Making Holes in the Second Symmetric Products of Dendrites and Some Fans

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Haciendo hoyos en los segundos productos simétricos de dendritas y algunos abanicos

Resumen. Sea X un continuo métrico tal que el segundo producto simétrico de X, $F_2(X)$ es unicoherente. Sea $A \in F_2(X)$, A se dice que hace un hoyo a $F_2(X)$, si $F_2(X) - \{A\}$ no es unicoherente. En este artículo, caracterizamos a los elementos $A \in F_2(X)$ tales que A hace un hoyo a $F_2(X)$, donde X es una dendrita o un abanico homeomórfico al cono sobre un espacio métrico compacto.

Palabras clave: continuo, producto simétrico, propiedad *b*), unicoherencia.

Abstract. Let X be a metric continuum such that the second sym¬metric product of X, $F_2(X)$, is unicoherent. Let $A \in F_2(X)$, A is said to make a hole in $F_2(X)$, if $F_2(X) - \{A\}$ is not unicoherent. In this paper, we characterize the elements $A \in F_2(X)$ such that A makes a hole in $F_2(X)$, where X is either a dendrite or a homeomorphic fan to the cone over a compact metric space.

Key words: continuum, symmetric products, property *b*), unicoherence.

Introduction

A connected topological space Z is *unicoherent* if whenever $Z = A \cup B$, where A and B are closed, connected subsets of Z, we have $A \cap B$ is connected. Let Z be a unicoherent topological space and let z be an element of Z, we say that z *makes a hole in* Z if $Z - \{z\}$ is not unicoherent. A *compactum* is a nondegenerate compact metric space. A *continuum* is a connected compactum. Given a continuum X, we define its *hyperspaces*: $F_n(X)$ as the set of all nonempty subsets A of X such that A has at most n points, for each positive integer n. Such hyperspaces are considered with the Hausdorff metric.

S. Macías, in (Macías, 1999. Theorem 8), proved that, if X is a continuum and n is an integer bigger than two, then $F_n(X)$ is unicoherent. E. Castañeda in (Castañeda, 1998) gave a unicoherent continuum X such that $F_2(X)$ is not unicoherent.

The following problem arises in (Anaya, 2007):

Problem. Let $\mathcal{H}(X)$ be a hyperspace of X. For which elements $A \in \mathcal{H}(X)$, A makes a hole in $\mathcal{H}(X)$.

Some partial solutions of this problem are presented in (Anaya, 2007), (Anaya, 2011) and (Anaya *et al.*, 2010). In the current paper, we are presenting the solution to this problem when X is either a dendrite or a fan homeomorphic to the cone over a compactum and $\mathcal{H}(X) = F_2(X)$.

1. Notation and auxiliary results

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the set of positive integers and the set of real numbers, respectively. Let Z be a topological space and let A be a subset of Z, the symbol $cl(A)$ denotes the closure of A in Z. An *arc* is any space homeomorphic to [0, 1]. A *free arc* in a continuum X is an arc pq, where p and q are the end points of pq, such that $pq - \{p, q\}$ is open in X. A point z in a connected topological space Z is a *cut point* of (*non-cut point of*) Z provided that $Z - \{z\}$ is disconnected (is connected). A map $f : Z \to S^1$, where Z is a topological connected space and S^1 is the unit circle in the Euclidean plane $\mathbb{R}^2$, *has a lifting* if there exists a map $h : Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the map of $\mathbb{R}$ onto S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. A connected topological space Z *has property b*) if each $f : Z \to S^1$ has a lifting.

Let Y be a continuum arcwise connected, by an *end point of* Y, we mean an end point in the classical sense, which means a point p of Y that is a non-cut point of any arc in Y that contains p, the set of all end points of Y is denoted by $E(Y)$. A point p of a continuum X is a *ramification point* provided that p is a point which is a common end point of three or more arcs in X that are otherwise disjoint and the set of all ramification point of X is denoted by $R(X)$.

A metric space X is called *Peano space* provided that for each $p \in X$ and each neighborhood V of p, there exists a connected open subset U of X such that $p \in U \subset V$. A Peano continuum X is said to be a *dendrite* if X contains no simple closed curve. A *fan* is an arcwise connected, hereditarily unicoherent continuum with exactly one ramification point (*hereditarily unicoherent* means each subcontinuum is unicoherent).

A subspace Y of a topological space Z is a *deformation retract* of Z if there exists a map $H : Z \times [0, 1] \to Z$ such that, for each $x \in Z$, $H(x, 0) = x$, $H(Z \times \{1\}) = Y$ and, for each $y \in Y$, $H(y, 1) = y$. We say that a topological space, Z, is *contractible* if there exists $z \in Z$, such that $\{z\}$ is a deformation retract of Z.

Let Z be a topological space. Given two subsets K_1 and K_2 of Z, we define $\langle K_1, K_2 \rangle = \{\{x, y\} \subset Z : x \in K_1$ and $y \in K_2\}$.

The following proposition is easy to prove.

Proposition 1.1. If X is a continuum and K_1, K_2 are subcontinua of X, then $\langle K_1, K_2\rangle$ is a subcontinuum of $F_2(X)$, and it does not have cut points, when K_1, K_2 are nondegenerate.

Proposition 1.2. Let X be a Peano continuum and let $p \in X$ be such that $X - \{p\}$ has at least three components. Then there exist two nondegenerate subcontinua Y_1 and Y_2 of X such that p is a cut point of either Y_1 or Y_2, $Y_1 \cap Y_2 = \{p\}$ and $Y_1 \cup Y_2 = X$.

Proof. Let F_0 be a component of $X - \{p\}$ and let $K_0 = \bigcup\{C \subseteq X - \{p\} : C$ is a component of $X - \{p\}$ and $C \neq F_0\}$. We consider $Y_1 = F_0 \cup \{p\}$ and $Y_2 = K_0 \cup \{p\}$. It is easy to see that Y_1 and Y_2 satisfy the required properties. □

Proposition 1.3. Let X be a Peano continuum and let p and q be different cut points of X. Then there exist three nondegenerate subcontinua Q_1, Q_2 and Q_3 of X such that p and q are non-cut points of Q_1, $p \in Q_2$, $q \in Q_3$, $Q_1 \cap Q_2 = \{p\}$, $Q_1 \cap Q_3 = \{q\}$, $Q_2 \cap Q_3 = \emptyset$ and $X = Q_1 \cup Q_2 \cup Q_3$.

Proof. Let C_0 be the component of $X - \{p\}$ such that $q \in C_0$. Notice that $C_0 \cup \{p\}$ is a subcontinuum of X. Since q is a cut point of X, q is a cut point of $C_0 \cup \{p\}$. Let D_0 be the component of $(C_0 \cup \{p\}) - \{q\}$ such that $p \in D_0$.

We consider $Q_1 = D_0 \cup \{q\}$, $Q_2 = \bigcup\{C \subset X - \{p\} : C$ is a component of $X - \{p\}$ and $C \neq C_0\} \cup \{p\}$ and $Q_3 = \bigcup\{D \subseteq (C_0 \cup \{p\}) - \{q\} : D$ is a component of $(C_0 \cup \{p\} - \{q\}$ and $D \neq D_0\} \cup \{q\}$. It is not difficult to show that Q_1, Q_2 and Q_3 satisfy the required properties. □

2. Dendrites

Given a dendrite X. It is known that $F_2(X)$ is unicoherent (see Ganea, 1954). In this section, we characterize those elements A of $F_2(X)$ such that A makes a hole in $F_2(X)$.

Given $x \in X$. The order of x in X we mean the Menger-Urysohn order, see (Kuratowski, 1968, §51, I, p. 274), or equivalently (see, for example Kuratowski, 1968, §51, I, p. 274) the classical sense, *i. e.*, the number of arcs emanating from x and disjoint out of x (see Charatonik, 1962, p. 229 and Lelek, 1961, p. 301), we will denote it by $\text{ord}(x, X)$. This number is equal to the number of components of $X - \{x\}$ (see Whyburn, 1942: 11, (1.1), (iv), p. 88). Notice that the points of order 1 are end points of X, the points of order 2 or more are cut points of X and the points of order 3 or more are ramification points of X. The symbol $R_2(X)$ denotes the set of all points of X of order 2. Notice that, $X = E(X) \cup R_2(X) \cup R(X)$.

Throughout this section, X will denote a dendrite.

Theorem 2.1. Let p be a ramification point of X. Then $\{p\}$ makes a hole in $F_2(X)$.

Proof. By Proposition 1.2, there exist two nondegenerate subcontinua Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \{p\}$ and p is a cut point of either Y_1 or Y_2. Suppose that p is a cut point of Y_1. Let $\mathcal{A}_1 = F_2(Y_1) - \{\{p\}\}$ and $\mathcal{A}_2 = (F_2(Y_2) \cup \langle Y_1, Y_2\rangle) - \{\{p\}\}$.

Note that $\mathcal{A}_1 = F_2(Y_1) \cap (F_2(X) - \{\{p\}\})$, $\mathcal{A}_2 = (F_2(Y_2) \cup \langle Y_1, Y_2\rangle) \cap (F_2(X) - \{\{p\}\})$. Then, by Proposition 1.1, $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed subsets of $F_2(X) - \{\{p\}\}$. Let $q \in Y_2 - \{p\}$. Then $\{p, q\} \in (F_2(Y_2) \cap \langle Y_1, Y_2\rangle) - \{\{p\}\}$.

Given $i \in \{1, 2\}$. Note that $F_2(Y_i) - \{\{p\}\} = \langle Y_i, Y_i\rangle - \{\{p\}\}$. Then, by Proposition 1.1, $F_2(Y_i) - \{\{p\}\}$ and $\langle Y_1, Y_2\rangle - \{\{p\}\}$ are connected. So $\mathcal{A}_1$ and $\mathcal{A}_2$ are connected. Clearly, $F_2(X) - \{\{p\}\} = \mathcal{A}_1 \cup \mathcal{A}_2$. Notice that $\mathcal{A}_1 \cap \mathcal{A}_2 = \langle\{p\}, Y_1\rangle - \{\{p\}\}$ and $\{p\}$ is a cut point of $\langle\{p\}, Y_1\rangle$. So $\mathcal{A}_1 \cap \mathcal{A}_2$ is not connected. Hence, $F_2(X) - \{\{p\}\}$ is not unicoherent. □

Theorem 2.2. Let p and q be different cut points of X. Then $\{p, q\}$ makes a hole in $F_2(X)$.

Proof. By Proposition 1.3, there exist three nondegenerate subcontinua Q_1, Q_2 and Q_3 of X such that p and q are non-cut points of Q_1, $p \in Q_2$, $q \in Q_3$, $Q_1 \cap Q_2 = \{p\}$, $Q_1 \cap Q_3 = \{q\}$, $Q_2 \cap Q_3 = \emptyset$ and $X = Q_1 \cup Q_2 \cup Q_3$.

We consider the sets $\mathcal{A}_1 = (F_2(Q_1) \cup F_2(Q_2) \cup F_2(Q_3)) - \{\{p, q\}\}$ and $\mathcal{A}_2 = (\langle Q_1,Q_2\rangle \cup \langle Q_1,Q_3\rangle \cup \langle Q_2,Q_3\rangle) - \{\{p, q\}\}$. Clearly, $F_2(X) - \{\{p, q\}\} = \mathcal{A}_1 \cup \mathcal{A}_2$. Using Proposition 2.1, it can be proved that $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed and connected subsets of $F_2(X) - \{\{p, q\}\}$. Since $\mathcal{A}_1 \cap \mathcal{A}_2 = (\langle\{p\}, Q_1 \cup Q_2\rangle \cup \langle\{q\}, Q_1 \cup Q_3\rangle) - \{\{p, q\}\}$, $\mathcal{A}_1 \cap \mathcal{A}_2$ is disconnected. Therefore $F_2(X) - \{\{p, q\}\}$ is not unicoherent. □

Theorem 2.3. Let p be an end point of X and let $q \in X$. If $p \notin cl(R(X))$, then $\{p, q\}$ does not *make a hole* in $F_2(X)$.

Proof. Since $p \notin cl(R(X))$, there exists an arc I contained in $R_2(X) \cup \{p\}$ such that $p \in E(I)$ and $I \neq X$. Let $v \in E(I) - \{p\}$. Let $f : I \to [0, 1]$ be a homeomorphism such that $f(p) = 1$ and $f(v) = 0$. We define the function $g : I \times [0, 1] \to I$ by $g(x, t) = f^{-1}((1 - t)f(x))$. Notice that g is a map such that, for each $t \in [0, 1]$, $g(v, t) = v$, for each $x \in I$, $g(x, 0) = x$ and $g(x, 1) = v$. Moreover,

$$g(x, t) = p \text{ if and only if } x = p \text{ and } t = 0. \quad (1)$$

We define the function $h : X \times [0, 1] \to X$ by

$$h(x, t) = \begin{cases} g(x, t), & \text{if } x \in I, \\ x, & \text{if } x \in cl(X - I). \end{cases}$$

Since $cl(X - I) = (X - I) \cup \{v\}$, it is easy to prove that h is a map. Notice that, for each $y \in cl(X - I)$, $h(y, 1) = y$ and, for each $x \in X$, $h(x, 0) = x$ and $h(x, 1) \in cl(X - I)$. Then $cl(X - I)$ is a deformation retract of X.

Now, we define $G : (F_2(X) - \{\{p, q\}\}) \times [0, 1] \to F_2(X) - \{\{p, q\}\}$ by

$$G(\{x, y\}, t) = \{h(x, t), h(y, t)\}.$$

By (1), G is well defined. It is easy to prove that G is continuous. Notice that, for each $\{x, y\} \in F_2(X) - \{\{p, q\}\}$, $G(\{x, y\}, 0) = \{x, y\}$, $G((F_2(X) - \{\{p, q\}\}) \times \{1\}) = F_2(cl(X - I))$ and, for each $\{u, v\} \in F_2(cl(X - I))$, $G(\{u, v\}, 1) = \{u, v\}$. Hence, $F_2(cl(X - I))$ is a deformation retract of $F_2(X) - \{\{p, q\}\}$. Since $cl(X - I)$ is a dendrite (see Nadler 1992, 10.6 Corollary, p. 167), $F_2(cl(X - I)$ is unicoherent. So, $F_2(X) - \{\{p, q\}\}$ is unicoherent (see Eilenberg, 1936, §3. Theorem 7, p. 73). □

Given a continuum X, 2^X and $C(X)$ will denote the hyperspace of all closed and nonempty subsets of X and the hyperspace of all nonempty subcontinua of X, respectively. Let $\mathcal{K}(X) \subseteq 2^X$. A *Whitney map for* $\mathcal{K}(X)$ is a map $\mu : \mathcal{K}(X) \to [0, 1]$ that satisfies the following two conditions:

1. or any $A, B \in \mathcal{K}(X)$ such that $A \subseteq B$ and $A \neq B$, $\mu(A) < \mu(B)$;
2. $\mu(A) = 0$ if and only if $A \in \mathcal{K}(X) \cap \{\{x\} : x \in X\}$.

Let $w \in X$. Then X is *arc-smooth at* w provided that there exists a continuous function $\alpha_w : X \to C(X)$ that satisfies the following conditions:

1. $\alpha_w(w) = \{w\}$,
2. for each $y \in X - \{w\}$, $\alpha_w(y)$ is an arc from w to y, and
3. if $x \in \alpha_w(y)$, then $\alpha_w(x) \subseteq \alpha_w(y)$.

Theorem 2.4. Let p be an end point of X and let $q \in X$. If $p \in cl(R(X))$, then $\{p, q\}$ does not make a hole in $F_2(X)$.

Proof. We can consider $w_0 \in X$ such that it is either any point of $R(X)$, if $p = q$, or an element of $R(X)$ such that p and q belong to different components of $X - \{w_0\}$, if $p \neq q$. Let I be the arc joining p and w_0 in X. Given $w \in R(X) \cap I - \{w_0\}$. Since X is arc-smooth at w (see Illanes and Nadler 1999. p. 226), there exists α_w as above. Let $W(w)$ be the subcontinuum of X such that $W(w) \cap I = \{w\}$ and $X - W(w)$ has two component. Let μ be a Whitney map for $C(X)$ (see Illanes and Nadler 1999. Theorem 13.4, p. 107). We define the following functions $L_w : W(w) \to [0, 1]$ and $g_w : W(w) \times [0, 1] \to W(w)$ by $L_w(x) = \mu(\alpha_w(x))$ and

$$g_w(x, t) = \begin{cases} x, & \text{if } L_w(x) \leq 1 - t, \\ \text{the unique point } y \in \alpha_w(x) & \\ \text{such that } L_w(y) = 1 - t, & \text{if } L_w(x) \geq 1 - t, \end{cases}$$

respectively. We are going to prove that g_w is continuous. Let $\{(x_n, t_n)\}_{n=1}^{\infty}$ be sequence in $W(w) \times [0, 1]$ and let $(x_0, t_0) \in W(w) \times [0, 1]$ be such that $\lim(x_n, t_n) = (x_0, t_0)$. Taking subsequences if necessary, we may consider the following two cases:

Case 1. $L_w(x_n) \leq 1 - t_n$ for each n.

Since L_w is a map and $\lim t_n = t_0$, $L_w(x_0) \leq 1 - t_0$. Hence, $\lim g_w(x_n, t_n) = \lim x_n = x_0 = g_w(x_0, t_0)$.

Case 2. $L(x_n) \geq 1 - t_n$ for each n.

Let $y_n \in \alpha_w(x_n)$ be such that $L_w(y_n) = 1 - t_n$. Taking subsequences if necessary, we may suppose that there exists $y_0 \in X$ such that $\lim y_n = y_0$. Then $\alpha_w(y_0) = \lim \alpha_w(y_n) \subseteq \lim \alpha_w(x_n) = \alpha_w(x_0)$. Hence, $y_0 \in \alpha_w(x_0)$. Since L_w is a map, $L_w(x_0) \geq 1 - t_0$ and $L(y_0) = \lim L(y_n) = \lim(1 - t_n) = 1 - t_0$. Therefore $g_w(x_0, t_0) = y_0 = \lim y_n = \lim g_w(x_n, t_n)$.

We conclude that g_w is a map. Notice that, for each $x \in W(w)$, $g_w(x, 0) = x$ and $g_w(x, 1) = w$, and for each $t \in [0, 1]$, $g_w(w, 1) = w$. Then $\{w\}$ is a deformation retract of $W(w)$.

Let Y be the subcontinuum of X such that $Y \cap I = \{w_0\}$ and $X - Y$ has one component. We define the function $g : X \times [0, 1] \to X$ by

$$g(x, t) = \begin{cases} g_w(x, t), & \text{if } x \in W(w) \text{ for some } w \in R(X) \cap I) - \{w_0\}, \\ x, & \text{if } x \in Y \cup I. \end{cases}$$

Clearly, g is well defined. In order to proof that g is continuous, we consider $\{(x_n, t_n)\}_{n=1}^{\infty}$ a sequence in $X \times [0, 1]$ and $(x_0, t_0) \in X \times [0, 1]$ such that $\lim(x_n, t_n) = (x_0, t_0)$. Taking subsequences if necessary, we may consider the following three cases:

Case 1. $x_n \in Y \cup I$ for each n.

Since $Y \cup I$ is a closed subset of X, $x_0 \in Y \cup I$. So, $\lim g(x_n, t_n) = g(x_0, t_0)$.

Case 2. There exists $w \in (R(X) \cap I) - \{w_0\}$ such that $x_n \in W(w)$ for each n.

Since $W(w)$ is a closed subset of X, $x_0 \in W(w)$. Then $\lim g(x_n, t_n) = \lim g_w(x_n, t_n) = g_w(x_0, t_0) = g(x_0, t_0)$.

Case 3. For each n, there exists $w_n \in (R(X) \cap I) - \{w_0\}$ such that $x_n \in W(w_n)$ and $w_n \neq w_m$, if $n \neq m$.

We may assume that there exist $A \in C(X)$ and $y \in I$ such that $\lim W(w_n) = A$ and $\lim w_n = y$. Then $y \in A$. We prove that $A = \{x_0\}$. Let $z \in A$. Then there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of X such that $\lim z_n = z$ and $z_n \in W(w_n)$ for each n. We consider α_y and α_z as above (see Illanes and Nadler 1999. p. 226). By the continuity of α_y and α_z, $\{y\} = \alpha_y(y) = \lim \alpha_y(w_n)$ and $\lim \alpha_z(z_n) = \alpha_z(z) = \{z\}$. Since X is a dendrite, $\alpha_y(w_n) \subseteq \alpha_z(z_n)$ for each n. Hence, $z = y$. Since $x_0 \in A$, $A = \{x_0\}$. Then $\lim W(w_n) = \{x_0\}$. Since $g_{w_n}(x_n, t_n) \in W(w_n)$ for each n, $\lim g(x_n, t_n) = \lim g_{w_n}(x_n, t_n) = x_0 = g(x_0, t_0)$.

Hence, g is a map. Since for each $x \in X$, $g(x, 0) = x$ and $g(x, 1) \in Y \cup I$, and for each $y \in Y \cup I$, $g(y, 1) = y$, $Y \cup I$ is a deformation retract of X. Notice that

$$g(x, t) = p \text{ if and only if } x = p \qquad (2)$$

and

$$g(y, t) = q \text{ if and only if } y = q \qquad (3)$$

We consider the function $G : F_2(X) - \{\{p, q\}\}) \times [0, 1] \to F_2(X) - \{\{p, q\}\}$by

$$G(\{x, y\}, t) = \{g(x, t), g(y, t)\}.$$

By (2) and (3), G is well defined. It can be proved that G is a map. Notice that, for each $\{x, y\} \in F_2(X) - \{\{p, q\}\}$, $G(\{x, y\}, 0) = \{x, y\}$, $G((F_2(X) - \{\{p, q\}\}) \times \{1\}) = F_2(Y \cup I)$ and, for each $\{u, v\} \in F_2(Y \cup I)$, $G(\{u, v\}, 1) = \{u,v\}$. Then $F_2(Y \cup I) - \{\{p, q\}\}$ is a deformation retract of $F_2(X) - \{\{p, q\}\}$. Since $Y \cup I$ is a dentrite (see Nadler 1992, 10. 6 Corollary, p. 167) and $p \notin cl(R(Y \cup I)$, by Theorem 2.3, $F_2(Y \cup I) - \{\{p, q\}\}$ is unicoherent. Therefore $F_2(X) - \{\{p, q\}\}$ is unicoherent (see Eilenberg, 1936, §3, Theorem 7, p. 73). □

The proof of the following Corollary follows from the Theorems 2.3 and 2.4.

Corollary 2.5. Let p be an end point of X and let $q \in X$. Then $\{p, q\}$ does not make a hole in $F_2(X)$.

Theorem 2.6. Let $p \in X$ such that $ord(p, X) = 2$. Then $\{p\}$ does not make a hole in $F_2(X)$.

Proof. It can be proved that there exist two nondegenerate subcontinua F and K of X such that $p \in E(F) \cap E(K)$ and $X = F \cup K$. So, $F_2(X) - \{\{p\}\} = (F_2(F) - \{\{p\}\}) \cup (F_2(K) - \{\{p\}\}) \cup (\langle F, K\rangle - \{\{p\}\})$. By Corollary 2.5, $F_2(F) - \{\{p\}\}$ and $F_2(K) - \{\{p\}\}$ are unicoherent.

We can prove that $\langle F, K\rangle - \{\{p\}\}$ is homeomorphic to $F \times K - \{(p, p)\}$. Since F and K are dendrites (see Nadler 1992, 10.6 Corollary, p. 167) and $p \in E(F) \cap E(K)$, it can be proved that $F - \{p\}$ and $K - \{p\}$ are contractibles. Then $F - \{p\}$ and $K - \{p\}$ are unicoherent (see Eilenberg, 1936, §3, Theorem 7, p. 73). By Theorem 5 of (Eilenberg, 1936, §3, p. 72), $(F - \{p\} \times (K - \{p\})$, $(F - \{p\}) \times \{p\}$ and $\{p\}\times (K - \{p\})$ are unicoherent. Hence, $((F - \{p\} \times (K - \{p\})) \cup ((F - \{p\}) \times \{p\}) \cup (\{p\} \times (K - \{p\})) = F \times K - \{(p, p)\}$ is unicoherent (see Eilenberg, 1936, §3, Theorem 4, p. 72).

Since $F_2(F) - \{\{p\}\}$, $F_2(K) - \{\{p\}\}$ and $\langle F, K\rangle - \{\{p\}\}$ are unicoherent, $F_2(X) - \{\{p\}\}$ is unicoherent (see Eilenberg, 1936, §3, Theorem 4, p. 72).

2.1. Classification

Theorem 2.7. Let X be a dendrite and let $\{x, y\} \in F_2(X)$. Then $\{x, y\}$ makes a hole in $F_2(X)$ if and only if $x \neq y$ and neither x nor y is an end point of X, or $x = y$ and x is a ramification point.

Proof. Necessity, let x and y be elements of X such that $\{x, y\}$ makes a hole in $F_2(X)$. If $x = y$, by Theorem 2.6 and Corollary 2.5, x is a ramification point of X. On the other hand, if $x \neq y$, by Corollary 3.5, neither x nor y is an end point of X.

The sufficiency follows from Theorems 2.1 and 2.2. □

3. Fans

In this section, we characterize those elements A of $F_2(X)$ such that A makes a hole in $F_2(X)$, when X is a fan homeomorphic to the cone over a compactum.

The unique ramification point of a fan X is called the *top of* X, τ always denotes the top of a fan.

Whenever X is a fan homeomorphic to the cone over a compactum, S. Macías proved that, for each $n \geq 2$, $F_n(X)$ is homeomorphic to the cone over a continuum (see Macías, 2003). Then, for each $n \geq 2$, $F_n(X)$ is contractible (see Rotman, 1998. Theorem 1.11, p. 23). Hence, given $n \geq 2$, $F_n(X)$ has property *b*) (see Anaya, 2007. Proposition 9, p. 2001), and so it is unicoherent (see Eilenberg, 1936, Theorems 2 and 3, pp. 69 and 70).

Let X be a fan which is homeomorphic to the cone over a compactum. We may assume that X is embedded in $\mathbb{R}^2$ (see Eberhart, 1969, Corollary 4, p. 90 and Charatonik, 1967, Theorem 9, p. 27), $\tau = (0, 0)$ is the top of X, the legs of X are convex arcs of length one (see Macías and Nadler 2002, 4.2). Let $E(X) = \{e_\lambda\}_{\lambda \in \Lambda}$. Then X is the cone over $E(X)$ (see Macías and Nadler 2002, 4.2). Given two points a and b of $\mathbb{R}^2$, $[a, b]$ denotes the convex arc in $\mathbb{R}^2$ whose end points are a and b, and $||a||$ denotes the norm of a in $\mathbb{R}^2$. Each leg of X, $[\tau, e_\lambda]$, is parameterized for $\{re_\lambda : r \in [0, 1]\}$. Note that for $r = 0$, $re_\lambda = \tau$.

A fan X with top τ is said to be *smooth* provided that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to a point $x \in X$, then the sequence $\{\tau x_n\}_{n=1}^{\infty}$ of the arcs in X converges, in the hyperspace of subcontinua of X, to the arc τx. For example, the cone over a compact totally disconnected metric space is easily seen to be a smooth fan. Conversely, it is shown in (Eberhart, 1969) that every smooth fan is homeomorphic with a subcontinuum of the Cantor fan.

Throughout this section, X will denote a fan which is homeomorphic to the cone over a compactum.

We denote $Cut(X) = \{x \in X - \{\tau\} : x \text{ is a cut point in } X\}$ and $NCut(X) = \{x \in X - E(X) : x \text{ is a non-cut point in } X\}$.

It is convenient to have the following proposition in order to write and prove some of the subsequent theorems below.

Proposition 3.1. If $p \in X - (\{\tau\} \cup E(X))$, then:

a) $p \in Cut(X)$ if and only if p belongs to some free convex arc of length one in X,

b) $p \in NCut(X)$ if and only if p belongs to some convex arc of length one, Z, such that each $q \in Z - \{\tau\}$ is a limit point of $X - Z$.

Proof. The proof follows from X is a fan homeomorfic to the cone over a compactum. □

The proof of the following two theorems may be modeled from the proofs of Theorems 2.1 and 2.2, respectively.

Theorem 3.2. $\{\tau\}$ makes a hole in $F_2(X)$.

Theorem 3.3. If p and q are different elements of $Cut(X)$, then $\{p, q\}$ makes a hole in $F_2(X)$.

Theorem 3.4. If $p \in Cut(X)$, then $\{p\}$ does not make a hole in $F_2(X)$.

Proof. Consider a free arc Z in X such that $\{p\}$ belongs to the interior of $F_2(Z)$ in $F_2(X)$. Thus, $F_2(Z)$ is a closed neighborhood of $\{p\}$. It is not difficult to prove that the boundary of $F_2(Z)$ in $F_2(X)$ is connected. Since $F_2(Z)$ is homeomorphic to a 2-cell and $\{p\}$ is an element of its manifold boundary, then $F_2(Z) - \{\{p\}\}$ is contractible. Then $F_2(Z) - \{\{p\}\}$ has property *b*) (see Anaya, 2007. Proposition 9, p. 2001). By Proposition 2.4 of (Anaya, 2011), we have $F_2(X) - \{\{p\}\}$ has property *b*). □

Theorem 3.5. If $p \in Cut(X)$, then $\{\tau, p\}$ makes a hole in $F_2(X)$.

Proof. Let Z be a leg of X such that $p \in Z$. Let $\mathcal{A}_1 = F_2(Z) - \{\{\tau, p\}\}$ and $\mathcal{A}_2 = (F_2(Y) \cup \langle Y, Z\rangle) - \{\{\tau, p\}\}$, where $Y = (X - Z) \cup \{\tau\}$.

It is easy to prove that $\mathcal{A}_1 \cup \mathcal{A}_2 = F_2(X) - \{\{\tau, p\}\}$, and $\mathcal{A}_1$, $\mathcal{A}_2$ are connected closed subsets of $F_2(X) - \{\{\tau, p\}\}$. Since $\mathcal{A}_1 \cap \mathcal{A}_2$ is the set $\{\{\tau, x\} : x \in Z - \{p\}\}$ and it is disconnected, $F_2(X) - \{\{\tau, p\}\}$ is not unicoherent. □

Theorem 3.6. If $A_0 \in F_2(X)$ is such that $A_0 \cap E(X) \neq \emptyset$, then A_0 does not make a hole in $F_2(X)$.

Proof. By Theorem 3.1 from (Macías, 2003), $F_2(X)$ is homeomorphic to the cone over the set $\mathcal{B} = \bigcup\{\{A \in F_2(X) : e_\lambda \in A\} : \lambda \in \Lambda\}$. Therefore, $F_2(X) - \{A_0\}$ is homeomorphic to the cone over the set $\mathcal{B}$ minus the element $\{(A_0, 0)\}$. Then, $F_2(X) - \{A_0\}$ is contractible. Hence, $F_2(X) - \{A_0\}$ has property *b*). Therefore $F_2(X) - \{A_0\}$ is unicoherent (see Eilenberg, 1936, Theorems 2 and 3, pp. 69 and 70). □

The last result is true for any hyperspace of X that appears in Theorem 3.1 from (Macías, 2003).

Lemma 3.7. Let X be a continuum. We suppose that there exists a connected subset $\mathcal{A}$ in 2^X that has property *b*), a sequence of subcontinua $\{\mathcal{A}_n\}_{n=0}^{\infty}$ in $\mathcal{A}$, $B \in \bigcap_{n=0}^{\infty} A_n$ and a sequence $\{A_n\}_{n=0}^{\infty}$ of 2^X such that $A_0 = \lim A_n$ and, for each $n \in \mathbb{N} \cup \{0\}$, $A_n \in \mathcal{A}_n$.

If $f : \mathcal{A} \to S^1$ is a map, $t_0 \in \exp^{-1}(f(B))$ and, for each $n \in \mathbb{N} \cup \{0\}$, there exists a map $h_n : \mathcal{A}_n \to \mathbb{R}$ such that $f|_{\mathcal{A}_n} = \exp \circ h_n$ and $h_n(B) = t_0$, then $h_0(A_0) = \lim h_n(A_n)$.

Proof. Since $\mathcal{A}$ has property *b*), there exists a map $h : \mathcal{A} \to \mathbb{R}$ such that $f = \exp \circ h$ and $h(B) = t_0$. Given $n \in \mathbb{N} \cup \{0\}$. Since h_n and $h|_{\mathcal{A}_n}$ are liftings of $f|_{\mathcal{A}_n}$ and $h|_{\mathcal{A}_n}(B) = h_n(B)$, by (Greenberg and Harper, 1981, 5.1), $h|_{\mathcal{A}_n} = h_n$. Hence, $h_0(A_0) = \lim h_n(A_n)$. □

Theorem 3.8. If $p \in NCut(X)$, then $\{p\}$ does not make a hole in $F_2(X)$.

Proof. By Proposition 8 of (Anaya, 2007), we only need to prove that there exist two connected and closed subsets of $F_2(X) - \{\{p\}\}$, $\mathcal{A}$ and $\mathcal{D}$, which have property *b*) and the intersection of them is connected.

Suppose that $p = \frac{3}{4}e_{\lambda_0}$, for some $e_{\lambda_0} \in E(X)$. For any λ, $\gamma \in \Lambda$, let

$$\mathcal{A}_{\lambda,\gamma} = \left\langle \left[\frac{1}{2}e_\lambda, e_\lambda\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right]\right\rangle \cup \left\{\{te_\lambda, te_\gamma\} : t \in \left[0, \frac{1}{2}\right]\right\}$$

and

$$\mathcal{A} = \bigcup_{\lambda,\gamma \in \Lambda} \mathcal{A}_{\lambda,\gamma} - \{\{p\}\}.$$

Let $Y = \bigcup_{\lambda\in\Lambda} [\tau, \frac{1}{2} e_\lambda]$, for any $\lambda \in \Lambda$, let $\mathcal{D}_\lambda = \langle Y, [\frac{1}{2}e_\lambda, e_\lambda]\rangle$ and let

$$\mathcal{D} = F_2(Y) \cup \left(\bigcup_{\lambda\in\Lambda} \mathcal{D}_\lambda\right).$$

Clearly, $F(X) - \{\{p\}\} = \mathcal{A} \cup \mathcal{D}$.

Since, for any $\lambda, \gamma \in \Lambda$, $\{\tau\} \in \mathcal{A}_{\lambda,\gamma}$, we obtain that $\mathcal{A}$ is connected. Also, since for any $\lambda \in \Lambda$, $\{\frac{1}{2}e_\lambda\} \in \mathcal{D}_\lambda \cap F_2(Y)$, we have that $\mathcal{D}$ is connected.

Notice that

$$\mathcal{A} \cap \mathcal{D} = \bigcup_{\lambda,\gamma\in\Lambda} \left(\left\{\{te_\lambda, te_\gamma\} : t \in \left[0, \frac{1}{2}\right]\right\} \cup \left\langle \left\{\frac{1}{2}e_\lambda\right\}, \left[\frac{1}{2}e_\gamma, e_\gamma\right]\right\rangle\right).$$

Hence, $\mathcal{A} \cap \mathcal{D}$ is connected.

Since X is embedded in $\mathbb{R}^2$, with $\tau = (0, 0)$ and its legs are convex arcs of length one, it is easy to prove that $\mathcal{A}$ and $\mathcal{D}$ are closed subsets of $F_2(X) - \{\{p\}\}$.

We are going to prove that $\mathcal{D}$ has property *b*). Since Y is a fan homeomorphic to the cone over a compactum, $F_2(Y)$ is homeomorphic to the cone over a continuum (see Macías, 2003). By Theorem 1.11 of (Rotman, 1998), $F_2(Y)$ is contractible. Therefore, $F_2(Y)$ has property *b*). Since $F_2(Y)$ is a deformation retract of $\mathcal{D}$, $\mathcal{D}$ has property *b*) (see Anaya, 2007. Proposition 9).

We will prove that $\mathcal{A}$ has property *b*). Let $f : \mathcal{A} \to S^1$ be a map. We do the proof in two steps. First we are going to prove that there exist two contractible subsets of $\mathcal{A}$ whose union is $\mathcal{A}$ and after that we will define the lifting of *f*.

Let $\mathcal{B} = (\mathcal{A} - \mathcal{A}_{\lambda_0, \lambda_0}) \cup \{\{\tau\}\}$.

Notice that $\mathcal{A} = \mathcal{B} \cup (\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\})$.

Now we are going to show that $\mathcal{B}$ and $\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}$ are contractible. Notice that

$$\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\} = \left(F_2\left(\left[\frac{1}{2}e_{\lambda_0}, e_{\lambda_0}\right]\right) \cup \left\{te_{\lambda_0}\right\} : t\in\left[0, \frac{1}{2}\right]\right) - \left\{\left\{\frac{3}{4}e_{\lambda_0}\right\}\right\},$$

$$F_2\left(\left[\frac{1}{2}e_{\lambda_0}, e_{\lambda_0}\right]\right) \cap \left\{te_{\lambda_0}\right\} : t \in \left[0, \frac{1}{2}\right] = \left\{\left\{\frac{1}{2}e_{\lambda_0}\right\}\right\}$$

and $\{\frac{3}{4} e_{\lambda_0}\}$ belongs to the manifold boundary of $F_2([\frac{1}{2} e_{\lambda_0}, e_{\lambda_0}])$. Hence, $\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}$ is contractible.

If we define the following map $H : \mathcal{B} \times [0, 1] \to \mathcal{B}$ by:

$$H(\{re_\lambda, se_\gamma\}, t) = \begin{cases} \{\alpha(r, t)e_\lambda, \alpha(s, t)e_\gamma\}, & \text{if } t\in[0,\frac{1}{2}] \text{ and } r, s\in[\frac{1}{2}, 1] \\ \{\beta(\frac{1}{2}, t)e_\lambda, \beta(\frac{1}{2}, t)e_\gamma\}, & \text{if } t\in[\frac{1}{2},1] \text{ and } r, s\in[\frac{1}{2}, 1] \\ \{re_\lambda, se_\gamma\}, & \text{if } t\in[0,\frac{1}{2}] \text{ and } r = s\in[0, \frac{1}{2}] \\ \{\beta(r, t)e_\lambda, \beta(s, t)e_\gamma\}, & \text{if } t\in[\frac{1}{2},1] \text{ and } r = s\in[0, \frac{1}{2}] \end{cases}$$

where $\alpha, \beta : [0, 1] \times [0, 1] \to [0, 1]$ defined by $\alpha(x, t) = 2x(\frac{1}{2} - t) + t$ and $\beta(x, t) = 2x(1 - t)$, we have $\{\{\tau\}\}$ is a deformation retract of $\mathcal{B}$. Then $\mathcal{B}$ is contractible.

Now, we will define a lifting of the map *f*. Since $\mathcal{B}$ and $\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}$ have the property *b*), there exist two liftings $h_1 : \mathcal{B} \to \mathbb{R}$ and $h_2 : \mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\} \to \mathbb{R}$ of $f|_{\mathcal{B}}$ and $f|_{\mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}}$, respectively, such that $h_1(\{\tau\}) = h_2(\{\tau\})$. We define $h : \mathcal{A} \to \mathbb{R}$ by:

$$h(\{x, y\}) = \begin{cases} h_1(\{x, y\}), \text{ if } \{x, y\} \in \mathcal{B}, \\ h_2(\{x, y\}), \text{ if } \{x, y\} \in \mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}. \end{cases}$$

We will prove that h is a lifting of *f*. Clearly, $\exp\circ h = f$.

In order to prove that h is continuous, we consider a sequence $\{\{x_n, y_n\}\}_{n=1}^{\infty}$ in $\mathcal{A}$ and an element $\{x_0, y_0\}$ of A such that $\{x_0, y_0\} = \lim\{x_n, y_n\}$. We only need to consider three cases:

Case 1. $\{\{x_n, y_n\} : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{B}$. Since h_1 is continuous, $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$.

Case 2. $\{\{x_n, y_n\} : n \in \mathbb{N} \cup \{0\}\} \subset \mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}$.

Since h_2 is continuous, $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$.

Case 3. $\{x_0, y_0\} \in \mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}, \{\tau\}\}$ and, for each $n \in \mathbb{N}$, $\{x_n, y_n\} \in \mathcal{B}$. We want to use Lemma 3.7, to prove that $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$. Given $n \in \mathbb{N} \cup \{0\}$, there exist $t_n, s_n \in [0, 1]$ such that $x_n = t_n e_{\lambda_n}$ and $y_n = s_n e_{\gamma_n}$. We can suppose that, $\{\lambda_n, \gamma_n\} \neq \{\lambda_m, \gamma_m\}$, if $n \neq m$. Since X is a fan homeomorphic to the cone over a compactum, we have $x_0 = \lim x_n$, $y_0 = \lim y_n$, $\lim e_{\lambda_n} = e_{\lambda_0} = \lim e_{\gamma_n}$, $s_0 = \lim s_n$ and $t_0 = \lim t_n$.

We are going to prove that, for each $n \in \mathbb{N} \cup \{0\}$, there exists an arc α_n in $\mathcal{A}$ such that $\alpha_n \subset \mathcal{B}$, if $n \neq 0$, $\alpha_0 \subset \mathcal{A}_{\lambda_0, \lambda_0} - \{\{p\}\}$, $\alpha_n \cap \alpha_m = \{\{\tau\}\}$, if $n \neq m$, $\{x_n, y_n\}$ and $\{\tau\}$ are the end points of α_n, $\{p\} \notin \bigcup_{n=0}^{\infty} \alpha_n$, $\alpha_0 = \lim \alpha_n$ and $\bigcup_{n=0}^{\infty} \alpha_n$ is contractible. We consider two cases:

Case 1. For each $n \in \mathbb{N} \cup \{0\}$, $t_n, s_n \in [0, \frac{1}{2}]$.

Then $t_n = s_n$ and, for each $n \in \mathbb{N} \cup \{0\}$, we consider

$$\alpha_n = \{\{re_{\lambda_n}, re_{\gamma_n}\} : 0 \leq r \leq t_n\}$$

Case 2. For each $n \in \mathbb{N}$, $t_n, s_n \in (\frac{1}{2}, 1]$ and $s_0, t_0 \in [\frac{1}{2}, 1]$. Given $n \in \mathbb{N} \cup \{0\}$, consider

$$\alpha_{1,n} = \left\{\left\{x_n, \left(\left(\frac{1}{2} - s_n\right) r + s_n\right) e_{\gamma_n}\right\} : 0 \leq r \leq 1\right\},$$

$$\alpha_{2,n} = \left\{\left\{\left(\left(\frac{1}{2} - t_n\right) r + t_n\right) e_{\lambda_n}, \frac{1}{2}e_{\gamma_n}\right\} : 0 \leq r \leq 1\right\},$$

$$\alpha_{3,n} = \left\{\{re_{\lambda_n}, re_{\gamma_n}\} : 0 \leq r \leq \frac{1}{2}\right\}$$

and

$$\alpha_n = \alpha_{1,n} \cup \alpha_{2,n} \cup \alpha_{3,n}$$

(notice that $\gamma_0 = \lambda_0$).

Since $p \in NCut(X)$, it can be shown that $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the required properties.

Notice that $\bigcup_{n=0}^{\infty} \alpha_n$ has property b). If we consider, for each $n \in \mathbb{N}$, $\Phi_n = h_1|_{\alpha_n}$ and $\Phi_0 = h_2|_{\alpha_0}$, by Lemma 3.7, $h_2(\{x_0, y_0\}) = \lim h_1(\{x_n, y_n\})$. Hence h is a map. Therefore $\mathcal{A}$ has property b). □

Theorem 3.9. If $p \in NCut(X)$ and $q \in X - E(X)$, with $q \neq p$, then $\{p, q\}$ does not make a hole in $F_2(X)$.

Proof. In light of Proposition 8 of (Anaya, 2007), it suffices to prove that there exist two connected and closed subsets $\mathcal{A}$ and $\mathcal{D}$ of $F_2(X) - \{\{p, q\}\}$, which have property b) and the intersection of them is connected.

We may assume that $p = \frac{1}{2}e_{\lambda_0}$ for some $e_{\lambda_0} \in E(X)$ and $0 \leq \|q\| < \frac{1}{4}$. Since X is the cone over a compactum, there exist two disjoint subsets Λ_0 and Λ_1 of Λ such that $\lambda_0 \in \Lambda_0$, $\Lambda = \Lambda_0 \cup \Lambda_1$, $\{e_\lambda : \lambda \in \Lambda_0\}$ and $\{e_\lambda : \lambda \in \Lambda_1\}$ are open and closed in $E(X)$. Let $\mathcal{A}$ the set:

$$\left[F_1(Y) \cup \bigcup_{(\lambda,\gamma)\in\Lambda\times\Lambda_0}\left(\left\langle\left[\tau, \frac{1}{4}e_\lambda\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right]\right\rangle \cup \left\langle\{e_\gamma\}, \left[\frac{1}{4}e_\gamma, e_\gamma\right]\right\rangle\right)\right]$$

minus the point $\{p, q\}$, where $Y = \bigcup_{\lambda\in\Lambda_0} \tau e_\lambda$, and $\mathcal{D}$ the set:

$$\left[F_2(X_1) \cup \bigcup_{(\lambda,\gamma,\nu)\in R}\left(\left\langle\left[\frac{1}{4}e_\lambda, e_\lambda\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right]\right\rangle \cup \left\langle\left[\frac{1}{2}e_\gamma, e_\gamma\right], \left[\frac{1}{4}e\nu, e\nu\right]\right\rangle\right)\right]$$

minus the point $\{p, q\}$, where $X_1 = \{re_\lambda : (r, \lambda) \in [0, 1] \times \Lambda_1\} \cup \{re_\lambda : (r, \lambda) \in [0, \frac{1}{2}] \times \Lambda_0\}$ and $R = \Lambda_1 \times \Lambda_0 \times \Lambda_0$.

It is easy to see that $\mathcal{A}$ and $\mathcal{D}$ are connected and closed subsets of $F_2(X) - \{\{p, q\}\}$.

We prove that $F_2(X) - \{\{p, q\}\} = \mathcal{A} \cup \mathcal{D}$. Let $\{x, y\} \in F_2(X) - \{\{p, q\}\}$.

We can suppose that $\{x, y\} \notin \mathcal{D}$. Then $\{x, y\} \not\subseteq X_1$. Without loss of generality we may assume that $y \notin X_1$. Hence, there exists $\gamma \in \Lambda_0$ such that $y \in [\frac{1}{2}e_\gamma, e_\gamma]$ and, by the definition of $\mathcal{D}$, $x \in \bigcup_{\lambda\in\Lambda} [\tau, \frac{1}{4}e_\lambda]$. So, $\{x, y\} \in \mathcal{A}$.

We are going to prove the connectedness of $\mathcal{A} \cap \mathcal{D}$. Let:

$$\mathcal{L}_{\lambda,\gamma} = \langle[\tau, \tfrac{1}{4} e_\lambda], \{\tfrac{1}{2} e\gamma\}\rangle \cup \langle\{\tfrac{1}{4} e_\lambda\}, [\tfrac{1}{2} e_\gamma, e_\gamma]\rangle \cup \langle\{e_\gamma\}, [\tfrac{1}{4} e_\gamma, e_\gamma]\rangle,$$

for each $(\lambda, \gamma) \in \Lambda \times \Lambda_0$.

First, we prove that

$$\mathcal{A} \cap \mathcal{D} = (F_1(Y) \cup (\bigcup\{\mathcal{L}_{\lambda,\gamma} : (\lambda, \gamma) \in \Lambda \times \Lambda_0\})) - \{\{p, q\}\}.$$

Clearly, $F_1(y)$ and $\langle\{e_\gamma\}, [\frac{1}{4}e_\gamma, e_\gamma]\rangle$ are subsets of $\mathcal{A} \cap \mathcal{D}$ for each $\gamma \in \Lambda_0$.

Moreover, $\langle[\tau, \frac{1}{4}e_\lambda], [\frac{1}{2}e_\gamma, e_\gamma]\rangle \cap \mathcal{D}$ is the set

$$\left\langle\left[\tau, \frac{1}{4}e_\lambda\right], \left\{\frac{1}{2}e_\gamma\right\}\right\rangle \cup \left\langle\left\{\frac{1}{4}e_\lambda\right\}, \left[\frac{1}{2}e_\gamma, e_\gamma\right]\right\rangle - \{\{p, q\}\},$$

for each $(\lambda, \gamma) \in \Lambda \times \Lambda_0$. Then

$$\mathcal{A} \cap \mathcal{D} = (F_1(Y) \cup (\bigcup\{\mathcal{L}_{\lambda,\gamma} : (\lambda, \gamma) \in \Lambda \times \Lambda_0\})) - \{\{p, q\}\}.$$

Now, let

$$\mathcal{B} = F_1(Y) \cup (\mathcal{A} \cap \mathcal{D} - (\bigcup\{\mathcal{L}_{\lambda,\gamma} : (\lambda, \gamma) \in \Lambda \times \Lambda_0 \text{ and } \lambda_0 \in \{\lambda, \gamma\}\})).$$

Using that $\mathcal{L}_{\lambda,\gamma} \cap F_1(Y) \neq \emptyset$ and $\mathcal{L}_{\lambda,\gamma}$ is connected for each $(\lambda, \gamma) \in \Lambda \times \Lambda_0$ such that $\lambda_0 \notin \{\lambda, \gamma\}$, we have that $\mathcal{B}$ is connected. Now, consider $(\lambda, \gamma) \in \Lambda \times \Lambda_0$ such that $\lambda_0 \in \{\lambda, \gamma\}$. Since $p \in NCut(X)$, $\mathcal{L}_{\lambda,\gamma} - \{\{p, q\}\} \subset Cl_{F_2(X)}(\mathcal{B})$. So $\mathcal{A} \cap \mathcal{D} \subset Cl_{F_2(X)}(\mathcal{B})$. Then $\mathcal{A} \cap \mathcal{D}$ is connected.

Now, we will prove that $\mathcal{A}$ and $\mathcal{D}$ have property b). In order to prove that $\mathcal{A}$ has property b), it suffices to show that there exists a deformation retract $\mathcal{K}$ of $\mathcal{A}$ such that $\mathcal{K}$ has property b) (see Anaya, 2007. Proposition 9, p. 2001).

Consider the following sets:

$$\mathcal{C}_\gamma = F_1(\tau e_\gamma) \cup \langle[\tau, e_\gamma], \{e_\gamma\}\rangle,$$

$$\mathcal{K}_\gamma = \mathcal{C}_\gamma \cup \left\langle \bigcup_{\lambda\in\Lambda}\left[\tau, \frac{1}{4}e_\lambda\right], \{e_\gamma\}\right\rangle,$$

for each $\gamma \in \Lambda_0$,

$$\mathcal{M} = \bigcup_{\gamma\in\Lambda_0} \mathcal{C}_\gamma \text{ and } \mathcal{K} = \bigcup_{\gamma\in\Lambda_0} \mathcal{K}_\gamma.$$

Notice that $\mathcal{M}$ is a closed subset of $\mathcal{K}$ and $\mathcal{K}$ is a closed subset of $\mathcal{A}$. We need to prove that $\mathcal{C}_\gamma$, $\mathcal{K}_\gamma$, $\mathcal{M}$ and $\mathcal{K}$ have property b).

Using the smoothness of X, it can be shown that $\mathcal{M}$ is homeomorphic to Y. Thus $\mathcal{M}$ is contractible and, so it has property b) (see Anaya, 2007. Proposition 9, p. 2001). Let $\gamma \in \Lambda_0$. Since $\mathcal{C}_\gamma$ is an arc, it has property b). We see that $\left\langle\bigcup_{\lambda\in\Lambda} [\tau, \frac{1}{4}e_\lambda], \{e_\gamma\}\right\rangle$ is homeomorphic to X. Hence, $\left\langle\bigcup_{\lambda\in\Lambda}[\tau, \frac{1}{4}e_\lambda], \{e_\gamma\}\right\rangle$ is contractible and, so it has property b) (see Anaya, 2007. Proposition 9, p. 2001). Notice that $\mathcal{C}_\gamma \cap \left\langle\bigcup_{\lambda\in\Lambda} [\tau, \frac{1}{4}e_\lambda], \{e_\gamma\}\right\rangle = \left\langle[\tau, \frac{1}{4}e_\gamma], \{e_\gamma\}\right.$. Then, $\mathcal{K}_\gamma$ has property b) (see Anaya, 2007. Proposition 8, p. 2001).

Now, we are going to prove that $\mathcal{K}$ has property b).

Let $g : \mathcal{K} \to S^1$ be a map and let $t_0 \in \exp^{-1}(g(\{\tau\}))$. Since, for each $\gamma \in \Lambda_0$, $\mathcal{K}_\gamma$ has property b), there exists a map $h_\gamma : \mathcal{K}_\gamma \to \mathbb{R}$ such that $g|_{\mathcal{K}_\gamma} = \exp\circ h_\gamma$ and $h_\gamma(\{\tau\}) = t_0$. We define $h : \mathcal{K} \to \mathbb{R}$ by $h(\{x, y\}) = h_\gamma(\{x, y\})$, if $\{x, y\} \in \mathcal{K}_\gamma$. We prove that h is a lifting of g. Notice that, for each $\gamma_0, \gamma_1 \in \Lambda_0$ such that $\gamma_0 \neq \gamma_1$, we have that $\mathcal{K}_{\gamma_0} \cap \mathcal{K}_{\gamma_1} = \{\{\tau\}\}$. Then h is well defined. Clearly, $\exp\circ h = g$. In order to prove that h is continuous, let $\{\{x_n, y_n\}\}_{n=1}^{\infty}$ be a sequence of $\mathcal{K}$ and let $\{x_0, y_0\} \in \mathcal{K}$ such that $\{x_0, y_0\} = \lim\{x_n, y_n\}$. We will prove that $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$. We consider two cases.

Case 1. For each $n \in \mathbb{N}$, $\{x_n, y_n\} \in \mathcal{M}$.

Since $\mathcal{M}$ has property *b*), there exists a map $h_1 : \mathcal{M} \to \mathbb{R}$ such that $g|_{\mathcal{M}} = \exp \circ h_1$ and $h_1(\{\tau\}) = t_0$. Notice that, given $\gamma \in \Lambda_0$, $h_\gamma|_{\mathcal{C}_\gamma}$ and $h_1|_{\mathcal{C}_\gamma}$ are liftings of $g|_{\mathcal{C}_\gamma}$ and $h_\gamma|_{\mathcal{C}_\gamma}(\{\tau\}) = t_0 = h_1|_{\mathcal{C}_\gamma}(\{\tau\})$. Thus, $h_\gamma|_{\mathcal{C}_\gamma} = h_1|_{\mathcal{C}_\gamma}$ (see Greenberg and Harper, 1981, 5.1). Hence, $h_1 = h|_{\mathcal{M}}$.

Since $\mathcal{M}$ is a closed subset of $\mathcal{K}$, $\{x_0, y_0\} \in \mathcal{M}$. Then $h(\{x_0, y_0\}) = h|_{\mathcal{M}}(\{x_0, y_0\}) = h_1(\{x_0, y_0\}) = \lim h_1(\{x_n, y_n\}) = \lim h|_{\mathcal{M}}(\{x_n, y_n\}) = \lim h(\{x_n, y_n\})$.

Case 2. For each $n \in \mathbb{N}$, $\{x_n, y_n\} \notin \mathcal{M}$.

Given $n \in \mathbb{N}$, let $\eta_n \in \Lambda$ and $\gamma_n \in \Lambda_0$ such that $\eta_n \neq \gamma_n$ and $\{x_n, y_n\} \in \langle[\tau, \frac{1}{4}e_{\eta_n}], \{e_{\gamma_n}\}\rangle$. Since $\{e_\gamma : \gamma \in \Lambda_0\}$ and $E(X)$ are compact, taking subsequences if neccesary, we may assume that there exist $e_{\gamma_0} \in \{e_\gamma : \gamma \in \Lambda_0\}$ and $e_{\eta_0} \in E(X)$ such that $e_{\gamma_0} = \lim e_{\gamma_n}$ and $e_{\eta_0} = \lim e_{\eta_n}$.

Let $\mathcal{U} = \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{U}_n$, where $\mathcal{U}_n = \mathcal{C}_{\gamma_n} \cup \langle[\tau, \frac{1}{4}e_{\eta_n}], \{e_{\gamma_n}\}\rangle$ for each $n \in \mathbb{N} \cup \{0\}$.

Using the smoothness of Y, it is easy to prove that $\mathcal{U}_0 = \lim \mathcal{U}_n$. Then $\{x_0, y_0\} \in \mathcal{U}_0$. We need to consider two subcases.

Subcase 1. For each $m, n \in \mathbb{N}$, $e_{\gamma_n} = e_{\gamma_m}$.

Then, for each $n \in \mathbb{N}$, $e_{\gamma_0} = e_{\gamma_n}$. Hence, $\mathcal{U} \subset \mathcal{K}_{\gamma_0}$. By the definition of h, $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$.

Subcase 2. For each $m, n \in \mathbb{N}$, $e_{\gamma_n} \neq e_{\gamma_m}$

Using the smoothness of Y, it is easy to prove that $\mathcal{U}$ is contractible. Then $\mathcal{U}$ has property *b*) (see Anaya, 2007. Proposition 9, p. 2001). If we consider $g|_{\mathcal{U}} : \mathcal{U} \to S^1$ and, for each $n \in \mathbb{N} \cup \{0\}$, $\Phi_n = h_{\gamma_n}|\mathcal{U}_n$, by Lemma 3.7, $\Phi_0(\{x_0, y_0\}) = \lim \Phi_n(\{x_n, y_n\})$. Hence, $h(\{x_0, y_0\}) = \lim h(\{x_n, y_n\})$.

This proves that $\mathcal{K}$ has property *b*).

Now, in order to prove that $\mathcal{K}$ is a deformation retract of $\mathcal{A}$, let

$$\mathcal{A}_1 = \left(\bigcup_{(\lambda,\gamma) \in \Lambda \times \Lambda_0} \langle[\tau, \tfrac{1}{4}e_\lambda], [\tfrac{1}{2}e_\gamma, e_\gamma]\rangle\right) - \{\{p, q\}\}$$

and let $\Psi : \mathcal{A} \times [0, 1] \to \mathcal{A}$ defined by:

$$\Psi(\{se_\lambda, re_\gamma\}, t) = \begin{cases} \{se_\lambda, re_\gamma\}, & \text{if } \{se_\lambda, re_\gamma\} \in \mathcal{K}, \\ \{se_\lambda, ((1-r)t + r)e_\gamma\}, & \text{if } \{se_\lambda, re_\gamma\} \in \mathcal{A}_1. \end{cases}$$

Notice that $\Psi|_{\mathcal{A}_1 \times [0,1]}$ and $\Psi|_{\mathcal{K} \times [0,1]}$ are continuous, $\mathcal{A}_1$ is a closed subset of $\mathcal{A}$ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{K}$. Moreover, if $(\{se_\lambda, re_\gamma\}, t) \in (\mathcal{A}_1 \times [0, 1]) \cap (\mathcal{K} \times [0, 1])$, then $r = 1$ and $\Psi|_{\mathcal{A}_1 \times [0,1]}(\{se_\lambda, re_\gamma\}, t) = \Psi|_{\mathcal{K} \times [0,1]}(\{se_\lambda, re_\gamma\}, t)$. So Ψ is continuous. Thus $\mathcal{A}$ has property *b*).

Now, in order to prove that $\mathcal{D}$ has property *b*), we are going to show that $F_2(X_1) - \{\{p, q\}\}$ is a deformation retract of $\mathcal{D}$ and it has property *b*) (see Anaya, 2007. Proposition 9, p. 2001).

Let:

$$\mathcal{F}_0 = F_2(X_1) - \{\{p, q\}\},$$

$$\mathcal{F}_1 = \bigcup_{(\lambda,\gamma) \in \Lambda_1 \times \Lambda_0} \left\langle \left[\frac{1}{4}e_\lambda, e_\lambda\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right] \right\rangle,$$

$$\mathcal{F}_2 = \bigcup_{(\nu,\gamma) \in \Lambda_0 \times \Lambda_0} \left\langle \left[\frac{1}{4}e_\nu, \frac{1}{2}e_\nu\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right] \right\rangle,$$

and

$$\mathcal{F}_3 = \bigcup_{(\nu,\gamma) \in \Lambda_0 \times \Lambda_0} \left\langle \left[\frac{1}{2}e_\nu, e_\nu\right], \left[\frac{1}{2}e_\gamma, e_\gamma\right] \right\rangle.$$

Notice that each $\mathcal{F}_i$ is a closed subset of $\mathcal{D}$ and $\mathcal{D} = \bigcup_{i=0}^{3} \mathcal{F}_i$. We define $F : \mathcal{D} \times [0, 1] \to \mathcal{D}$ by:

$$F(\{re_\lambda, se_\gamma\}, t) = \begin{cases} \{re_\lambda, se_\gamma\}, & \text{if } \{re_\lambda, se_\gamma\} \in \mathcal{F}_0, \\ \{re_\lambda, x(t, s)e_\gamma\}, & \text{if } \{re_\lambda, se_\gamma\} \in \mathcal{F}_1, \\ \{re_\lambda, x(t, s)e_\gamma\}, & \text{if } \{re_\lambda, se_\gamma\} \in \mathcal{F}_2, \\ \{x(t, r)e_\lambda, x(t, s)e_\gamma\}, & \text{if } \{re_\lambda, se_\gamma\} \in \mathcal{F}_3. \end{cases}$$

where $x(t, s) = \frac{1}{2} + (1 - t)s$ for each $t, s \in [0, 1]$.

Clearly, each $F|_{\mathcal{F}_i \times [0,1]}$ is continuous. Notice that:

$$\mathcal{F}_0 \cap \mathcal{F}_1 = \bigcup_{(\lambda,\gamma) \in \Lambda_1 \times \Lambda_0} \left\langle \left[\frac{1}{4}e_\lambda, e_\lambda\right], \left\{\frac{1}{2}e_\gamma\right\} \right\rangle,$$

$$\mathcal{F}_0 \cap \mathcal{F}_2 = \bigcup_{(\nu,\gamma) \in \Lambda_0 \times \Lambda_0} \left\langle \left[\frac{1}{4}e_\nu, \frac{1}{2}e_\nu\right], \left\{\frac{1}{2}e_\gamma\right\} \right\rangle,$$

$$\mathcal{F}_0 \cap \mathcal{F}_3 = \left\{\left\{\frac{1}{2}e_\nu, \frac{1}{2}e_\gamma\right\} : \nu, \gamma \in \Lambda_0\right\},$$

$$\mathcal{F}_2 \cap \mathcal{F}_3 = \bigcup_{(\nu,\gamma) \in \Lambda_0 \times \Lambda_0} \left\langle \left\{\frac{1}{2}e_\nu\right\}, \left[\frac{1}{2}e_\gamma, e_\gamma\right] \right\rangle$$

and

$$\mathcal{F}_1 \cap F_2 = \mathcal{F}_1 \cap \text{F3} = \emptyset.$$

Clearly, if $\mathcal{G}_1, \mathcal{G}_2 \in \{\mathcal{F}_i : i \in \{0, 1, 2, 3\}\}$ such that $\mathcal{G}_1 \cap \mathcal{G}_2 \neq \emptyset$, then $F|_{\mathcal{G}_1 \times [0,1]}(\{x, y\}, t) = F|_{\mathcal{G}_2 \times [0,1]}(\{x, y\}, t)$ for each $(\{x, y\}, t) \in (\mathcal{G}_1 \cap \mathcal{G}_2) \times [0, 1]$. Hence, F is continuous. Then $\mathcal{F}_0$ is a deformation retract of $\mathcal{D}$.

By the definition of X_1 and since p is an end point of X_1, it can be shown that there exists a natural homeomorphism $f : X_1 \to X$ such that $f(p) \in E(X)$. Then $\mathcal{F}_0$ is contractible (see the proof of Theorem 3.6). By Proposition 9 of (Anaya, 2007. p. 2001), $\mathcal{F}_0$ has property *b*).

Therefore, $F_2(X) - \{\{p, q\}\}$ has property *b*). □

3.1. Classification

Theorem 3.10. Let X be a fan which is homeomorphic to the cone over a compactum and $\{x, y\} \in F_2(X)$. Then $\{x, y\}$ makes a hole in $F_2(X)$ if and only if $x = y$ and x is the top of X or $y \neq x$ and $x, y \in Cut(X) \cup \{\tau\}$.

Proof. Necessity, let $\{x, y\} \in F_2(X)$ be such that $\{x, y\}$ makes a hole in $F_2(X)$. In the case that $x = y$, by Theorems 3.4 and 3.8, we have $x = \tau$. On the other hand, if $x \neq y$, by Theorems 3.6 and 3.9, $x, y \in Cut(X) \cup \{\tau\}$. The sufficiency follows from Theorems 3.3, 3.2 and 3.5. □

Conclusions

Intuitively, a connected topological space is unicoherent if it does not have holes. K. Kuratowski was the first author which used the unicoherence to obtain topological caracterization of the sphere (see Kuratowski, 1926 and Kuratowski, 1929). In (Borsuk, 1931), K. Borsuk introduce to use maps from a given space on S^1 to study unicoherence. This tecnique was developed by the authors in (Eilenberg, 1936), (Eilenberg, 1935), (Ganea, 1952a) and (Ganea, 1952b)).

Unicoherence has been useful to distinguish topological space. In (Illanes, 2002, 7, Lemmas 2.1 and 2.2, p. 348 and 349) the author showed that $C_2([0, 1]) - \{A\}$ is unicoherent for each $A \in C_2([0, 1])$ while $C_2(S^1) - \{S^1\}$is not unicoherent. As a consequence A. Illanes obtain that $C_2([0, 1])$ and $C_2(S^1)$ are not homeomorphics; this is in contrast to the fact that $C([0, 1])$ and $C(S^1)$ are homeomorphic. Thus the following problem arises: if X is a continuum and $\mathcal{H}(X)$ is a hyperspace of X, for which elements $A \in \mathcal{H}(X)$, A makes a hole in $\mathcal{H}(X)$.

In this paper, we continue the works in the items (Anaya, 2007), (Anaya *et al.*, 2010) and (Anaya, 2011). We obtain the characterization of the elements $A \in F_2(X)$ such that A makes a hole in $F_2(X)$, when X is either a dentrite or the cone over a compactum. Notice that dendrites and the cone over a compactum are smooth dendroids. So, we consider the following questions:

1. Is it possible to obtain a characterization of those $A \in F_2(X)$ such that A makes a hole in $F_2(X)$ when X is a smooth dendroid?

2. Is there a particular arc-smooth dendroid for which it is impossible or very dificult to answer the Question 1?

3. What happens with $F_n(X)$, for $n \geq 3$. Is it possible to find an element $A \in F_n(X)$ such that A makes a hole in $F_n(X)$?

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