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A Model-Theoretical Generalization of Steinitz's Theorem

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Abstract. Infinitary languages are used to prove that any strong isomorphism of substructures of isomorphic structures can be extended to an isomorphism of the structures. If the structures are models of a theory that has quantifier elimination, any isomorphism of substructures is strong. This theorem is a partial generalization of Steinitz's theorem for algebraically closed fields and has as special case the analogous theorem for differentially closed fields. In this note, we announce results which will be proved elsewhere.

Keywords: Strong isomorphism; infinitary languages; isomorphism extension; quantifier elimination.

Dedicated to Newton C.A. da Costa
on his 80th birthday, 2010, December.

Strong isomorphisms

Let $\mathscr{L}$ be a language and let $\mathscr{E}_1$ be an $\mathscr{L}$-structure defined on a domain D_1. Given two infinite cardinals $\alpha, \beta, \beta \leq \alpha$, and assuming that α is a regular cardinal, we shall denote by $\mathscr{L}_{\alpha\beta}$ the infinitary language which has the same symbols as $\mathscr{L}$ and which allows sequences of conjunctions of less than α formulas and sequences of instantiations of less than β variables. We consider $\mathscr{E}_1$ also as a $\mathscr{L}_{\alpha\beta}$-structure and call $\mathscr{L}_{\alpha\beta}$ the extended language of $\mathscr{L}$.

Consider a second $\mathscr{L}$-structure $\mathscr{E}_2$ defined on a domain D_2 and let $\mathscr{H}_i$ be $\mathscr{L}$-substructures of $\mathscr{E}_i$, defined on the domains $F_i \subset D_i, i = 1,2$. The cardinals α, β being fixed, we shall denote by $[\varphi]_i \subset D_i^\gamma$ the relation defined by a formula φ of the language $\mathscr{L}_{\alpha\beta}$, of arity γ, in the $\mathscr{L}_{\alpha\beta}$-structure $\mathscr{E}_i, i = 1,2$. For the definition of relation defined by a formula of an infinitary language see, for instance, Rodrigues et al. 2010.

Definition 1. An isomorphism $h : F_1 \to F_2$ is *$\mathscr{L}_{\alpha\beta}$-strong* if for any relation $R \subset D_1^\gamma$, which is defined by a formula φ of arity $\gamma < \alpha$ of the language $\mathscr{L}_{\alpha\beta}$ we have

$$h^\gamma([\varphi]_1 \cap F_1^\gamma) = [\varphi]_2 \cap F_2^\gamma,$$

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where $h^\gamma : F_1^\gamma \to F_2^\gamma$ is the extension of the map h to the power γ of the domain F_1. If h is $\mathscr{L}_{\omega\omega}$-strong, we say simply that h is a *strong isomorphism*.

Theorem 2. *Let $h : F_1 \to F_2$ be a strong isomorphism of the substructures $\mathscr{H}_1$ and $\mathscr{H}_2$. Then, h is $\mathscr{L}_{\alpha\beta}$-strong for any choice of α and β.*

Proof. (Sketch). The proof is by induction on definition of formulas. If φ is an atomic formula, φ is finitary and the result holds by hypothesis. For negation and conjunction clauses the proves are straightforward. The only non trivial case is the instantiation which is proved by transfinite induction on the arity of formulas. □

If $h : F_1 \to F_2$ admits an extension $\tilde{h} : D_1 \to D_2$ which is an isomorphism of the structures $\mathscr{E}_1$ and $\mathscr{E}_2$, then $\tilde{h}^\gamma([\varphi]_1) = [\varphi]_2$, implying that h is $\mathscr{L}_{\alpha\beta}$-strong. Theorem 3 below shows that if α and β are large enough, the converse of this statement holds. For the main ideas involved in the proof of theorem 3, see Sebastião e Silva 1985, fundamental theorem I, p. 109, and da Costa & Rodrigues 2007, theorem 7.1, p. 23. The notion of strong isomorphisms seems due to Sebastião e Silva.

In the theorem below, d^+ denotes the first cardinal greater than the cardinal d of D_1.

Theorem 3. *Assume that the $\mathscr{L}$-structures $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic and let $h : F_1 \to F_2$ be an $\mathscr{L}_{d^+d^+}$-strong isomorphism of $\mathscr{H}_1$ and $\mathscr{H}_2$. Then, there exists an isomorphism $\tilde{h} : D_1 \to D_2$ of $\mathscr{E}_1$ and $\mathscr{E}_2$ which extends h, that is, $\tilde{h} \restriction F_1 = h$.*

Proof. (Sketch). Let $\tilde{g} : D_1 \to D_2$ an isomorphism of $\mathscr{E}_1$ and $\mathscr{E}_2$. Consider an enumeration $p : \lambda \to F_1$ of F_1 and prolong p to $q : d \to D_1$ an enumeration of D_1, i.e., $\lambda < d$ and $q(i) = p(i)$ for $i \in \lambda$. Let G be the automorphism group of $\mathscr{E}_1$. Let ρ and $\tilde{\rho}$ be the G-orbit of p and q, respectively. Since that in $\mathscr{L}_{d^+d^+}$ every G-orbit of arity less than d^+ is definable (see, for instance, Rodrigues et al. 2010, theorem 3.7, p 127), there are formulas φ and $\tilde{\varphi}$ defining ρ and $\tilde{\rho}$, i.e., $\rho = [\varphi]_1$ and $\tilde{\rho} = [\tilde{\varphi}]_1$. We may assume that the free variables of φ and $\tilde{\varphi}$ are $Var(\phi) = \{x_i : i \in \lambda\}$ and $Var(\tilde{\phi}) = \{x_i : i \in d\}$. Now, consider the formula

$$\psi((x_i)_{i\in\lambda}) := \exists\eta(\tilde{\varphi} \wedge \varphi).$$

where η is the sequence of variables $\eta : i \in d - \lambda \mapsto x_i \in Var$. It is easy to see that $p \in [\psi]_1$ and then $\rho \subset [\psi]_1$. Since h is $\mathscr{L}_{d^+d^+}$-strong, then

$$h^\lambda([\psi]_1 \cap F_1^\lambda) = [\psi]_2 \cap F_2^\lambda.$$

As $p \in [\psi]_1 \cap F_1^\lambda$, we have $h^\lambda(p) \in [\psi]_2 \cap F_2^\lambda$. Therefore, $h^\lambda(p) \in [\psi]_2$ and then for every $i \in d - \lambda$, there exists $c_i \in D_2$ such that $((h(p_i))_{i\in\lambda}, (c_i)_{i\in d-\lambda}) \in [\tilde{\varphi}]_2$. Since $[\tilde{\varphi}]_2$ is the orbit of $\tilde{g} \circ q$ in $\mathscr{E}_2$, there exists $g \in G$ such that $\tilde{g} \circ g \circ q = ((h(p_i))_{i\in\lambda}, (c_i)_{i\in d-\lambda})$. We define $\tilde{h} = \tilde{g} \circ g$ and it is easy to see that $\tilde{h}$ is an isomorphism of $\mathscr{E}_1$ and $\mathscr{E}_2$ such that $\tilde{h} \restriction F_1 = h$. □

Theorem 4 follows from theorems 2 and 3.

Theorem 4. *If the $\mathscr{L}$-structures $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic, any strong isomorphism $h : F_1 \to F_2$ of two substructures $\mathscr{H}_1$ and $\mathscr{H}_2$ can be extended to an isomorphism $\tilde{h} : D_1 \to D_2$ of $\mathscr{E}_1$ and $\mathscr{E}_2$.*

Quantifier elimination and Extensions of Isomorphisms

Theorem 4 simplifies when applied to models of a theory that has quantifier elimination.

Theorem 5. *If $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic model of an $\mathscr{L}$-theory that has quantifier elimination, any isomorphism of substructures $\mathscr{H}_1$ of $\mathscr{E}_1$ and $\mathscr{H}_2$ of $\mathscr{E}_2$ is a strong isomorphism.*

Proof. (Sketch). By induction on the definition of formulas. Let $h : F_1 \to F_2$ be an isomorphism of substructures $\mathscr{H}_1$ of $\mathscr{E}_1$ and $\mathscr{H}_2$ of $\mathscr{E}_2$. If φ is atomic we have the result, for isomorphisms preserve primitive relations of the structures involved. If φ is a negation or a conjunction and the arity of φ is n, then the result follows from the fact that h^n preserves boolean operations. If φ is an instantiation, as the $\mathscr{L}$-theory has quantifier elimination, the result follows from the same reason. □

The following theorem is an immediate consequence of theorems 4 and 5.

Theorem 6. *If $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic model of an $\mathscr{L}$-theory that has quantifier elimination, any isomorphism of substructures $\mathscr{H}_1$ of $\mathscr{E}_1$ and $\mathscr{H}_2$ of $\mathscr{E}_2$ admits an extension to an isomorphism of $\mathscr{E}_1$ and $\mathscr{E}_2$.*

Theorem 6 is a model-theoretical generalization of well known theorems of the theories of algebraically closed fields and differentially closed fields of fixed characteristic.

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Resumo. Linguagens infinitárias são utilizadas para provar que qualquer isomorfismo forte de subestruturas de estruturas isomorfas pode ser estendido para um isomorfismo das estruturas. Se as estruturas são modelos de teorias que admitem eliminação de quantificadores, qualquer isomorfismo de subestruturas é forte. Este teorema é uma generalização parcial do teorema de Steinitz para corpos algebricamente fechados e tem como caso especial o teorema análogo para os corpos diferencialmente fechados. Nesta nota, anunciamos resultados que serão demonstrados em um trabalho posterior.

Palavras-chave: Isomorfismo forte; linguagens infinitárias; extensão de isomorfismo; eliminação de quantificadores.