

ON THE CONCEPTS OF FUNCTION AND DEPENDENCE

ANDRÉ BAZZONI

Abstract. This paper briefly traces the evolution of the function concept until its modern set-theoretic definition, and then investigates its relationship to the pre-formal notion of variable dependence. I shall argue that the common association of pre-formal dependence with the modern function concept is misconceived, and that two different notions of dependence are actually involved in the classic and the modern viewpoints, namely effective and functional dependence. The former contains the latter, and seems to conform more to our pre-formal conception of dependence. The idea of effective dependence is further investigated in connection with the notions of function content and intensionality. Finally, the relevance of the distinction between the two kinds of dependence to mathematical practice is considered.

Keywords: Function concept; variable dependence; philosophy of mathematics; history of mathematics.

The notion of function in mathematics, like many others, evolved from particular concrete examples to a general abstract definition that eventually raised them to the status of mathematical objects.

Also, the central place of the function concept in mathematics as a whole seems to be universally acknowledged; suffice it to say with Kleiner (1989, p.282) that it “is one of the distinguishing features of ‘modern’ as against ‘classical’ mathematics.”¹

Ancient mathematicians such as Archimedes were already dealing with geometrical problems that would later contribute to the development of the modern idea of a function as an arbitrary correspondence. However, the formal similarities underlying particular methods were not yet sufficiently emphasized (cf. Gardiner 1982). For that matter, it is widely agreed that the turning point in the evolutionary process was the rise of Newton and Leibniz’s calculi in the 17th-century. This will be, therefore, our chronological starting point.

The main object of this paper is an investigation on the relationship between the function concept as it was gradually developed throughout its historic evolution, and the pre-formal notion of dependence—especially variable dependence.

It seems reasonable to suggest (cf. e.g. Bos 1980; Kleiner 1989) that an important step towards the isolation of functions as autonomous mathematical objects was accomplished by shifting attention from the incidental role of variables in algebraic expressions to more essential and general many-one correspondence patterns.

Since the ‘algebraic era’, the relationship between the variables of a function has been more or less explicitly understood as a relation of dependence. According to

Principia 19(1): 1–15 (2015).

Published by NEL — Epistemology and Logic Research Group, Federal University of Santa Catarina (UFSC), Brazil.

such a perspective, the expression “ y is a function of x ” may be explained by saying that the value of y depends on the value of x .

History further brought forth the events that culminated in the modern set-theoretic formulation of the function concept, but the association with the pre-formal idea of dependence survived, if not as an explicit keyword in the formal definitions, at least as a fair description of what the nature of functions really amounts to.

I shall attempt to show that such an association is misconceived, and that modern functions are *not* essentially related to variable dependence.

To this end, two notions of dependence will be distinguished: *effective* dependence, and *functional* dependence. The former is intrinsic to the classic conception, but it is lost in the modern standpoint. The latter is almost a circular one—it is defined according to the modern formulation of the function concept, and is then said to characterize it. I shall argue that the pre-formal notion of dependence is best understood as effective dependence, and that functional dependence does not correspond to the way we typically think of variable dependence.

We are then naturally led to a further examination of the deep mechanics of the effective dependence relation. We shall see that one interesting way of understanding effective dependence is through the notion of intensionality, as opposed to the extensionality involved in the set-theoretic definition of function. This is related to a discussion brought forth by Shapiro on the notion of pre-formal effectiveness.

I shall conclude by pointing out that, mathematically, there seems to be no trouble in associating the notions of function and dependence, for mathematicians deal primarily with functions, hence they can inadvertently account for both kinds of dependence according to specific practical needs. However, if dependence is taken primarily, it would be advisable to carefully examine which notion of dependence is at stake before going on to formalize it through mathematical functions.

1. The mathematical concept of function: from the concrete to the abstract

As it happened with other vital concepts, functions were in the beginning implicit objects gradually emerging from explicit practices.

The multiplication of curves and formulas in mathematical experience paved the way little by little toward the question of what the general structure of such objects was, regardless of their particular presentations. And from a more global perspective, it seems that this is precisely how we reached the famous foundational era that characterized mathematics from the 19th-century up to the present days: after an enormous accumulation of methods and techniques, the need was felt to systematize the field and put it on more solid grounds.

The geometrical matters that motivated the development of the infinitesimal calculus—namely those related to finding certain tangents and areas—focused on the notion of geometrical variables rather than on functions. The use of the word ‘function’ is credited to Leibniz, but in his work variables actually range over geometrical objects: “a tangent is a function of a curve” Struick (1969, p.272; cf. Kleiner 1989). The immense success of the techniques of the infinitesimal calculus is probably the reason why further investigations in the nature of such variables and their general mutual relationship (and in the formulation of those problems in algebraic terms, cf. Gardiner 1982; Kleiner 1989) were eventually postponed.

The famous notation $f(x)$ seems to come from the works of Clairaut and Euler around 1734 (see Eves 1983, p.154). Euler in 1748 is by the way one of the first (Johann Bernoulli before him in 1718) to have formulated the function concept in terms of an arbitrary (and no longer specific algebraic) correspondence between variables and constants (Rüthing 1984):

A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.

This definition already belongs to the algebraic context that superseded the geometrical standpoint, as indicated by Euler’s phrase “analytical expression.”

On the other hand, even though variables ceased to range over geometrical objects and occurred now in specific algebraic formulas, they still prevailed (over some more abstract notion of functional correspondence between variables) as the basic elements of investigation. Yet formulas suited more to the unveiling of general frames of relations between variables than the old geometrical objects, and constituted thus an important step in the evolutionary process.

Indeed, the evolution was effervescent by that time. In 1749 (cf. Lützen 2003, p.469), the same Euler held that “the truth of the differential calculus is based on the generality of the rules it includes,” illustrating the increasing focus on general patterns; and in 1755 he put forward the first definition (cf. Rüthing 1984) of a function in terms of dependence:

If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities.

It was by no accident that he now dropped reference to analytical expressions altogether: the general relation between variables was eventually emerging from the particular algebraic instances.

Dirichlet is often credited the first definition of a function in terms of an arbitrary correspondence. This definition, dated from 1837 (see Lützen 2003), also mentions explicitly the notion of dependence:

If every x gives a unique y in such a way that when x runs continuously through the interval from a to b then $y = f(x)$ varies little by little, then y is called a continuous function of x in this interval. It is not necessary that y depends on x according to the same law in the entire interval. One does not even need to think of a dependence that can be expressed through mathematical operations.

The historical evolution of the function concept thus pointed towards a general tendency to abstract away from the role of particular algebraic expressions. In this connection, Dirichlet himself had already proposed in 1829 his well-known *Dirichlet function*, “the first explicitly stated function that was not given through one or several analytic expressions” (Lützen 2003, p.472).

We can see from the statements above that, as formal similarities between algebraic expressions emerged to the mathematical consciousness, the abstract correspondence between variables suggested itself as an autonomous object of investigation. The pre-formal notion of dependence, which was already present in the algebraic framework, carried over to the incipient set-theoretic function concept as a natural characterization of that kind of abstract correspondence between variables.

The function concept changed its skin, whereas the pre-formal notion of dependence remained preserved in its flesh. But how deep was the latter rooted in the former?

As it happens, not every explicit definition of a function is written down in terms of dependence. The essential component of such a shift to the abstract that characterized the transition to the ‘set-theoretic era’ was in fact the idea of an arbitrary many-one correspondence. Fourier’s 1822 definition does not mention dependence:

In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x , there are an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity.

At the same time, dependence was still thought of as the main intuitive idea behind the abstract functional correspondence, and this seems to be the case right up to our present day. For example, in an article from 2003 Lützen presents Fourier’s definition (Lützen 2003, p.473) by saying that he “consistently insisted that functions were given as a dependence between two variables.”

Dirichlet’s definition is essentially what is now commonly accepted as the modern, formal set-theoretic conception of function² in terms of a set of ordered pairs. The following is taken from MacLane (1986, p.129):

A function f on the set X to the set Y is a set $S \subset X \times Y$ of ordered pairs which to each $x \in X$ contains exactly one ordered pair $\langle x, y \rangle$ with first component x . The second component of this pair is the value of the function f at the argument x , written $f(x)$. We call X the *domain* and Y the *codomain* of the function f .

Such a function is typically noted $f : X \rightarrow Y$, or alternatively $f : x \mapsto y$.

MacLane further observes that this “provides a formal definition which, in plausible ways, does match the intent of the various preformal descriptions of a function. Specifically, it does provide y , depending on x .” This is again an example of a recent author who understands that pre-formal dependence is duly captured by the modern function concept.

The pre-formal notion of dependence has little place now in rigorous definitions of a function, but it still remains as a backstage intuition on the role of functions as arbitrary many-one correspondences between sets.

The following question arises: Is the pre-formal notion of dependence really preserved under the transition from the concrete to the abstract conceptions of function?

2. Effective dependence

Let us take two arbitrary variables x and y , and consider the following conditions: (i) all other possible parameters being fixed, for some variation in the value of x , we observe a variation also in the value of y —or in more colloquial words, y is not always indifferent to x —; and (ii) no variation of y is possible without a corresponding variation of x —this is the standard many-oneness condition on functions. Let us call the conjunction of (i) and (ii) *effective dependence*.

The many-oneness condition seems indisputable. The point, however, is that condition (i) is likewise reasonable: assuming the many-oneness condition, if (all other possible parameters being fixed) the value of y remains constant as we freely vary the value of x , would we be willing to say that y depends on x ? If one apple costs one dollar regardless of the weight of the apple, would we be willing to say that the price depends on the weight of the apple?

2.1. Effective dependence and algebraic functions

Do functions as algebraic expressions feature effective dependence?

To begin with, let us notice that effective dependence is explicitly stated in Euler’s 1755 definition quoted above, in an informal way: he explained “ y function of x ” in terms of the variation in the value of y under variation of the value of x . Dirichlet’s definition is similar in that regard. The function concept was thus intimately related

to effective dependence by the times of its algebraic formulation and the transition to the abstract conception.

For y to be a function of x , one required (besides the many-oneness condition) that some change in the value of x produced a change in that of y . This seems to be the general case as regards algebraic expressions, indeed.

We could try to conceive of a counterexample, that is, of some algebraic expression containing a variable x such that the value of the expression undergoes no variation, no matter how we make the value of x vary. This amounts to finding some algebraic expression featuring a vacuous, redundant variable of which it is a function.

It is true that, strictly speaking, counterexamples of that kind can be easily produced through the explicit notation of eliminable variables. Consider for instance the following formula:

$$(1) \quad y = x(2x) \cdot \frac{z^3}{x^2}$$

According to the Euler-Dirichlet definition of a function, y is a function of z only, but not of x , because variations in the value of z produce variations in the value of y , whereas *no* variation in the value of x (for any fixed z) produces *any* variation in the value of y whatsoever.

Of course, algebraic expressions account for vacuous variables through the standard rules of variable elimination. Thus in (1) x actually cancels out, finally yielding:

$$(2) \quad y = 2z^3$$

Now it is noteworthy that the algebraic form of y was crucial to the elimination of the redundant variable of (1). So what about the set-theoretic conception of function, in which algebraic forms are abstracted away?

2.2. Effective dependence and correspondence

Let us note the function related to (1) as $f : (x, z) \mapsto y$, thus abstracting away from the algebraic form and adopting the arbitrary correspondence standpoint.

We may alternatively define the associated function $f : \mathbb{R}^* \times \mathbb{R} \rightarrow \mathbb{R}$ in the usual set-theoretic way in terms of a set of triples of the general form $\langle x, z, y \rangle$. In this way if we fix $z := 2$, we have for $x := 1$ the triple $\langle 1, 2, 16 \rangle$; for $x := 10^{10}$ the triple $\langle 10^{10}, 2, 16 \rangle$; and for $x := \pi\sqrt{2}$ the triple $\langle \pi\sqrt{2}, 2, 16 \rangle$; and in general for any arbitrary $x \in \mathbb{R}^*$, we obtain the triple $\langle x, 2, 16 \rangle$.

The upshot is that now we have a well-defined function f of variables x, z such that f does not depend on x in the sense of effective dependence. Now given its set-theoretic definition only, there is no way of eliminating x , because there is no rule, let alone algebraic expression, on which one could base the elimination.

To put it differently, the set-theoretic definition abstracts away from algebraic expressions, and with it, from the algebraic method of elimination of redundant variables. As a consequence, the effective notion of dependence is left out of the evolution step leading from the concrete to the abstract concepts of function.

3. Functional dependence

One might just turn the tables and hold that what the set-theoretic definition shows after all, is that effective dependence is not in general the right type of pre-formal dependence described by the function concept, and that it was only the limited algebraic viewpoint that suggested pre-formal dependence as effective dependence.

In fact (to push the reasoning), as mentioned above, dependence was a secondary notion even with respect to the algebraic perspective, and not every algebraic-like definition of function was formulated in terms of dependence. The underlying key idea is in fact correspondence, and as it stands, correspondence does not entail dependence: to take again the example of $y = f(x, z)$, what the set-theoretic definition states is a many-one correspondence between the values of x, z , and the value of y , but it may very well be the case that the values of x , although they do determine (together with the values of z) the values of y , do not make any difference in this determination procedure.

The core notion of correspondence (so the argument goes) had to wait for the modern function concept to eventually emerge to the surface. In the end, therefore, we should define pre-formal dependence in terms of many-one correspondence rather than effective dependence.

Let us dub this version of dependence *functional dependence*.³

This is a fairly literal move: to say that y depends on x and z means that there exists a many-one correspondence between the values of x and z on the one hand, and the values of y on the other hand, without thereby implying that the values of x and z cannot be redundant in such a correspondence.

Since effective dependence contains this condition as (i) above, it follows that effective dependence implies functional dependence.

3.1. Dependence and many-oneness

I suggest, however, that the pre-formal notion of dependence is in fact effective dependence.

Evidence is already provided by the informal explanations given by mathematicians throughout the history of the function concept, as illustrated by the passages quoted in section 1 above. The condition stated in those passages that some change

in the value of x affect also the value of y is not fulfilled by functional dependence, because the latter allows for redundant variables.

Consider again (1). Just as we eliminated x in that expression, we might as well introduce an arbitrary number of redundant variables, for example as follows:

$$y = x(2x) \cdot \frac{z^3}{x^2} + u - \frac{\cos(v)}{w/\tan(v)} \left[\sin(v) \cdot \frac{w}{1 - \cos^2(v)} \right] - \frac{u^2}{\ln e^u} + \lim_{x \rightarrow +\infty} \left(1 - \frac{\pi}{1+x} \right).$$

This can be written as $f : (x, z, u, v, w) \mapsto y$ according to the set-theoretic standpoint, but its algebraic form is again reducible to (2) through variable elimination.

If we take a more careful look at the many-oneness condition, we quickly realize that the fact that equal values of x are associated with different values of y does not entail that no many-one correspondence involving x and y is possible altogether. In other words, the negation of functional dependence merely states that y is not functionally dependent *only* on x , but it is (almost) always possible to associate x and y in a many-one correspondence, provided other variables are involved in such a correspondence.

For example, from the two xy -series (i.e., series of values of the variables x and y , respectively) $\langle 0, 0 \rangle$ and $\langle 0, 1 \rangle$, we can infer that y does not functionally depend *only* on x , but we cannot infer that y does not functionally depend on x *simpliciter*. Indeed, we could expand our series so as to include information about z , such that we obtain, say, the two xzy -series $\langle 0, 0, 0 \rangle$ and $\langle 0, 1, 1 \rangle$. Now y functionally depends on x *and* z , relative to the expanded series.

It is not plainly trivial to make some y functionally depend on some x (when y does not functionally depend only on x) by such a method of series expansion, because in so proceeding we must include at least one variable such that it varies when y does, as illustrated by the series related to the variation of z above. On the other hand, given the weak assumption that y functionally depends on something (whatever it may be),⁴ then we can always make it functionally depend on that something, *and* x . Therefore, given this weak assumption, we can make y functionally depend on any variables we please.

I might say that my salary is determined by my academic position, *and* the speed of expansion of the universe, *and* the world population, *and* the price of the yen, and so on indefinitely, while my salary actually varies only when my academic position does. Yet according to the functional perspective, we would still say that my salary depends on all the above (among an *infinite* number of other) parameters.

This does not seem to be the way we understand dependence, though. Whenever I add a vacuous variable to the relevant correspondence, I add nothing to the information of how the values of the function vary. This information, however, seems to be crucial to our pre-formal idea of dependence.

To take another hypothetical daily-life example, functional dependence would lead us to conclude, from the observation that anytime a tsunami occurs people are found dead, that human death is dependent on the existence of tsunamis—indeed, what is required by some many-one correspondence from x to y is that for any equal values of x , the respective values of y be the same. Now suppose that one day people are found dead, although no tsunami occurred. Then human death would still functionally depend on tsunamis.

The point is thus: intuitively, for y to depend on x , we should observe what happens not only when x is given the same value, but also when it is given different ones. This is effective dependence, which takes care of the many-oneness condition and meets at the same time the non-redundancy requirement.

Let us now summarize the discussion so far. If we want to describe the modern set-theoretic function in terms of dependence, we should adopt functional rather than effective dependence. However, functional dependence is not pre-formal dependence, and if so, there is no reason to stick to the notion of dependence to explain the modern function concept. As a matter of fact, we only called ‘functional dependence’ the many-oneness correspondence between variables to keep using ‘dependence’ at the terminological level. Once functional dependence is seen not to correspond to pre-formal dependence, mere terminology does not really matter. Therefore, it would be preferable to call it just ‘many-one correspondence’, and leave (pre-formal effective) dependence to the algebraic standpoint.

Contrary to what most authors throughout the history of the function concept let it be understood, the notion of dependence is *not* essential to the set-theoretic approach.

4. Function and content

What are the intrinsic differences between the two kinds of functions, algebraic and set-theoretic?

One of the powers of the abstract set-theoretic conception is that it allows for the definition of a number of particular functions that were, if not undefinable, at least much less easily accessible in the old algebraic conception (e.g. Dirichlet’s function defined piecewise in a non-algebraic way).

On the other hand, it seems that no function in the mathematical practice includes any redundant variable at all. If this is right, as we have seen, the generality of the set-theoretic conception leaves out an important practical feature of the notion of function. What exactly is this feature?

When a certain kind of objects is singled out, the next theoretic stage consists typically in setting identity criteria for objects of that kind.

On which grounds should we say that two functions are identical? We have well-defined identity criteria for set-theoretic functions, according to which two functions are the same if and only if they have the same domain, the same codomain, and the same set of ordered pairs. Consequently, functions algebraically reducible to one and the same function do not describe set-theoretically the same function, if only because they have different domains.

Such a question does not so literally arise from the point of view of algebraic functions. As we have seen, if one expression is reducible to another through variable elimination, then the two expressions are viewed as equivalent. In this way, all of the three expressions for y above are considered, if not strictly speaking the same expression, at least in some sense equivalent. There is a clear intimate connection between two functions that are reducible to the same function in the algebraic sense, and as we have already observed, this connection is left out of the set-theoretic perspective.

Given the (equivalence) class $F = \{f_1^{(n_1)}, \dots, f_k^{(n_k)}\}$ of all equivalent (in the sense above) n_i -ary functions ($0 < i \leq k$), is it not the case that each $f_i \in F$ somewhat conveys a common *content*? In other words, if some $f \in F$ has redundant arguments in its set-theoretic definition, it conveys in some suitable sense the same mathematical content as the function $\bar{f} \in F$ defined exactly as f except for the fact that its domain does not contain the redundant arguments of f (and accordingly for the set of ordered pairs).

If so, what is to count as the common content of all the members of F ?

One natural answer is the $f^{(n)} \in F$ such that $n = \min \{i : f^{(i)} \in F\}$, that is, the function in F with the least arity; or in still other terms, the unique irreducible function of F . Such a function exists by well-ordering of \mathbb{N} (“every nonempty subset of \mathbb{N} has a smallest element”); it is irreducible, for otherwise there would be another function in F with lesser arity; and finally it is unique, for otherwise we would have two different functions defined in the same domain and codomain and with the same ordered pairs (by their equivalence). Thus we may speak of *the set-theoretic reduction* \bar{f} of an f .

To state it briefly, the content of an f is its set-theoretic reduction \bar{f} , whose arguments are variables that \bar{f} effectively depends upon. Effective dependence is thus brought back to the picture through the notion of function content.

Rigorously, effective dependence is not inherent to the algebraic definition of a function, for the latter only states that variable correspondence is expressible through algebraic expressions. The many-oneness requirement is intrinsically met, but not the non-redundancy condition, as shown by (1) above. In practice, however, algebraic expressions make possible the application of the standard variable elimination rules, whereby the explicit expression of the relevant function content may be obtained.

4.1. Effectiveness and intensionality

Now since these rules are practical in nature and, as such, depend on the mathematician's ability for algebraic manipulations, there seems to be an *intensional*, or *epistemic* aspect crucially involved in the notion of function content.

Shapiro (2006) (cf. also Shapiro 1985) makes a similar point with respect to the notion of effectiveness as related to computability. He observes that the standard notion of a computable function has an extensional emphasis, whereas the pre-formal idea of effectiveness seems to involve a crucial intensional component: it would be of limited interest to establish that some function is computable, if we cannot *know* how to compute such a function. It then follows that its set-theoretic definition (its extension) is of no help once its domain is infinite, because then in order to know that the function is computable, we would have to define it in terms of other (computable) functions, that is, through some particular presentation of the originally set-theoretically defined function.

MacLane (1986) p.127 also writes that “[f]ormulas’ depend on the symbolism, but functions depend upon the facts,” hence formulas depend on the *knowledge* of the symbolism. For a mathematician to operate a variable elimination in some function, (s)he must be in possession, in the general case, of an algebraic expression for that function. In the general case, because in the particular case in which the domain is finite, set-theoretic reduction, though less practical—it might be not easily perceptible ‘with the naked eye’—than algebraic reduction, *is* in principle feasible: it suffices to conceive of an algorithm that goes along through every argument value of the function inspecting whether there is some of those arguments for which, fixing all the others, the value of the function remains constant. Whenever the answer is positive, this argument is eliminated and the function is redefined for the remaining ones. However, as in the computability case pointed out by Shapiro, such a reduction method would be of no help when applied to infinite domains.

The analogy between effective dependence and effective (that is, pre-formal) computability is a remarkable one. For some set-theoretically defined f with finite domain, both effective computability and effective dependence are algorithmically solvable. For infinite domains, both involve knowledge of some other algebraic presentation of f : in the former case, an algebraic redefinition of f in terms of other effectively computable functions; in the latter, an algebraic reduction of f .

Shapiro's words below (2006, p.45) seem equally applicable to both senses of effectiveness:

[...] the pre-formal notion of effectiveness is pragmatic, or epistemic. It is not a property of sets or functions themselves, independent of the way they are presented.

[...] the pre-formal notion of effectiveness is an intensional concept. It is

not a property of sets or functions themselves. Perhaps effectiveness can be thought of as a property of *presentations*, interpreted linguistic entities that denote sets and functions.

The notion of function content is left at an informal level here for reasons of space and scope, but it seems a promising line for future research. We might ask for example what the relation is, if any, between the notions of function content and function intension, as contrasted to the set-theoretic extensional function. Shapiro's aforementioned works contain by the way illuminating discussions on the relevance of function intension to logic and mathematics. My only aim in this paper is to draw attention to the fact that pre-formal (i.e., effective) dependence relates to something left uncaptured by the set-theoretic function concept.

5. Two examples from mathematical theories

Working mathematicians do not have to bother with such a distinction between effective and functional dependence. To begin with, they use the abstract notion of function to study general properties of functions (e.g. continuity, differentiability) in which function content has no bite, not to address questions of variable dependence. Moreover, when it comes to the concrete manipulation of functions (e.g. integrating particular functions, solving particular equations), where function content does make a difference, they implicitly account for effective dependence through algebraic methods of variable elimination.

Let us consider two cases in which the notion of dependence is explicit in mathematical theories. Unsurprisingly, they both involve effective dependence.

The first is the definition of *linear independence* in linear algebra. A family $V = \{v_1, \dots, v_n\}$ of vectors is said to be linearly independent if for scalars $\alpha_1, \dots, \alpha_n$, we have

$$\alpha_1 v_1 + \dots + \alpha_n v_n = \mathbf{0}$$

if and only if $\alpha_i = 0$ for every $0 < i \leq n$.⁵ In other terms, v_1, \dots, v_n are linearly independent if none of them is expressible by means of any combination of the others.

It follows that it is *not* the negation of *functional* dependence that is at issue here. The direct way of seeing this is by noticing that, even if V is linearly independent, we may very well have v_i functionally dependent on v_j , for any $v_i, v_j \in V$, because there is always a function taking v_j as an argument, and giving v_i . We may write such a function simply as $v_i = \alpha_k v_k + v_j - v_j$, for some convenient vector $v_k \notin V$ and scalar α_k .

As in the case of algebraic expressions in general, the definition of linear independence seems to implicitly carry the assumption that redundant arguments are previously ruled out of the analysis.

The second case is the notion of *mutually independent events* in probability theory. One way of introducing the notion without going too deep into the formal apparatus of probability theory is to state that event B is independent (or stochastically independent) of event A if the following is the case:

$$(3) \quad P(B|A) = P(B)$$

where $P(B|A)$ stands for the “probability of B given A ”—i.e., the probability of event B on the hypothesis that event A takes place.

However, the fact that $P(B|A)$ is the same as $P(B)$ means that event B is indifferent to event A . This indicates that it is effective dependence that is denied again: varying the value of $P(A)$ never affects the value of $P(B)$.

On the other hand, we may still have functional dependence even though B is independent of A , for it suffices to write the full expression of (3), which is (for $P(A) > 0$):

$$(4) \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

and then to recall that another way of defining “ B independent of A ” is through the following equation:

$$(5) \quad P(B \cap A) = P(A) \cdot P(B)$$

Now (4) reads:

$$(6) \quad P(B|A) = \frac{P(A) \cdot P(B)}{P(A)}$$

or:

$$(7) \quad P(B) = \frac{P(A) \cdot P(B|A)}{P(A)}$$

In (7), $P(B)$ is functionally dependent on $P(A)$.

6. Conclusion

The place of the notion of dependence as the background pre-formal idea underlying the function concept remained preserved under the transition from the classic to the modern approaches, as indicated in the first section of this paper by various statements by authors of that period, as well as from contemporary ones. This was shown to be a wrong move: the notions of function and dependence are not essentially associated in modern mathematics.

As I attempted to show, the pre-formal notion of dependence that was used to characterize the concrete, algebraic conception of functions in classical mathematics was effective dependence, which does not allow for redundant variables. In contrast,

the modern, abstract function concept is based on the weaker idea of a many-one correspondence, thus allowing for redundancy. As a consequence, many-one correspondence does not carry the notion of effective dependence. If we wish to keep the association between function and dependence in the modern set-theoretic framework, we must redefine dependence as functional dependence. But then, as we have seen, this would be a kind of terminological acrobatics lacking conceptual content.

We have also examined the relationship between effective dependence and the intensional concept of function content, and how closely the latter is related to Shapiro's observations about the modern notion of computability and recursion in connection with the pre-formal idea of effectiveness. I suggested that this line of investigation could be a promising one towards a better understanding of our pre-formal idea of dependence.

We have finally concluded that there seems to be no practical conflict in assimilating pre-formal and functional dependences, for mathematics does not substitute the modern concept for the old practice. Instead, both the set-theoretic and the algebraic concepts are in use, according to the specific needs of particular mathematical theories. As pointed out by Stein (1988), the function concept evolved toward its abstract set-theoretic version not because of prior philosophical inquiry, but rather because of 'on-the-spot' mathematical urges. In other words, mathematicians came upon the abstract concept of function because of practical needs for dealing with structures of a more general kind. It would be hard to imagine how mathematicians could in our present time study the notion of continuity of a function in its full generality without an abstract function concept allowing them to analyze functions *as such*, instead of particular instances given by algebraic expressions.

If function is primary, as it is in mathematics, both types of dependence are accounted for, each in the relevant (abstract or concrete) framework. If we are interested in specific questions in which effective dependence is relevant, as in the examples above, we just come back to the old algebraic expressions—the set-theoretic conception by no means replace the algebraic use. However, if dependence is primary, that is, if one has a prior, pre-formal notion of dependence and wishes to make it precise through the set-theoretic function concept, as in foundational and modeling contexts in general, the automatic use of functions to formalize dependence might lead to undesirable and harmful misconceptions. It is then advisable to ask first which kind of dependence one seeks to formalize.

References

- Bos, H. 1980. Mathematics and rational mechanics. In: G. S. Rousseau; R. Porter (eds.) *Ferment of Knowledge*, p.327–55, Cambridge University Press: Cambridge.

- Eves, H. 1983. Great Moments in Mathematics (After 1650), Dolciani Mathematical Expositions No. 7. *The Mathematical Association of America*.
- Gardiner, A. 1982. *Infinite Processes*. Springer-Verlag.
- Kleiner, I. 1989. Evolution of the function concept: A brief survey. *The College Mathematics Journal* (Mathematical Association of America) **20**(4): 282–300.
- Lützen, J. 2003. Between rigor and applications: Developments in the concept of function in mathematical analysis. In: M. J. Nye (ed.), *The Cambridge History of Science Vol. 5: The Modern Physical and Mathematical Sciences*. Cambridge University Press: Cambridge.
- MacLane, S. 1986. *Mathematics, Form and Function*. Springer.
- Rüthing, D. 1984. Some definitions of the concept of function from Bernoulli, Joh. to Bourbaki, N., *Mathematical Intelligencer* **6**(4): 72–7.
- Schaaf, W. L. 1930. Mathematics and World History. *Math. Teacher* **23**: 496–503.
- Shapiro, S. (ed.) 1985. *Intensional Mathematics*. Elsevier.
- . 2006. Effectiveness. In: J. van Benthem; G. Heinzmann; M. Rebuschi; H. Visser (eds.) *The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today*, p.37–49, Springer: Dordrecht.
- Stein, H. 1988. *Logos, logic, and logistiké*: Some philosophical remarks on the nineteenth century transformation of mathematics. In: W. Aspray; P. Kitcher (eds.) *History and Philosophy of Modern Mathematics*, pp.238–259, Minnesota Studies in the Philosophy of Science, Volume 11, University of Minnesota Press, Minneapolis.
- Struik, D. G. 1969. *A Source Book in Mathematics, 1200-1800*. Harvard University Press: Cambridge, MA.

ANDRÉ BAZZONI
 University of California Berkeley
 Philosophy Department
 andrebazzone@gmail.com

Notes

¹ Schaaf (1930, p.500) goes so far as to state that

[t]he key note to Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts—it is rather a complete emancipation from such notions.

² Tension subsisted, however, among the different trends. For example, Weierstrass somewhat tried to revive Euler's algebraic conception by defining an analytic function as a collection of power series (see Lützen 2003, p.479). And there were other ways, in addition to the set-theoretic method, in which the function concept alternatively evolved (e.g. the mathematical notion of distributions as generalized functions; cf. Lützen (2003) for details).

³ This term is also used with the same meaning in database theory.

⁴ If y is a constant, it can be made trivially dependent on anything.

⁵ $\mathbf{0}$ is the null vector.