford, 2001b, p. 83) among those who construct them in social activity.

In the next structural phase of contextual generalization, reference to particular stages, rhythm, and pointing are all replaced by "linguistic-objectifying" actions that are performed not on a concrete stage but at the abstract level. For example, the terms "add," "the figure," and "the next figure" in reference to the stages in pattern 1 of Figure 16 replace factual actions that depend heavily on context. In contextual generalizations, "the abstract object appears as being objectified through a refined term pointing to a non-materially present concrete object through a discursive move that makes the structure of relevant events visible" (Radford, 2001b, p. 84). However, such abstract objects still remain context-based on the particularities of the relevant concrete objects, hence, the use of the category "abstract deictic objects." While the operational schemes have undergone objectification and the objects have transitioned into their abstract form, however, they are still connected to both the positional features and the individual(s) that made them possible in the first place.

The third, and final, structural phase of symbolic generalization exemplifies what Radford classifies as an algebraic generalization. Contra contextual generalizations, symbolic or algebraic generalizations have overcome their spatial-temporal character, that is, they are "unsituated and temporal" and disembodied or desubjectified objects (Radford, 2001b, p. 86). The use of letters marks an entry into algebraic generalization. For example, in pattern 1 of Figure 16, a student in Radford's study suggested and justified the formula $(n+n)+1$, in which case the student employed the variable n within the context of an impersonal voice that has overcome the spatial, temporal, and positional constraints that characterize such variables in a contextual generalization. Further, desubjectification or disembodiment marks a further phase in algebraic generalization, where symbolic expressions $(n+n)+1$ and $(n+1)+n$ in the context of pattern 1 in Figure 16 are not seen as different but equivalent actions. The full stage, in fact, necessitates a total decentration of such actions, where the dual relation between subject and object is "shattered" and the variables seen as "objects in a different way" and bearing "a different kind of existence" that is not "haunted" by the "phantom of the students' actions" (Radford, 2001b, p. 87).

Radford notes that the entry into, and presence of, variable-based generalizations do not necessarily convey meaningful algebraic generalizations. This claim underscores his much larger view in which the presence and use of letters do not necessarily "amount to doing algebra," that "just as not all symbolization is algebraic, not all patterning activity leads to algebraic thinking" (Radford, 2006, p. 3). There are students, for example, who produce symbolic generalizations on the basis of some "procedural mechanism" following a "trial-and-error" heuristic that they are sometimes unable to explain beyond the response "Uh… because it works!" on the basis of a number of additional extensional generalizations (Radford, 2000, p. 82). Radford (2006) categorizes this process as *naïve induction*. Also, there are those who produce recursively additive formulas (in the form Next = Current + Common Difference, which Radford (2006) classifies as arithmetical generalizations), however, they are unable to use them correctly when confronted with a far generalization task. Even in situations when some students are able to verbalize regularities and talk about the general through the particular, in some cases their underlying understanding of the pattern under investigation appears to remain at either the iconical or indexical level in which case they fail to establish equivalent structures. Suffice it to say, PGs that yield symbolic variable-based generalizations depend significantly on the context in which individual learners understand variables and, more generally, sign use.

We close this particular section by noting recent work that seems to indicate that a student's embodied apprehension of objects in a pattern can also either hinder or support representational understanding (Samson & Schäfer, 2011). Samson (2011) and Rivera and Becker (2011) note the "inherent [and subtle] ambiguities in [the structure of] symbolic expressions of generality" (Samson, 2011, p. 28). In the second year of Rivera and Becker's (2011) three-year study with middle school students involving linear patterns, a number of US Grade 7 students (mean age of 12 years) who acquired more understanding of the commutative property for multiplication interpreted the expressions $a \times b$ and $b \times a$ to be referring to the same grouping of objects, which confused especially those who were generalizing numerically from a data table when they tried to justify their structures. Drawing on their study with a group of Grade 9 students in South Africa, Samson and Schäfer (2011) point out that the variable expression $2(n+1)$ in reference to the pattern in Figure 16 can be interpreted either as two groups of $(n+1)$ segments or as $(n+1)$ copies of two segments. The recommendation, of course, is not to prevent such situations from occurring but to encourage classroom discourse in ways that enable learners to "validate multiple visually mediated interpretations" of such patterns as a consequence of the "always-already" embodied nature of generalizing. Hence, perception in representing generalizations is a central cognitive aspect in any account of structural generalization formation, that is, in Samson and Schäfer's (2011) words, while perception is "critically related to the *manner* of one's interaction with perceptual objects, it also remains sensitive to both the phenomenological and semiotic aspects of the generalization process" (p. 42).

Contexts in Generalizing

Context matters. Ainley et al. (2003) investigated the significance of context and calculations in the representations of PGs among two high school groups of UK students (ages 11 to 12 years) who participated in clinical interviews that were conducted near the end of their first year of schooling. Figure 17 shows the PG task that they analyzed in some detail in their reported study.

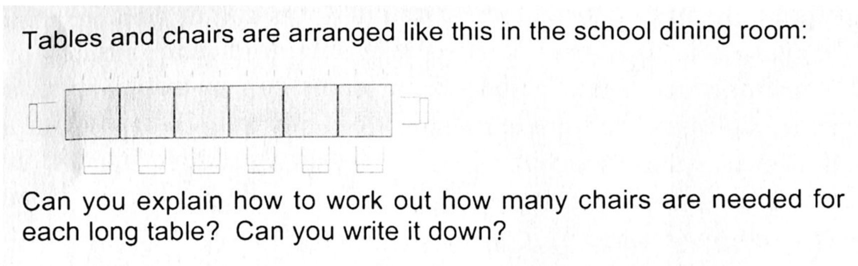


Figure 17. The tables and chairs patterning task (Ainley et al., 2003, p. 11)

The authors found that the students' generalizations fell under two different categories. One set of verbal statements provided general descriptions of an inferred context (e.g., "two on each table except for the ends, which is three," "for every table there's two chairs plus the other two that are on the end"). The other set of verbal statements utilized arithmetical calculations in order to obtain the total number of chairs that were needed for any number of tables (e.g., "just how many tables double that, and then, plus two for the ends," "you take the tables and you times it by two and then plus two"). The author note that students who generalize on context alone might experience difficulty in obtaining a total count because the complexity of the accompanying verbal descriptions might not easily translate into algebraically useful expressions. Further, such cases involving context-based responses tend to be narrowly confined to favored ways of seeing; it is, thus, likely that they will find other possibilities to be confusing and distracting. The authors also note that students who generalize by drawing on both context and calculations are easily able to articulate justifiable direct expressions. Thus, "generaliz[ing a] context is not sufficient to enable students to express relationships [symbolically] in algebra-like notation(s), and that generaliz[ing] the calculations that are required is a significant bridge that [can] support [them] in constructing meaning for a symbolic expression of the [inferred] relationships" (Ainley et al., 2003, p. 13). The authors suggest that any generalizing from context to calculation can be initiated through, say, "adding an element to the task which signals clearly the need to describe a calculation (such as 'Could you tell the caretaker how to work out how many chairs should get out of the storeroom?')" (p. 15).

Ellis (2007) also addressed the issues of context and domain in generalizing in her work with seven US Grade 7 prealgebra students. The students initially explored linear relationships in two everyday situations involving speed and gear ratios. They used physical gears to perform the ratio activity and the SimCalcMathworlds program to make sense of speed. Based on the students' responses and the teacher's prompts in various aspects of the two activities, their PGs underwent a conceptual evolution from thinking about them as number patterns (in discrete terms) to situating them in the context of quantitative relationships (in continuous terms) that eventually supported and "encouraged the development of more powerful general principles related to linearity" (Ellis, 2007, p. 223). For Ellis, asking students to prove, appropriate to their level of proficiency, can be used "to help [them] generalize more effectively, rather than as an act that necessarily follows generalization" (Ellis, 2007, p. 224). Further, Ellis underscores the importance of problem solving situations that "focus on relationships between quantities instead of number patterns or procedures alone" since quantitative relationships are more likely to support powerful and productive PGs than number patterns.

Nathan and Kim (2007) investigated US Grades 6-8 students' performance on discrete and continuous linear functions that were conveyed to them in graphical and verbal contexts. Discrete linear function tasks pertain to problem situations in which individual given instances are explicitly stated and assumed to be related in some way, while continuous linear functions convey problem situations that have a continuous relationship among the relevant quantities. They found that the Grade 6 group was adept at far prediction, the Grade 7 group dealt with both far prediction and formula construction on similar levels of proficiency, and the Grade 8 group produced more correct formulas than far predictions. Further, they saw that while both the Grades 6 and 7 groups employed similar generalizing processes when they dealt with both far prediction and formula construction tasks, the Grade 8 group found formula construction tasks easier to accomplish than calculating far prediction tasks in both discrete and continuous contexts. The authors noted as well that across grade level the students were more successful in situations when function tasks appeared to them in continuous form (i.e., as verbal rules and line graphs) over situations when similar tasks were presented in discrete form (i.e., in the form of a list of cases and point-wise graphs). Their findings seem to suggest that when students are provided with both verbal and graphical representations for representing linear function contexts, then they may achieve better PGs in comparison with contexts that show either representation alone.

Conclusion: Towards a Distributed View of Pattern Generalization Processing

Performing PG on a single task involves tapping a complex set of predictable and unexpected factors that influences the manner in which interpreted structures are constructed, expressed, and justified. Each time learners such as Skype in the introduction are confronted with a PG task, they will always find themselves navigating through different components of the conceptual framework shown in Figure 8, and any route (i.e., sequential or narrowly connected paths) is never smoothly linear but graded due to learning (i.e. more training) and experience. If there is one converging observation that can be inferred from the studies discussed and highlighted in this article, it may very well be the case that the difference between an easy or familiar PG task and a nonroutine or unfamiliar one is fundamentally a matter of complex routes of connectedness that individual learners construct and reconstruct over time. Multiple entry points in PG are possible, which significantly and ultimately depend on individual learners' (semiotic) contexts and other factors that help them build structures that they consider meaningful and appropriate. Instruction, of course, is a type of cultural mediation that can be employed to make PG processing more equitable for all learners.

Complications in PG ability happen because any interpreted structure is basically a function of its emergence in an individual learner's complex system, from his or her perceptions of objects in a pattern to apprehensions and meanings that he or she infers on the limited external representations that often characterize PG activity. Hence, the primary issue in PG activity is not about whether students can generalize but how to design and sustain learning experiences that enable them to have the agility to choose sufficiently optimal connections between and among factors that bear on PG processing. Hence, a distributed view of PG situates the emergence of an interpreted structure in terms of "cooperative and competitive" (Plaut, McClelland, Seindenberg, & Patterson, 1996, p. 56) factors. When individual learners appear to exhibit stable PG processing on a few tasks, it means that the "weights on connection between" factors are strong. However, adjustments and changes always occur based on "the statistical structure of the environment [that] influences the behavior of the network" (Plaut et al., 1996, p. 56) of factors that influence PG processing. Further, the following point below made by Plaut et al. (1996) applies to PG processing if we think of "items" in terms of "factors."

> *[T]here is no sharp dichotomy between the items that obey the rules and the items that don't. Rather, all items coexists within a single system whose representations and processing reflect the relative degree of consistency in the mappings for different items.* (Plaut et al., 1996, p. 56)

Hence, consistency is a central feature in PG processing, which depends on several factors and ultimately on choices that an individual learner makes each time he or she is confronted with a PG task.

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