On the vertical rotation of a sand heap

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Abstract. This work is a phenomenological attempt to predict the dynamic response of a sand heap due to rotation about its vertical axis. We have made experiments and we've developed a model in order to describe the effect of the rotation on the pile's surface from a dimensionless force balance equation using the Coulomb yield law. We got a very good correspondence between the experimental patterns and the theory, depending on the material through the solid friction angle and we formulated a plausible suggestion for the way in which the pile's history is determined by dynamic (Froude number) and material (friction coefficient) parameters.

Introduction

The study of granular media has received great attention in the literature in the last years, due to its related fascinating phenomena, such as dilatance (Onoda and Liniger, 1990), arching (Edwards and Oakeshott, 1989), segregation (Jullien, *et. al.*, 1992; Britgewater, *et. al.*, 1985) or fluid-like behavior (Haff, 1983; Nederman, *et. al.*, 1982) which give origin to peculiar effects not occurring in other aggregation states such as liquids and solids (Wieghardt, 1975; Baxter, *et. al.*, 1993). These latter phenomena make difficult any theoretical description valid over a wide range of parameters. Even with an ideal granular cohesionless media made up of monodisperse rigid grains, the analysis presents such complexity that attempts to describe its dynamical regimes are of limited success (Mehta, 1992).

This paper deals with the free surface deformation of dry, noncohesive granular material during an axisymmetrical vertical rotation (parallel to gravity force). This phenomenon is interesting because the dynamical response of these materials composed of dense collections of solid grains, are not well understood. In order to study this problem, we made a theoretical approach using a force balance equation which includes the Coulomb yield law and compared both the theoretical and experimental results, showing a very good agreement between them. The experiments were performed using ideal granular material like Ottawa sand, made up of round grains that are more or less uniformly sized. Though our experiments were made using near two-dimensional bins, they showed interesting facts, which deviates from those found in newtonian fluids. We obtained multi-valued steady state solutions in terms of the material parameter μ (the friction coefficient) and the Froude number, F_r (Landau y Lifshits, 1959; Zierep, 1971), to be defined later. These solutions showed strong differences in comparison with a liquid. In particular, the analysis gives different steady-state solutions (hysteresis) for a given value of the Froude number if we reach it slowly coming up or going down. In the next section (Edwards and Oakeshott, 1989) we describe the two-dimensional experimental set-up and results. In Section II we present the theoretical analysis based on the Coulomb's law. Comparison of experimental and theoretical results are shown in Section III. Finally in Section IV we present the final remarks and conclusions.

I. Experiments

In order to observe and characterize the surface patterns during rotation, we did experiments with sand and got the free surface profiles for different motion states. Figure 1 illustrates a schematic frontal view of the experimental set-up. Due to 3-D visualization difficulties of surfaces in circular cylinders we used thin rectangular bins of plexiglass with the following dimensions: 30 cm length (R = 15 cm), 0.4 cm width and 30 cm height. We used gra-

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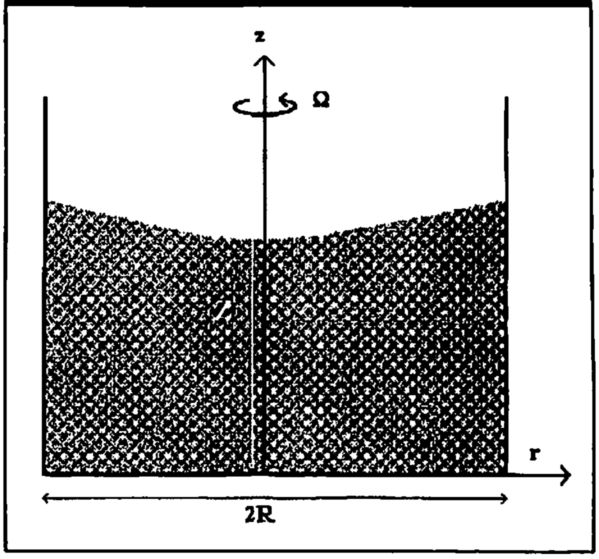
nular material composed of Ottawa sand with mean grain's size between 0.03–0.06 cm, bulk density $\rho =$ $grcm^{-3}$, mean value of solid friction angle $\phi_c = 31°$, and friction coefficient $\mu = \phi_c = 0.53$.

FIGURE 1. GEOMETRY OF THE SYSTEM



The bins were filled with granular material up to 14 cm height and rotated on the z − *axis* (parallel to the gravity vector) with a slowly change on the angular velocities, Ω, taking care of no introducing additional forces. The angular velocity varied from 0 to up 52.78 s^{-1}. The resulting Froude number, F_r, relating the centrifugal to gravity forces, is defined as:

$$Fr = \frac{\Omega^2 R}{g} \quad (1)$$

Where g is the magnitude of gravity acceleration. Therefore, in the experiments we use a Froude number such as $0 < F_r < 43$. Three dimensional effects were not observed in the thin bins (there was not any important kind of deformation in the azimutal φ-direction), and therefore the heaps can be treated to be two dimensional. The observations were made recording the motion with a CCD-camera to 500 frames per second.

We found several interesting patterns as we increased the Froude number, starting from zero: A first pattern, which conserves the initial free surface, occurs between $0 < F_r < 1.36$. As the Froude number is increased further, the free surface changes generating a peak at the center. This peak finally vanishes as the Froude number reaches a value of approximately $F_r \approx 26.14$. This shape is maintained for larger Froude numbers with increasing slopes.

Another patterns can be obtained when we reduce slowly the Froude number from the maximum value reached, $F_r \approx 43$. In this case, the surface shapes are clearly different than those obtained when we increased the Froude number. The final state is reached when $F_r = 0$, showing the free surface profiles represented by two symmetrical straight lines to an angle of 31° from the horizontal. All shapes obtained are reproducible showing an hysteretic behavior. For a given value of the Froude number we got different shapes depending if we arrive to this number coming up or going down. Otherwise, when the rotation device was not well fixed, vibrations appeared at large Froude numbers and a hole or crater was formed on the heap. With care, we can avoid this unwanted phenomenon. In Section III, we present some experimental results compared with the theory to be developed in the next Section.

II. Theoretical analysis

A common mechanical state of a cohesionless granular material is called the quasi-static regime, where low shear rates and high concentrations dominate.

An example of it is a sand pile whose free surface forms a plane with a constant slope with the horizontal: this one is subject to shear stress (τ) and normal stress (N) that tends to move the surface's pile in accordance with the Coulomb yield law (Coulomb, 1773) established two centuries ago:

$$|\tau| = N\mu|\alpha| \leq N\mu = N\tan\phi_c \quad (2)$$

This formula expresses that the slope does not change if the shear stress is less than the product of the normal stress and the friction coefficient μ. When the yield condition (equality) is reached, that is $\alpha = \pm 1$, the pile's surface yields, occurring a granular flow (Sokolovskii, 1965). This situation is also called the critical state in soil mechanics (Schofield and Wroth, 1968). Relation 2 (Edwards and Oakeshott, 1989) has been used in studying the stability of slopes (Sokolovskii, 1965; Rajchenbach, 1990), but it also can be used here to explain the problem of rotation of cohesionless granular piles in containers.

The criteria that supports the use of equation 2 is the continuum point of view of granular media. In this case we can formulate a balance of forces equation for a small element of volume with density ρ at the free surface of the granular pile. Using cylindrical coordinates, this equation can be written as:

$$\rho\left[\Omega^2 r\cos\theta - g\sin\theta\right] = \rho\left[\Omega^2 r\sin\theta + g\cos\theta\right]\mu\alpha \quad (3)$$

Where θ corresponds to the angle related with the horizontal of the free surface. The value of α can be $-1 \leq \alpha \leq 1$, depending on the direction of the friction force, the Froude number, the history how we reached this value together with the initial conditions. Rearranging terms, scaling the coordinates (z and r) with the radius, R, of a cylindrical or rectangular container (as in the previous experiments), and introducing the Froude number, F_r, defined in previous section, we obtain from equation 2 a dimensionless differential equation for the surface slope as:

$$\tan\theta = \frac{dz}{dr} = \frac{F_r r - \mu\alpha}{1 + \mu\alpha F_r r} \qquad (4)$$

Assuming we increase slowly the Froude number from zero, there is a critical value of the Froude number, $F_r^+ = \mu$, below which the surface does not show any deformation. The superscript plus sign in the Froude number indicates that the motion state results for increasing F_r. As the Froude number increases, the critical state is obtained automatically ($\alpha = 1$) for $r \geq \mu/F_r^+$ due to the centrifugal force. However, this critical state diffuses towards the center and can be achieved also in regions where for $r_c \leq r \geq \mu/F_r^+$ due to micro-avalanches occurring to replace the granular material removed outwards in the critical region. The value of r_c can be obtained by using the overall mass conservation. In the non-critical region $0 \leq r \leq r_o$, the surface remains with the initial shape, which in this case is horizontal, with a value of the friction parameter $\alpha = r$. Therefore, the value of α jumps from r_c to 1 at $r = r_c$ and can be given in the whole range as:

$$\alpha = r + H(r - r_c)(1 - r) \qquad (5)$$

Where $H(\xi)$ represents the Heaviside function. Equation 4 with $\alpha = 1$ can be integrated through the critical region, $r_c \leq r \leq 1$, giving

$$z(r) - z_c = \frac{1}{\mu}(r - r_c) - \frac{1+\mu^2}{\mu^2 F_r}\ln\left(\frac{1+\mu F_r r}{1+\mu F_r r_c}\right) \qquad (6)$$

Here, z_c, corresponds to the value of z at $r = r_c$. For the subcritical region, $0 \leq r \leq r_c$, we get $z = z_0 = H/R$. The maximum slope of the free surface can be obtained in the limit of very large Froude number from equation 6, resulting:

$$\left.\frac{dz}{dr}\right|_{max} = \frac{1}{\mu}$$

Overall mass conservation of granular material assuming an incompressible medium, can be written as:

$$\int_{r_c}^{1}\left[z(r) - z_0\right]dr = 0 \qquad (7)$$

Introducing equation 6 into equation 7, we get:

$$I = (z_c - z_0)(1 - r_c) + \frac{1}{2\mu}(1 - r_c)^2 - \frac{1+\mu^2}{F_r^{+2}\mu^3} \left[(1 + F_r^+\mu)\ln(1 + F_r^+\mu) - (1 + F_r^+\mu r_c)\ln(1 + F_r^+\mu r_c) - F_r^+\mu(1 - r_c)\right] = 0 \qquad (8)$$

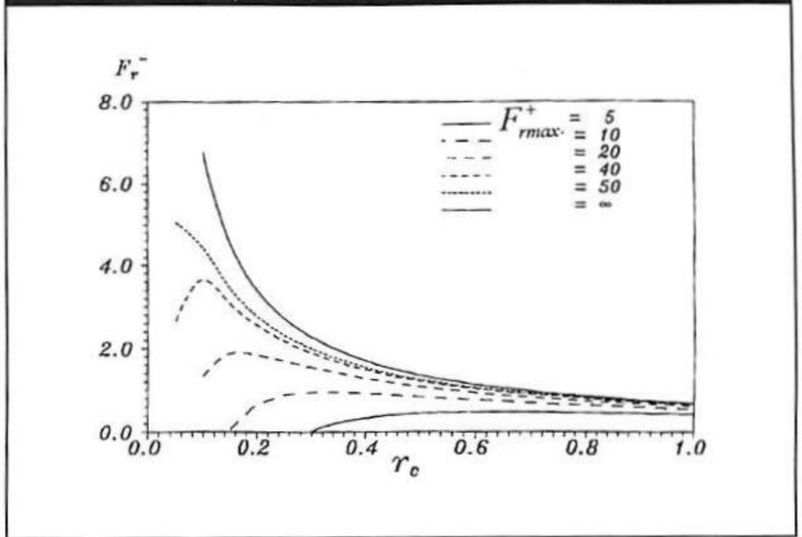
Equation 8 gives a relationship of the form,

$$I(z_c, r_c, F_r^+) = 0$$

For a Froude number such as $\mu \leq F_r^+ \langle F_{r1}^+$, there is an unperturbed subcritical region, resulting $z_c = z_0$, with $r_c(F_r^+)$ deduced from equation 8. There is a critical value of the Froude number, F_{r1}^+ that makes the whole region to be critical, that is $r_c = 0$, generating a peak at the center with slopes equal to $-\mu$. This critical value is obtained from equation 8 resulting a transcendental equation of the form $I(z_0, 0, F_{r1}^+) = 0$. An asymptotic relationship gives $F_{r1}^+ \approx 3\mu$, for $\mu \to 0$. In our case we get $F_{r1}^+ =$ 1.7918 for $\mu = 0.53$. Increasing further the Froude number, the value of z_c and the height of peak decreases vanishing the latter in the limit $F_r^+ \to \infty$. In the limiting case of $F_r^+ \to \infty$, z_c reaches a minimum value of $z_{cmin} = z_0 - 1/(2\mu)$, and the surface is represented by the straight lines with a slope of μ^{-1}. The surface profiles show a minimum at: $r = r_m = \mu/F_r^+ > r_c$. Practically, the peak disappears as r_m becomes of the magnitude order of the non-dimensional particle size, that is $F_r^+ = O(\mu R/d)$, where d corresponds to the particle diameter.

On the other hand, after reaching a maximum value of F_r^+, $F_{r\max}^+$, we decrease the Froude number slowly reaching again the zero Froude number. In this case, there are two different regions. The critical conditions now occur at the center of the bin, diffusing outwards as the Froude number, F_r^- de-

FIGURE 2. CRITICAL REGION RADIUS AS A FUNCTION OF THE ACTUAL FROUDE NUMBER, F_r^-, FOR DIFFERENT VALUES OF F^+_{rmax}

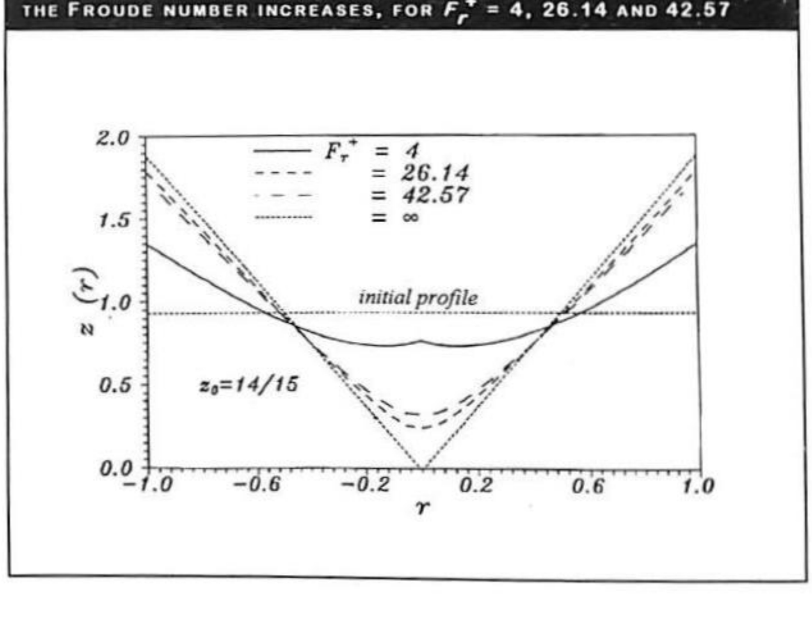


creases. This critical state is reached by micro-avalanches produced because the centrifugal force is unable to support the grains at the surface. In the now subcritical region, r > r_c, the slope of the surface does not change from the value obtained for the maximum Froude number. In the critical region, the slope of the surface can be obtained from equation 4, but with a value of α = -1. The position of r_c can be obtained by equating the slopes from equation 4 as follows:

$$\frac{F_r^- r_c + \mu}{1-\mu F_r^- r_c} = \frac{F^+_{rmax} r_c - \mu}{1+\mu F^+_{rmax} r_c} \tag{9}$$

Giving a quadratic equation for r_c. For finite values of $F^+_{r\,max}$, we can get as a result two values of r_c for a given value of. Criticality is reached automatically in the region $r_{c1} < r < r_{c2}$. However the central core

FIGURE 3. FREE SURFACE PROFILES OBTAINED FROM THE ANALYSIS AS THE FROUDE NUMBER INCREASES, FOR F_r^+ = 4, 26.14 AND 42.57



achieves also the critical state due to micro-avalanches from the critical region. Therefore, the critical region goes from r = 0 to r = r_c = r_{c2}. From equation 9 we can obtain the Froude number as a function of r_c as:

$$F_r^- = \frac{F^+_{rmax} r_c \left(1-\mu^2\right) - 2\mu}{2F^+_{rmax} r_c^2 \mu + r_c\left(1-\mu^2\right)} \tag{10}$$

In the limit $F^+_{r\,max} \to \infty$, equation 10 can be reduced to give r_c as a function of the actual Froude number as:

$$r_c = \frac{1-\mu^2}{2\mu F_r^-} \tag{11}$$

Figure 2 shows the value of r_c as a function of F_r^- for different values of $F^+_{r\,max}$. For r > r_c, the profile is the same as obtained with the maximum Froude number. For r < r_c the surface equation can be obtained from equation 4 with α = -1, resulting:

$$z(r) - z_c = \frac{1}{\mu}\left(r_c - r\right) - \frac{1+\mu^2}{\mu^2 F_r^-}\ln\left(\frac{1-\mu F_r^- r}{1-\mu F_r^- r_c}\right) \tag{12}$$

Where z_c can be obtained in the same form as before, using the overall mass conservation. Therefore, for the same Froude number reached from both sides, we get in this case two different surface's equations (*i. e.*, same as found experimentally). In general, it terms will be infinite number of possibilities depending on the history of how we got to that given Froude number, showing the non-linear character of the problem.

On the basis of this result, we can show some shapes of the surface of piles resulting from rotation. Assuming we that start the motion from rest with an initial flat horizontal surface, we obtain a peak at the center with decreasing height as the Froude number increases. Figure 3 shows two dimensional projections of surfaces generated by slowly increasing F_r^+. We take a value of μ = 0.53 (ϕ_c = 31°) in order to compare the theory with the experiments made with Ottawa sand. The values cf the chosen Froude number were: F_r^+ = 4.0, where a clear central peak is noted, F_r^+ = 26.14 and F_r^+ = $F^+_{r\,max}$ = 42.57. Figure 4 shows the height of the pile's center as a function of the Froude number. On the other hand, if we decrease the Froude number from $F^+_{r\,max}$ =

42.57, we get another type of solutions for the surface equation. Figure 5 shows the solutions for $F_r^- = 26.16$, $F_r^- = 4.0$ and finally, for $F_r^- = 0$, where we get the final state for the surface with a constant slope $\mu(\phi_c = 31°)$.

From equation 4 we recover also the newtonian fluid behavior in the case of $\mu = 0$ and $F_r \neq 0$, the solution that can be given in dimensionless form as:

$$z - z_0 = \frac{F_r}{2} r^2 \qquad (13)$$

III. Comparison between theory and experiments

A direct comparison between previous analysis and experimental results is given in figure 6, where we show the free surface shapes for $F_r^+ = 4$, $F_r^+ = 26.14$ and $F_r^+ = 42.57$. In the case of decreasing Froude numbers we also show in figure 7 a comparison between theory and experiment for the case of $F_r^- = 26.14$, $F_r^- = 4$ and $F_r^- = 0$. In all cases presented here, there is a very good agreement between theory and experiment, which confirms that the present model describes correctly the phenomenology of the experiment.

We should comment that the friction angle actually does not have an unique value. It fluctuates within a small range (Morales, *et. al.*, 1993), which in our experiments was: $\phi_c \pm \delta$, where $\delta \sim 1°$. For each Froude number, the experimental results deviate a few per cent (less than 2%) in the surface shape profiles.

IV. Remarks and conclusions

The problem of the rotation of granular material on the vertical axis, in general; is a very complex phenomenon because it takes into account not only the gravitational as well the centrifugal force, but also the history of the motion through the friction force. However, the history or memory effect disappears for continuously slow increasing or decreasing rotation, as the grain achieves the critical state everywhere. In this case, from a continuum point of view, this problem can be understood and a simple analysis can describe correctly the motional behavior. Even in the case of slowly changing the Froude number, the system shows hysteresis indicating the

Figure 4. Critical region height for increasing Froude numbers

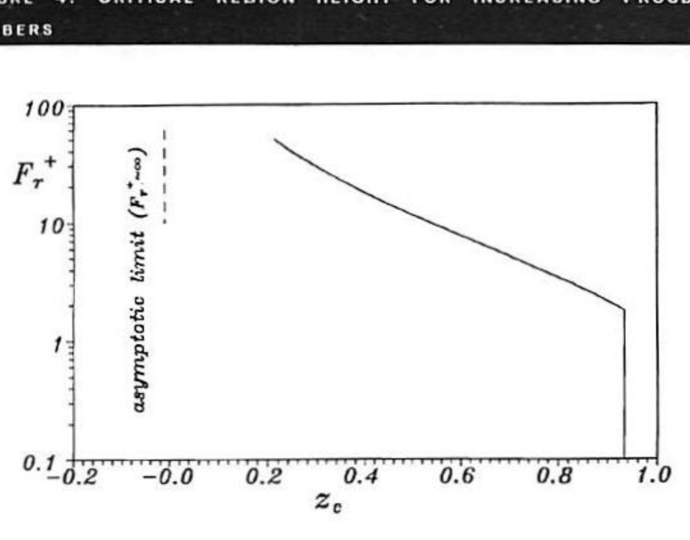


Figure 5. Free surface profiles obtained from the analysis as the Froude number decreases, for F_r^- = 0, 1, 4 and 26.14

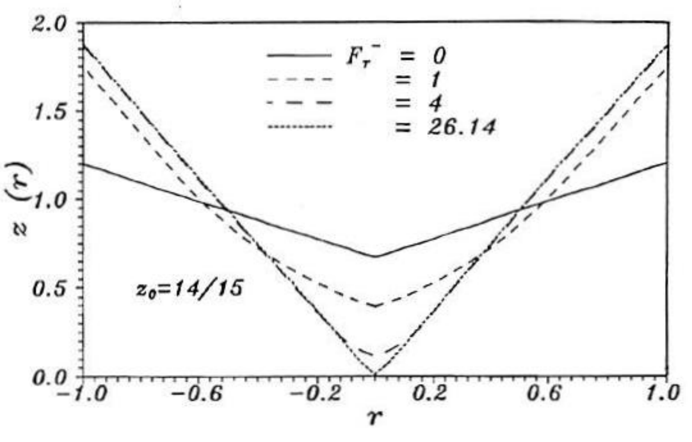


Figure 6. Comparison of experimental results and theory for different values of F_r^+

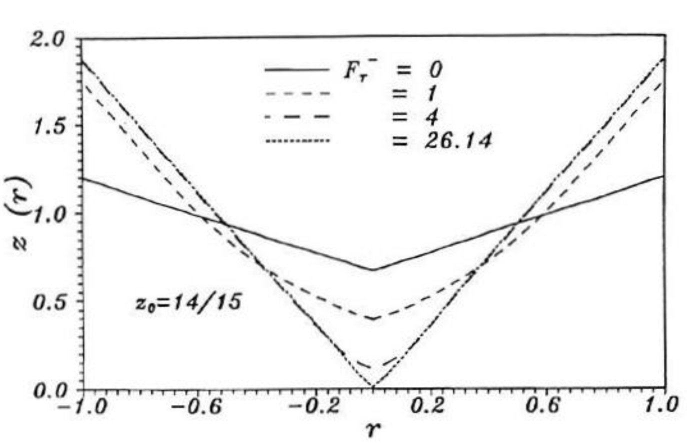
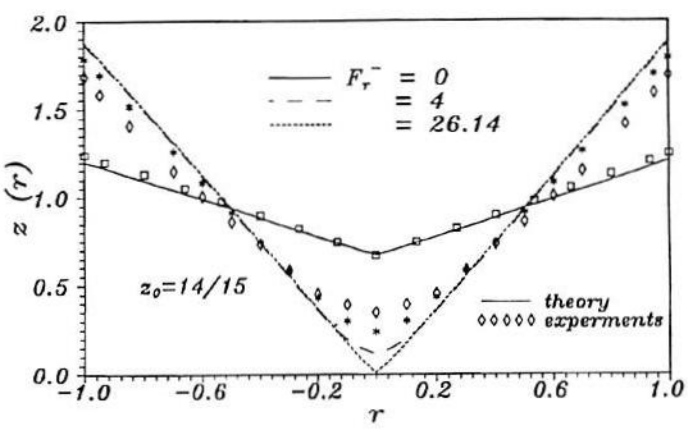


FIGURE 7. COMPARISON OF EXPERIMENTAL RESULTS AND THEORY FOR DIFFERENT VALUES OF F_r^-



existence of multiple steady-state solutions. In the more general case of reaching a given value of the Froude number though any arbitrary way (rapid step type changes), is possible to obtain any number of steady-state solutions. Hysteresis in avalanche processes is related to the changes in the slope near the maximum angle and the frictional and packing factors inside of piles. In the problem described in this work the hysteresis behavior is related with thesefactors but additionally the initial and boundary conditions. ◆

BIBLIOGRAPHY

Bagnold, A. (1966). "The shearing and dilatation of dry sand and the *singing* mechanism", *Proc. R. Soc.* London Ser. A 225, 219.

Baxter, W.; Leone, R. and Behringer, R. (1993). "Experimental Test of time scales in flowing sand", *Europhys. Lett.* 21, 569.

Bayly, B. (1991). "Complexity in fluids", in Nadel, L. and Stein, D. (Eds.) *Lectures in Complex Systems, SFI Studies in Sciences of Complexity,* Lect. Vol. III., Addison-Wesley.

Bridgewater, J.; Foo, S. and Stephens, J. (1985). "Fundamental powder mechanisms", *Powder Technology* 41, 147.

Coulomb, A. (1773). *Memoires de Mathématiques et de Physique Présentés à l'Académie Royale des Sciences par divers Savants et Lus dans les Assémbles,* L'imprimerie Royale, Paris p. 343.

Das Gupta, S.; Kharkhar, V. and Bhatia, K. (1991). "Axial segregation of particles in a horizontal rotating cylinder", *Chem. Eng. Sci.* 46, 1513.

Edwards, F. and Oakeshott, S. (1989). "The transmission of stress in an aggregate", *Physica* D 38, 88.

Haff, K. (1983). "Grain flow as a fluid-mechanical phenomenon", *J. Fluid Mech.* 134, 401.

Jackson, R. (1983). "Some mathematical and physical aspects of continuum models for the motion of granular materials", in Meyer, E. (Ed.) *Theories of Dispersed Multiphase Flow,* Academic Press.

Landau, D. and Lifshits, M. (1959). *Fluid Mechanics,* Pergamon Press, London.

Mehta, A. (1992). "Real sandpiles: hysteresis and cooperative dynamics", *Physica* A 86, 121.

Morales, E., *et. al.* (1993). "Avalanches in two-dimensional systems", *Phys. Rev. E.* 47, R2229.

Nederman, M.; Tuzun, U.; Savage, B. and Houlsby, T. (1982). "The flow of granular materials I: discharge rates from hoppers", *Chem. Eng. Sci.* 37, 1597.

Onoda, Y. and Liniger, G. (1990). "Random loose packings of uniform spheres and the dilatance onset", *Phys. Rev. Lett.* 64, 2727.

Rajchenbach, J. (1990). "Flow in powders: from discrete avalanches to continuous regime", *Phys. Rev. Lett.* 65, 2221 (1990).

Savage, B. (1989). "Flow of granular materials", in Germain, P. and Caillere, D. (Eds.) *Theoretical and applied mechanics,* Elsevier.

Schofield, A. and Wroth, P. (1968). *Critical State Soil Mechanics,* McGraw-Hill.

See for example: Jullien, R.; Meakin, P. and Pavlovitch, A. (1992). "Three dimensional model for particle-size segregation by shaking", *Phys. Rev. Lett.* 69 640. and Zik, O. *et. al.* (1994). "Rotationally induced segregation of granular materials", *Phys Rev. Lett.* 73, 644.

Sokolovskii, V. (1965). *Statics of Granular Media,* Pergamon Press. London.

Wieghardt, K. (1975). "Experiments in granular flow", *Ann. Rev. Fluid Mech.* 7, 89.

Zierep, J. (1971). *Simmilarity Laws and Modeling,* Marcel Dekker Publishing.