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A Phenomenological Approach to the Intuitive Aspect of Mathematical Practice

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> Il est temps de pénétrer plus avant et de voir ce qui se passe dans l' âme même du mathématicien.
>
> Henri Poincaré, *La science et l'hypothèse*.

RESUMEN

Proponemos una descripción fenomenológica de la parte intuitiva de la práctica matemática que tiene lugar en la investigación. Nuestra investigación se basa en el análisis de datos en primera persona procedentes de las respuestas ofrecidas por matemáticos profesionales en una encuesta realizada por la autora. Del análisis sistemático de esas respuestas, extraemos aquí invariantes de la práctica matemática que ponemos en relación con las obras filosóficas de H. Poincaré y J. Hadamard, así como con los testimonios de famosos matemáticos que han contribuido al desarrollo de una fenomenología de la práctica matemática con sus obras y su pensamiento. En último término proponemos una estructura intersubjetiva de la parte creativa del trabajo matemático.

PALABRAS CLAVE: *intuición, fenomenología, matemática, práctica matemática*.

ABSTRACT

We propose a phenomenological description of the intuitive part of mathematical practice in research. Our investigation is grounded in first person data coming from the answers of professional mathematicians to a survey conducted by the author. From a systematic analysis of those responses we extract invariants of mathematical practice that we put in line with philosophical works of H. Poincaré and J. Hadamard, as well as with testimonies of famous mathematicians, who have participated in the development of a phenomenology of mathematical practice. Finally we propose an intersubjective structure of the creative side of mathematical work.

KEYWORDS: *Intuition, Phenomenology, Mathematics, Mathematical Practice*.

INTRODUCTION

Texts on intuition, and on intuition in mathematics in particular, introduce this notion as a cornerstone for discovery in mathematics: without intuition no progress in mathematical knowledge or achievement would be possible [Poincaré (1908), Hadamard (1949), Hersh (2013), Guitart (1999)], even though intuition cannot be regarded as a justificatory criterion to decide whether a statement is true or false. Therefore, despite the consensus according to which a mathematical statement demands a proof to put it beyond all possible doubt, we think that the epistemology of mathematics might be concerned not only with the context of justification, but also with the investigation of its development as an ongoing social enterprise with common bonds of language and methodology [Gowers (2002), Guitart (1999), Hoyningen-Huene (2006), Poincaré (1905), Poincaré (1908), Rota (1997a), Siegel (1980)]. In this sense, we are in full agreement with F. Suppe's thesis according to which "the epistemology of scientific knowledge must take into account the context of discovery as well as the context of justification if it's to arrive at a satisfactory account of the epistemic features of scientific knowledge" [Suppe (1974)]. We have chosen a descriptive perspective on the development of mathematics and investigated the methodology adopted by mathematicians in the actual practice of their discipline. Our procedure entailed the gathering of a collection of data, from which we have extracted invariants of mathematical practice leading to discoveries. In other words, we study the phenomenology of living mathematics [Corfield (2004), Gowers (2002), Hersh (1999), (2013)], by focusing on the way mathematical knowledge advances, and search for common procedures in the manner mathematical problems are approached and solved [Brunschvicg (1912) – Introduction].

As we wanted by all means to avoid the tendency to base discussion on creative reconstructions of mathematical theories' development – on *make-believe* as W. Quine would have called it [Quine (1969)] – we made an empirical investigation to find out how advances in mathematics actually proceed, and questioned mathematicians about their personal methodology, with the goal of elucidating its intuitive aspect. We were inspired by the pioneering work made by J. Hadamard on mathematicians' psychology [Hadamard (1973)].

We made the hypothesis that a generic structure - or at least generic features – of the intuitive aspect of mathematical practice may exist. In order to assess the pertinence of this hypothesis, we established a survey that was submitted to contemporary professional mathematicians. The questionnaire, elaborated by the author, inquires about methodology, subjective criteria according to which mathematicians evaluate the correctness of a statement, and circumstances of discoveries in mathematics. The questions use a vocabulary commonly employed by researchers in mathematics. To give an example, we chose to use in the inquiry the term *discovery* and not *invention* because *discovery* was spontaneously used by the participants in the survey, *invention* seeming more appropriate to technical disciplines [for this matter see also the introduction of Hadamard (1973)]. Even though this choice can be considered as a central question in the philosophy of mathematics, in order to obtain descriptive rather than interpretative answers from the mathematicians, we intentionally did not bring up this key stake.

For the same reason, we carefully avoided asking the mathematicians about the issue of whether the mathematical entities they work with really exist in an ontological realm or not. In the execution of the survey, our aim was to stay away from known philosophical contexts in order to obtain spontaneous testimonies centered on *factual* descriptions (as opposed to *interpreted* ones) of effective research methodology. The adjective *factual* is to be understood in the sense that we asked the mathematicians to describe *how* they do mathematics instead of *why* they do such and such – even though their descriptions might not be devoid of interpretation.

The analysis of the collected answers aimed to extract an essence of mathematical practice. To reach this goal, the answers were categorized so as to identify common characteristics (though without ignoring exceptions), in order to put them in line with previous testimonies and philosophical works, both by mathematicians and philosophers. This allowed us to identify different aspects of the role of intuition in mathematical research and to propose an intersubjective description of creative mathematical work, as effectively lived by mathematicians.

All survey respondents are professors or researchers in universities, except one, a PhD student in the process of finishing his thesis. They work on different areas of mathematics, such as topology, algebra, stochastic analysis, theoretical computer science, analysis, dynamical systems, number theory, and differential geometry.

RESULTS

The survey consists in ten questions and the article focuses on the analysis of the following three questions, related to discovery and intuition in mathematics:

1. When a new result "appears" (either through proof or as the result of your intuition), could you identify and list the criteria allowing you to judge that the result is true?
2. Is the discovery of a new result a linear process?
3. If not (see Question 2), did you instead meet situations in which you knew the objective without knowing initially the means to reach it? In those cases, did you or do you trust your primary intuition?

Question 1 concerned both context of discovery and context of justification. The majority of the mathematicians answered spontaneously on the "intuitive side" of it and detailed on the aspect of their practice in which, once a new idea or statement is grasped, they go ahead after having proceeded only to quick verifications that are not full rigorous proofs. Section D is devoted to this aspect.

We agreed with the mathematicians, on the one hand, that *linear process* is to be understood as a purely deductive process and, on the other hand, that *intuition* corresponds to "a direct and immediate vision of a reality – immediate cognition of a truth without the use of reasoning" [*Dictionary of philosophy*, J. Ferrater Mora (2001), Loyola Ed.]. We deal here with a concept close to what H. Poincaré calls *illumination* in his chronological four-step model of mathematical work[1] [Poincaré (1908)] and that is a key reference for the phenomenology of mathematics.

The following paragraphs are a *categorization* followed by an *interpretation* of the collected data in response to Questions 1 to 3. Even though the answers are always in a certain context, we have abandoned the idea of reporting them integrally in order not to weaken or flood our argument with excessively long quotations.

A. *Occurrence of a New Result*

The collected answers converge and reveal that it is in a non-linear process that mathematical achievements are made. As mentioned earlier, *linear* is meant as sequential step-by-step process whereby the researcher starts with a series of hypotheses, follows deductive reasoning, and arrives at a solution.

We find here that linearity in effective mathematical practice belongs to the writing of the proof – subsequent to the discovery of a new result (A1) – and by no means characterizes mathematical research strictly speaking (A1 and A2).

(A1) "Not at all! The discovery of a mathematical result is everything but linear. When it is written in an article, all the process of discovery has been erased and only the proof remains."

(A2) "What is linear is the movement that starting from obvious hypotheses leads to a solution through a classical method. This doesn't have anything to do with research."

As another mathematician states it (A3), linearity would be suitable for school situations only.

(A3) "I would say that the problem-reasoning-solution process would be limited to what I would call school situations."

A deepening in the description shows that linearity is a characteristic of "predictable" cases that do not introduce new concepts (A4, A5, and A6), *i.e.* that lead to "not so deep" results (A7):

(A4) "It is possible to solve problems linearly, though this works only for problems that will not introduce new concepts."

(A5) "[Solving a problem] starting from what is known, is proceeding linearly and often this does not work."

(A6) "Only in very predictable situations I manage to identify a linear process! Most often there is back-and-forth."

(A7) "The process is linear only for very foreseeable cases – and thus not that deep. By "not that deep", I mean results for which the proof is essentially technical, where mastering usual techniques is enough to get to the result."

For what concerns the discovery of foreseeable results, the mathematicians agree on the fact that it is the consequence of a linear process. For the unpredictable ones, one mathematician states clearly that it is in a kind of passive approach – at least a not fully intentional one – that the most striking and innovative improvements are made (A8).

(A8) "But usually the most surprising and deepest results do seem to appear in a non-linear way, when our brain seems to make some unexpected connection."

Consequently, all the answers we gathered testify to this aspect of mathematical discovery: to start from a set of hypotheses and then follow only deductive reasoning cannot be the source of actual discoveries. By purely logical inferences no newness can be envisioned. In this, the testimonies of the mathematicians are in complete agreement with what H. Poincaré writes in *La valeur de la science* [Poincaré (1905)]: "La logique qui peut seule donner la certitude est l'instrument de la démonstration: l'intuition est l'instrument de l'invention"[2]. A contemporary mathematician, P. Halmos (1973), asserts: "Mathematics is never deductive in its creation. We make jumps to unwarranted conclusions. The mathematician becomes convinced of their truth long before he can write down a logical proof" in which the term "never" seems here positive.

Given this essential feature of discovery in mathematics, what, then, is the mathematical practice that *may* lead to it?

B. *How is a New Result Found?*

The descriptions given by the mathematicians of the "birth" of a new result show that discovery is characterized by a dendritic and multidirectional process that has ramifications, like the "unexpected connections" quoted previously (A8). A researcher has even used the expression "reasoning in network" (the response not reported here). The answers reveal a practice based on back-and-forth in reasoning, numerous attempts (B1 and B2), and the search for analogies (B3 and B4). We read:

(B1) "A discovery is most often an endless back-and-forth process, where we seem to be blocked, those hopeless moments being followed by sudden rays of light. What had seemed obscure for weeks, sometimes for years, becomes suddenly evident."

(B2) "It frequently happens that I tackle the problem in half a dozen different ways before finding the right path to the envisaged conclusion. Most often it is because the first attempts lead to technical difficulties I cannot circumvent."

(B3) "We try to find analogies with problems or situations of the same kind, contexts (meaning hypotheses) if not identical, at least close."

(B4) "Often the discovery of a new results is conditioned to the understanding of several facts that are more or less linked to the problem in question, facts that are sometimes useful for the proof."

The salient point here is that the mathematician tackles a problem in different manners at the same time, he searches to port his knowledge by adopting an analogical strategy. After G. Pólya, analogy is, along with induction, a variety of plausible reasoning [Pólya (1954)] On this matter, the powerfulness of analogical reasoning in mathematical practice, and especially in the case of invention/discovery, has been evidenced for several sounding mathematical achievements in the history of mathematics (see for a rich review Corfield (2004), chapter 4: The role of analogy in mathematics). In his time, G. Pólya went a step further by discussing in *Mathematics and Plausible Reasoning* the fact that common grounds exist between two mathematical domains and illustrated this in showing how analogy is used to pass from plane geometry to solid geometry [Pólya (1954), chapter II]. W. Thurston, a mathematician well known for having shared his conception of the *effective* practice of mathematics, underlines too the importance of analogies and connections in mathematical research: "People have amazing facilities for sensing something without knowing where it comes from (intuition): for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons (…). These facilities are quite important for mathematics" [Thurston (1994)].

Detailed descriptions of a mathematical practice leading to discovery allowed for an elucidation of the kind of goal the mathematician wants to reach. The answers reveal indeed two main metaphors: one game-like – the puzzle – and one allegorical – clarity. One being possibly the consequence of the other: once the puzzle is put together, the mathematician sees clearly.

(B5) "It's like cooking or putting a puzzle together, or making a proof in reverse: here are the possible proof techniques that I can apply, here are the objects I can apply them to, so let me just mix things around until they arrange into the proof of a non-trivial statement."

(B6) "Maybe like a puzzle: we have many pieces, many parts. In a puzzle there is the whole that represents something, for example a landscape, and there are smaller parts that can represent things in themselves (a house, a tree). In mathematics, we can manage to understand the tree without seeing the entire landscape."

(B7) "Suddenly all the pieces put themselves together and one sees what it was finally all about."

Or even:

(B8) "At the beginning, you have the impression to see some objects but you don't really know what you are seeing, there is a fog, it is complicated. You adjust the binoculars and suddenly, you see clearly, everything becomes intelligible."

It is worth noting that all these quotations converge towards the idea of a preexisting mathematical reality. For example, the use of the past tense in (B7) "what it was finally all about" implies that what was to be found already existed but the mathematician was not yet aware of it. We could almost dare say that for the mathematician, it is in a preexisting world that the as-yet-unassembled pieces lay and that the mathematician perceives through a fog fuzzy images of a preexisting world (B9). The role of the mathematician is then to try to assemble the puzzle; to take up the challenge this preexisting world proposes to him. As long as the mathematician does not succeed in solving the puzzle, in finding the path that leads rationally to the truth, he only gets access to a limited part of a bigger image that he cannot yet distinguish in its completeness (*B6*). In addition, the mathematicians use the term mathematical *facts*, also in favor of a subjacent ontology.

In *The Mathematician's Mind* [Hadamard (1973)], J. Hadamard was already speaking of the preexistence (at least for him) of mathematical reality: "Although the truth is not yet known to us, it preexists and inescapably imposes on us the path we must follow under penalty of going astray."

A mathematician deepens the description of the clarity after the mist:

(B9) "With mathematics, at least for me, there is not only this impression of clarity, there is also this impression of simplicity in comparison to all the time spent in the fog… The problem seemed so hard to solve, and once solved in an appropriate way, the solution imposes on us, as if it had been always there."

This is to a large extent identical to a description exposed by the Fields Medal-winning mathematician L. Lafforgue in his talk "The Invisible in Mathematics" [Lafforgue (2009)]. In it, L. Lafforgue underlines the transition from a state of confusion and vagueness to a state of clarity where everything becomes luminous – the mist being dispersed: "Cette réalité est d'autant plus étonnante et paradoxale qu'un mathématicien cherche toujours à mettre des mots sur les choses qu'il perçoit d'abord *confusément*, comme à travers un brouillard qu'il s'agit de déchirer, afin que la lumière du langage éclaire leurs contours et les rende peu à peu familières et tangibles."

These concepts of puzzle, fuzzy vision, and challenges that the mathematician takes up, not without emotion (see B1), are the substance of a detailed description written by one of the respondents to the survey. The practice of mathematics is described using the notion of percolation and a cascade of cartographical metaphors, like graph, arrows, path and labyrinth.

(B10) "I have developed on a graph an analogy between research and percolation. Let us assume mathematical facts are points, and the steps of the proof are arrows between the facts. The already-proved facts shape a big closely related component of the graph. Your goal is to prove some statement. For this you will try to construct a path between what is known (the big component) and your goal. You must add the missing arrows. Often an arrow is hard to prove. You will proceed without special order, but using your intuition. Starting from what is known is to proceed linearly, and often this does not work. (…) You can then draw a chain of facts that links the known to the goal. Once, at last, when you have managed to place the last arrow, the one that resisted you the most, it is this one you will remember, the one you will speak about to your colleague. It is not necessarily the most crucial arrow, but because it came as the last, you have the impression that it is the most important one. Just before laying it, you were in a state of anxiety. As you lay it, the current suddenly flows between the known and the goal. There is then a little moment of happiness."

This answer can be read as the summary of the convergence of the quotations reported above: linearity that "*often does not work*", the game-like character of mathematics (to solve a puzzle) and a multi-directional approach that reminds us of the "*reasoning in network*" mentioned earlier.

The impossibility to introduce new concepts by using *only* a deductive reasoning grounded in yet known and validated hypotheses, the need to approach a mathematical problem by an analogical method, to follow multiple approaches side by side, and finally, the presence of a mathematical reality that is at the beginning invisible or incomprehensible to the mathematician, all these seem to be characteristics of mathematical practice in research and parallel strongly with this mathematical truth "not known yet" cited by J. Hadamard [Hadamard (1973)].

The convergence and coherence in the responses lead us to the question of the access to this unassembled puzzle, to this primarily fuzzy picture that the mathematician perceives. We also would like to better comprehend the confidence [or the conviction, as in Halmos (1973)] that the mathematician has in the truth of a statement – the *élan* of confidence in which lies his motivation.

> (B10) "The confidence in what we want to prove as true is the force that motivates research."

For a new aspect emerges from the responses: the specific stimulation of the mind as a will to arrive at a clear vision. G.-C. Rota speaks about the strong determination and the intimate motivation responsible for some endless procedures in mathematics of proving and restating; those procedures ending only when mathematicians understand the reasons that lay concealed beneath a statement [Rota (1997a), (1997b)]. Indeed, mathematicians do not only want to *see*, they want to *see through*; they want to understand the sense of a statement. To use words of a survey respondent (B6), mathematicians want not only to see the tree, they want to see the whole picture. Indeed, a mathematician may have a proof, and therefore know that something is true, but nevertheless still find it hard to believe [Gouvêa (2011)][3]. Examples on different kind of proofs and their effectiveness on giving reason for believing and understanding the *why* of a statement are given by T. Gowers in *Mathematics: A Very Short Introduction* [Gowers (2002), see for instance pp. 48-51].

C. *Intuition: Putting Rationality to Rest*

Whether intuition is described as the perception of an unassembled puzzle, as a fuzzy vision, or more generally as the aim to be reached by the mathematician, the question of the means to access it/them is central. In what follows we attempt to give it an answer as from certain reports we were able to extricate what is experienced by mathematicians in a moment of intuition or insight. The sources of all the quotations reported in this section are responses to Question 3.[4]

A first recurrent aspect of the answers is the feeling of confidence that always accompanies the intuition of a new statement and that was introduced at the end of the previous section. One mathematician underlines that this confidence is the *only* source of newness (C1), while all respondents, when discussing their intuition say that they *always* (the term is strong) trust this intuition (C1, C2, and C3).

(C1) "I always trust my intuition, it is the only means I know to really do new things."

(C2) "I always have intuition of the goal to be reached, without knowing how to approach it. And a flawless confidence in it."

(C3) "It is always necessary to be confident in intuition, for in the end that is what motivates research."

A complete agreement of these quotations with the conclusions of Parts *A* and *B* is observed. We may here recall the citation of P. Halmos, quoted earlier: "The mathematician becomes convinced of their truth long before he can write down a logical proof" [Halmos (1973)], where an even stronger term than confidence is used (conviction). This converges with J. Brown's conclusions deduced from his discussion on diagrammatic reasoning, according to which mathematics is an inductive discipline: "Mathematical practice is inductive: we rely on intuitions and data to come up with new ideas and new formalisms" [Brown (1997)]. We may remember the words of H. Poincaré who "felt an absolute certainty at once" in his famous testimony about an illuminative instant in Caen [Poincaré (1908)].

Nonetheless we must be cautious here with the attribute of intuition that underlies the above quotations, as what is spoken about is not the insight or the illumination that gives the answer to a mathematical issue. Here, the respondents speak about the founding role of intuition in mathematics, the one that allows the apprehension of an aim and that is the ultimate constituent of discovery in mathematics. R. Guitart in *La pulsation mathématique* (*The Mathematical Pulsation* in English) [Guitart (1999)] summarizes this matter of fact as follows: "Il n'est pas d'autre solution fondatrice que de se confier à l'évidence du mouvement et de l'arrêt de notre pensée, ce qui s'appelle l'intuition", where the source of a new mathematical concept is to be found in intuition and not – at least in a first instance – in rationality ("l'arrêt de notre pensée").

When it comes to a description of the source of this inspiratory constituent of mathematical practice, and in which mathematicians have faith, we see that intuition is felt as guided and stimulated by experience. The survey respondents concur:

"Sometimes one does indeed have a feeling that something should be true. I think it is some sort of intuition that is guided by previous experiences and a growing familiarity with the subjects of one's investigations."

(C5) "Experience trains intuition, to a large extent."

(C6) "I don't think intuition comes out of a hat, like a magic trick."

Intuition does not surge *ex nihilo*, it is the fruit of practice, it is intimately linked with the essential experience of the researcher and his familiarity with his own domain of study; intuition is the *outcome* of expertise. In this, we find H. Poincaré's conception of mathematical work: the conscious preparatory work is an inescapable step in mathematical discovery [Poincaré (1908]. This preliminary work precedes an unconscious *incubation* phase, potentially followed by the occurrence of illumination/intuition. In his article about mathematical intuition, R. Hersh made a study case of H. Poincaré's experience in Caen and emphasized too the strong role of experience in intuition, speaking about the "faith – that experience is not a trap laid to mislead us – is the unstated axiom" [Hersh (2013)]. On the power of experience in intuition and *invention* (to employ Hadamard's terminology) J. Hadamard supported H. Poincaré's view and deepened the description of the importance of previous experience in mathematical intuition: during the incubation phase the unconscious mind does not work in a random manner, but according to certain patterns that only a prepared (*i.e.* an experienced) mind can accomplish [Hadamard (1949)].

Therefore, to the respondents, this is not a transcendent intuition that is faced but an access to a discovery mediated by prior work and experiences[5].

A subtlety, intimated by the following testimony, is that intuition goes beyond the experience and familiarity of the researcher with his domain; intuition is a full part of a preparatory activity:

(C7) "Intuition is almost never immediate. It is often the consequence of a process that consists of the refusal of the other possibilities."

This sentence could seem paradoxical at first glance, but in saying so, the mathematician does not only say that intuition is not transcendent: intuition is described as a process of elimination accompanying plausible reasoning, and close to the "guess and check" process advocated by G. Pólya [Pólya (1954)]. The solution is the non-immediate result of a preparatory work in which some envisaged solutions are ruled out. We might go even further and suggest that (C7) may not be understood as the description of a simple eliminative process of possible solutions, but like a statement supporting that the emergence of a new idea comes from the fact that the mathematician, pushed to his limits, *must* find the solution.

The last part of the present section is devoted to the description by the mathematicians of *their* intuition and the experience associated to a moment of insight. From their written reports, we were able to reveal the deep sensitive aspect of mathematical intuition, this later not being reducible to a psychological experience. This sensitive aspect is present in all the collected answers for which a description of intuition was given.

> (C8) "It seems to me that often I had the idea of the goal to reach, following an intuition rather than a vision… It is a sentiment, something that one feels: one feels it is going to be so, without being really able to explain it."

This response is very close to citation (C4) ("Sometimes one does indeed have the feeling that something should be true") and both refer to a felt sensation, a *sense of truth*, without apparent explanation. As a matter of fact this sensitive aspect has already been linked to intuition: intuition is accompanied by a feeling that permits one to get or to approach the solution. Charles S. Peirce speaks about "*guessing*" as a foundation of abduction [Peirce (1901)], as if, beyond reasoning and discursive processes, the human being has the ability to sense, to sensibly link himself to a truth.

In opposition to what is most often required for simple problems, for which the mathematician searches to put aside affectivity in order to concentrate on technical capacities, in his mathematical activity, the mathematician may sometimes leave it up to *sensitivity* (C9) rather than to deductive abilities.

> (C9) "Intuition: it is my affective abilities on which I have no control. It is a matter of putting on hold my conscious will and exacerbating my sensible affectivity."

W. Thurston (1994) described a practice of mathematics as *effectively* accomplished by the mathematician. He too mentions the necessity to put discursive rationality to rest, and underlines his will to listen to his intuitions. He even goes further by noticing the potentially harmful aspect of logic in the discovery phase. Listening to one's intuitions is, for the mathematician, to calm down his mind and concentrate on his sensitivity, in an effort of exacerbation of his affectivity: "Personally, I put a lot of effort into "listening" to my intuitions and associations. This involves a kind of simultaneous quieting and focusing of my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations."

Thus, it seems that in those moments of intuition, it is not anymore the technician who is working, but the being listening to his feelings and enlisted in a process that seems to escape his rationality. Once the sensitive content is grasped, it is subsequently interpreted and conceptualized in order to arrive to a fully validated statement. On the sensitive aspect of mathematical practice, R. Guitart also wrote: "l'objet des mathématiques est la rigueur, un affect intellectuel qui est le sentiment du tombé-pile entre une intuition et son écriture" [Guitart (1999)].

Finally, this idling of rationality that allows concentration on what is at first indistinctly perceived, is found in the following answer:

(C10) "I conceive intuition [of a new result] as the cliché of the crystal ball: at the beginning, everything is fuzzy but then, the more you concentrate, the more you see something appearing with clarity."

Here a vocabulary as un-rational as that of clairvoyance ("the crystal ball") is employed to describe the experience of the *moment of intuition,* for attention is focused on an image that is progressively appearing. The technical aspect, not being evoked here, is implicitly secondary.

D. *Subjective Criteria of Truth*

Aside from the verification step that validates any new mathematical result (and that is the subject of section E), we intend here to refer to the possible evaluation of the truth of a (new) statement made by the mathematician that is not pure justification (like the writing down of a complete proof would be). This is why we speak here about *subjective* criteria. Quotations reported here are mainly extracted from the answers to Question 1.[6]

(D1) "The best criterion is that the result is simple. To be simple is to be natural, to make sense in our mind."

(D2) "A result fits well with what we already know. It must be elegant and natural. It must fit well with our experience of mathematical things."

These two quotations are almost identical to a third one, formulated very concisely:

(D3) "Because there is an adequacy with everything I know on related questions."

The well-known idea of aesthetics in mathematics combines here with the notion of elegance. It parallels what L. Lafforgue describes as "(…) une impression éprouvée par la personne du mathématicien sensible à la profondeur de l'être et à la beauté" [Lafforgue (2009)][7]. However beauty is not what is repeatedly reported in the answers to the survey. The common denominator is rather the *natural*: a result or a statement might be true because it is natural and permits progress towards a completeness and towards the achievement of a *meaning* (see also quotation D4). This reinforces what was developed in Section C, as the word *natural* seems to reveal a capacity of judgement inherent to the mathematician, a predisposition of his mind that allows the perception (or the *reconnaissance*) of what is true and that is assisted by this growing familiarity mentioned in (C4). Mathematicians' sensibility and expertise permit to see things "correctly", to see things how they "really" are, to employ the inspired terminology of D. Corfield (2004), chapter 1, section 1.4.

Another subjective criterion was identified: the possibility of pushing further the boundaries of application of an already established result, giving here to mathematics a finalistic aspect. H. Poincaré wrote in *Intuition et logique en mathématiques* [Poincaré (1905)] that "on ne peut faire de conquête scientifique que par la généralisation". We seem to be in the presence of a *mise en abyme* of induction[8] as intuition allows one to encase levels of generality in mathematics. Indeed, from a general mathematical statement, one may generate an even more general one – like a *meta*-mathematical induction. The rightness of an intuition could be thus recognized or appreciated by the mathematician because the outcome of the intuition enlarges mathematical knowledge in "deepening" it (D5).

(D4) "[Because there is the] possibility to see in a result a natural generalization of a known one."

(D5) "It seems like, when an idea is correct, successive miracles occur, leading each time to a continuous deepening. The ab-

> sence of those miracles is the sign that the followed direction is wrong."

To perfect this part on the subjective side of evaluation of a new statement, we conclude with a response (D6) in which a mathematician synthesizes several aspects previously discussed. In addition to the notion of aesthetics ("nicely"), the mathematician is convinced of the veracity of a result because it fits with what already exists ("into the context") and it participates to the completeness of mathematics. Placing the last piece of the puzzle, the mathematician sees what is going on, he contacts that truth that he "was not knowing yet", to use J. Hadamard's formulation.
(D6) "It seems to fit nicely into the context, as if a missing piece of a puzzle."

E. *The Verification Step*

Even though we deliberately investigated the context of discovery in mathematics in adopting a descriptive perspective of the mathematical practice of living mathematics, we could check on the fact that all the respondents to the survey claim to be confident in the truth of a statement only after having a rigorous proof of it. Proof is the ultimate criterion to authenticate mathematical truth; at the end justification always "wins" over intuition. The outcome of our study on that matter is, as in the literature, positive. "Mathematical achievements may rest entirely on deductive evidence, but mathematical practice is based squarely on the inductive kind" as summarized by J. Brown [Brown (1997)].

The participants in our survey made theirs T. Gowers's words: "mathematicians are rarely satisfied with the phrase "it seems that". Instead they demand a proof that is an argument that puts a statement beyond all possible doubt" Gowers (2002)].

Here are a few excerpts from responses to the survey that we have selected for their concision. The quotation (E5) echoes nicely the last step of H.Poincaré's chronological model of mathematical work [Poincaré (1908)] that we referred to in note 1.

(E1) "In any case, to me a result is not true if I don't have a written proof."

(E2) "When a new result appears, I am fully convinced only when I have perfectly understood its proof."

(E3) "If the result is not proved, it is not a new result. What permits one to check on the veracity of a result, is indeed the study of its proof."

(E4) "Proof is the ultimate criterion to decide that a result is correct."

(E5) "The third and last step of my way of working: the effective verification step."

CONCLUSION

We think our study is a contribution to what G.-C. Rota [Rota (1997)] appealed for: "*a realistic description of what is going on in mathematics"*. Carrying it out was a matter of practicing a kind of *épochè*, suspending any of the beliefs we may have had on the practice of mathematical research and questioning directly professional mathematicians about their actual mathematical work. As mentioned in the introduction, we wanted to access the *how* mathematicians do mathematics, as opposed to the more commonly asked *why* they practice so and so. From descriptions of effective mathematical work, we intended to elucidate the role of intuition (understood as the immediate grasp of a truth) in mathematical practice and to identify a possible inherent structure of mathematical discoveries.

The outcome of a survey submitted to contemporary researchers in mathematics and working on different fields of mathematics, provided us with a collection of data about the individual practice of mathematics. Through the multiplicity of the answers, a *variational* method was made possible and invariants of the intuitive aspect of mathematical practice and advances were extracted. We evidenced that this search of invariants did not leave us with a fleshless skeleton. On the contrary, due to the convergence of the responses, together with previous works on the phenomenology of mathematical practice, we were able to check our hypothesis and propose an intersubjective scheme of discoveries in mathematics. In the context of discovery in mathematics, we give intuition its proper creative role and impact and access a description of its characteristics.

Intuition in mathematics is the fruit of experience, expertise, analogical practice, intense preparatory work, but also the outcome of the mathematician's ability to listen to his own sensitivity and to abandon directed thoughts. The moment of insight occurs during this particular attitude where the mathematician puts to rest led thinking and is attentive to his own affectivity. Intuition in mathematics can be seen as a movement where the mathematician oscillates between rigor and freedom; a movement that allows him to continually push the boundaries of his knowledge, *i.e.* of universal mathematical knowledge. Intuition in mathematics involves thus the *rigorous speculations* R. Guitart emphasized in *The mathematical pulsation* [Guitart (1999)], it does not come out of the blue, it is a matter of *guessing and checking* [Pólya (1954] where the mathematician is involved in an equilibristic attitude between discovery and validation. To have an intuition in mathematics means to have "*reasons to believe in it*" as a respondent to our survey positively claims.

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NOTES

[1] In this chronological model, H. Poincaré separates an intensive period of preparatory work that precedes an unconscious incubation time from the illumination/intuition that corresponds to the occurrence of the solution to the problem. The last step is the verification of the intuition.

[2] In Hadamard (1949), we read that "the object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there never was any other object for it".

[3] Interesting enough, F. Gouvêa discusses in his article the circumstances and context of Cantor's famous remark, "I see it but I don't believe it", and argues that Dedekind-Cantor's mathematical correspondence on that matter shows that Cantor was worried about the rightness of the proof. The author emphasized then the social dynamics that underlies mathematical work and its

importance to mathematical advances, giving so an important role to the context of discovery in epistemology.

[4] If not (see Question 2), did you instead meet situations in which you knew the objective without knowing initially the means to reach it? In those cases, did you or do you trust your primary intuition?

[5] In her proposed re-reading of Plato's *Meno*, V. Giardino [Giardino (2010)] contends that mathematical intuition cannot be envisaged as the direct vision of mathematical truths, but is instead an intuition guided by experience, as the one acquired by the slave while dialoguing with Socrates: "But mathematical intuition has also been related to the discovery of mathematical proofs: intuition would involve an unconscious preparation similar to a gestation, and afterwards an illumination by means of which we get to a new conclusion".

[6] When a new result "appears" (either through proof or as the result of your intuition), could you identify and list the criteria allowing you to judge the result is true?

[7] On the aesthetic aspect of mathematics and on what is called beauty in mathematics, we recommend to read "The phenomenology of mathematical beauty" by G.- C. Rota [Rota (1997b)].

[8] Recall that H. Poincaré described induction as a process that permits one to raise knowledge from a particular case to a general one [Poincaré (1905)].

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