Middle School Students' Conceptual Understanding of Equations: Evidence from Writing Story Problems

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Abstract

This study investigated middle school students' conceptual understanding of algebraic equations. 257 sixth- and seventh-grade students solved algebraic equations and generated story problems to correspond with given equations. Aspects of the equations' structures, including number of operations and position of the unknown, influenced students' performance on both tasks. On the story-writing task, students' performance on two-operator equations was poorer than would be expected on the basis of their performance on one-operator equations. Students made a wide variety of errors on the story-writing task, including (1) generating story contexts that reflect operations different from the operations in the given equations, (2) failing to provide a story context for some element of the given equations, (3) failing to include mathematical content from the given equations in their stories, and (4) including mathematical content in their stories that was not present in the given equations. The nature of students' story-writing errors suggests two main gaps in students' conceptual understanding. First, students lacked a robust understanding of the connection between the operation of multiplication and its symbolic representation. Second, students demonstrated difficulty combining multiple mathematical operations into coherent stories. The findings highlight the importance of fostering connections between symbols and their referents.

Keywords: conceptual understanding, algebra, equations, story problems, middle school.

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Comprensión Conceptual de Ecuaciones en Estudiantes de Secundaria: Evidencia en la Escritura de Problemas Historiados

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Resumen

Se investigó la comprensión conceptual de ecuaciones algebraicas en estudiantes de secundaria. 257 estudiantes de 6º y 7º grado resolvieron ecuaciones algebraicas y escribieron problemas que se correspondieran con ciertas ecuaciones. Aspectos sobre las estructuras de las ecuaciones, incluyendo el número de operaciones y la posición de la variable desconocida, influyeron en el rendimiento en ambas tareas. En la tarea de escritura de historias, el rendimiento en ecuaciones de dos funciones fue más pobre de lo esperado en base a su rendimiento en ecuaciones de una función. El alumnado cometió variedad de errores en esta tarea, incluyendo: (1) relatos que reflejan operaciones diferentes de las dadas en las ecuaciones, (2) fallos en ofrecer un contexto para algunos elementos de la ecuación dada, (3) fallos para incluir en sus historias contenido matemático de las ecuaciones dadas, e (4) inclusión de contenido matemático en las historias que no estaba en las ecuaciones dadas. La naturaleza de los errores de la escritura de historias sugiere dos lagunas centrales en la comprensión conceptual del alumnado: carecer de una comprensión robusta de la conexión entre la operación de multiplicación y su representación simbólica y dificultad combinando múltiples operaciones en historias coherentes. Los resultados subrayan la importancia de fomentar conexiones entre símbolos y sus referentes.

Palabras clave: comprensión conceptual, álgebra, ecuaciones, problemas historiados, escuela secundaria.

The teaching and learning of algebra has been a focus of reform recommendations over the past several decades (e.g., Kaput, 1998, 1999; Kilpatrick, Swafford, & Findell, 2001; RAND Mathematics Study Panel, 2003), prompting scholars to define algebra and identify aspects of algebraic reasoning that are accessible to students across the grades. Kaput (2008) identified two core aspects of algebra: (a) generalization and the expression of generalizations in increasingly systematic, conventional symbol systems and (b) syntactically guided action on symbols within organized systems of symbols.

Current reform recommendations are prompted in large part by high failure rates associated with the traditional treatment of algebra as an isolated high school course in which students manipulate symbols that hold no meaning for them. Indeed, Kaput's (2008) characterization of algebra highlights the importance of helping students become facile with the symbol system of algebra. Facility with the symbol system of algebra involves both *looking at* and *looking through* symbols (Kaput, Blanton & Moreno, 2008).

Looking through symbols involves maintaining a connection between symbols and their referents. *Looking at* symbols and acting on those symbols involves working with symbols as objects in their own right, without concern for their referents. In the context of instruction, students might be presented with a diagram, a table, a verbal description, or a physical enactment and be prompted to build oral, written, or drawn descriptions of the situation that are closely tied to the original situation. These descriptions can be further and further abstracted until a conventional symbolic representation (e.g., an algebraic equation) is reached. In each step of the symbolization process, one can *look through* the symbols and make a connection to the original context or a previous symbolization, or one can *look at* the symbols to take advantage of their compact form and be free of concern for their referents.

When students learn the procedures associated with *looking at* symbols without highlighting the referential connection to an associated situation or experience—a common occurrence in traditional algebra courses—difficulties can arise (Kaput et al., 2008). Indeed, the literature is replete with reports of middle and high school students' difficulties solving algebraic equations (e.g., Koedinger & Nathan, 2004), interpreting algebraic

equations (e.g., Stephens, 2003), and symbolizing mathematical situations (e.g., Clement, 1982; Heffernan & Koedinger, 1997; Kenney & Silver, 1997; McNeil et al., 2010). These difficulties might be construed as indicating gaps in students' conceptual understanding of algebraic symbols.

Understanding the meaning of algebraic symbols can be viewed as a form of conceptual understanding, in the sense that it reflects understanding of general principles or regularities within the domain (Crooks & Alibali, in press). Put another way, symbols have meanings that reflect general properties that apply across specific instances of the symbols. When students *look through* symbols, connecting them to their referents, these meanings are activated and they can inform students' behavior. However, when students *look at* symbols, for example, when operating on symbols without connecting them to their referents, these general meanings are not activated and therefore cannot guide students' behavior. Students' lack of conceptual understanding of algebraic symbols (or their failure to activate this understanding at a given moment) might lead them to misapply procedures learned by rote or to generate symbolic expressions that are syntactically incorrect or that do not appropriately capture the mathematical relations they wish to express.

How can students' conceptual understanding of symbolic algebraic equations be assessed? Measuring students' conceptual understanding presents researchers with many challenges (Crooks & Alibali, in press). In past work, researchers have asked students to solve algebraic equations (e.g., Herscovics & Linchevski, 1994) or to translate word problems into algebraic equations (e.g., Swafford & Langrall, 2000). However, for students who have had some exposure to instruction in the symbol system of algebra—even for those lacking the understanding to *look through* symbols—such tasks might be routine. Students can succeed on routine tasks without conceptual understanding if they have learned procedures by rote; therefore, students' performance on such tasks may not provide full information about their conceptual understanding of algebraic equations. Instead, novel tasks are needed to provide a more accurate view. Students given a novel task do not have readily available procedures for completing the task, and they must therefore rely on conceptual understanding to guide their approach to the task (Rittle-Johnson, Siegler & Alibali, 2001; Rittle-Johnson & Schneider, 2014).

In the present study, we asked middle school students to generate a story to correspond with a given equation, as a means to investigate their conceptual understanding of algebraic equations. Because the story writing task is novel to most learners, it has been used in previous studies to assess conceptual understanding in a range of mathematical domains and participant groups, including fraction division in late-elementary students (Sidney & Alibali, 2013) and teachers (Ma, 1999), and one- and two-operator algebraic equations in high school students (Stephens, 2003). We also asked students to solve a set of symbolic equations so we could assess the relationship between their conceptual understanding and their equation solving.

Method

Participants

Participants in the study were 257 students (213 6th-grade students and 44 7th-grade students) from a middle school in Boulder, Colorado. Students in both grade levels utilized the *Connected Mathematics* curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). All students had experience solving equations, but they had not been exposed in school to the novel task of writing a story that could be represented by a given equation. Due to absences, thirteen students did not complete the equation-solving assessment, and three students did not complete the story-writing assessment.

Materials

For the equation-solving task, students were asked to solve for n in each of 12 equations. The equations varied systematically along three parameters: position of the unknown (start vs. result), number of operations (one vs. two), and operation type (addition, subtraction, or multiplication for one-operator equations and addition-subtraction, multiplication-addition, or multiplication-subtraction for two-operator equations). The equations used are presented in the Appendix. Order was counterbalanced across two different test forms.

For the story-writing task, students were given a set of single-unknown algebraic equations and were asked to write corresponding stories. The

given equations were generated using the same three parameters as in the equation-solving task, resulting in a total of twelve types of equations. These equation types were divided into two sets, which we refer to as "versions," each of which contained three result-unknown equations and three start-unknown equations. Version A included result-unknown addition, result-unknown multiplication, result-unknown multiplication-subtraction, start-unknown subtraction, start-unknown addition-subtraction, and start-unknown multiplication-addition equations; Version B included result-unknown subtraction, result-unknown addition-subtraction, result-unknown multiplication-addition, start-unknown addition, start-unknown multiplication, and start-unknown multiplication-subtraction equations. In addition, for each equation type, two different number sets were used. Finally, each set was presented in forward and reverse order. The equations used in the story-writing task are presented in the Appendix.

To minimize demands on their creativity, students were provided with eight story scenarios that they could use when writing their stories. The scenarios were provided at the top of each page of the story writing booklet and were as follows: (1) Kevin lives on a farm, (2) Nicole is going shopping, (3) Ian collects CDs, (4) Emily is playing basketball, (5) Tara is saving to buy a bicycle, (6) Mike is baking cookies, (7) Alayna has some M&Ms, and (8) Beth is having a birthday party. Students were told that they did not have to use all eight of the scenarios when writing their stories and that they could use the same scenario more than once. To clarify the task, students were given an example equation, $22 - 8 = n$, accompanied by the example solution "Kevin lives on a farm. He had 22 pigs, but he sold 8 of them. How many pigs does he have left?"

Procedure
Students' classroom teachers administered the paper-and-pencil assessments. Each student was randomly assigned to one of the two equation-solving forms and one of the eight story-writing forms. One of the two participating teachers administered both assessments on the same day; the other administered them on two consecutive days. All students completed the story-writing assessment before the equation-solving assessment.

Students were instructed to show all of their work, to draw a circle around their final answers (on the equation-solving form), to not use a calculator, and to not erase any work. The teachers collected the forms at the end of each testing session.

Coding

Equation solving. Students' solutions to the equation-solving tasks were given a score of 1 if they were correct or if they showed evidence of a correct procedure with a computational error. Solutions that were otherwise incorrect were given a score of 0.

Story writing. Students' solutions to the story-writing tasks were given a score of 1 if they were well-formed story problems that corresponded with the numbers and operations in the given equation, and a score of 0 if they were incorrect attempts or if no attempt was made. Cases in which students solved a given equation for n and then integrated that solution into the story rather than pose a question were also treated as correct, as long as they did not also include other errors. For example, for the equation $19 + 33 = n$, one student wrote, "Ian has 19 CDs one month. The next month, he collected 33 more. Now he has 52 CDs"; this story was considered correct because it correctly corresponds with the given numbers and operations.

Each incorrect solution was assigned one or more codes describing the nature of the students' errors. Error categories and accompanying examples are presented in Table 1.

Table 1
Errors types and examples

Type of Error	Equation	Student Example	Percent of Problems
No response		(Student leaves problem blank)	3.1
Incomplete story	$6 \times n + 23 = 89$	Ian collects CDs. He was trying to figure out how many he has.	1.4
Wrong operation	$63 + n - 13 = 91$	Alayna has 63 M&Ms and she gives some to a friend. Then another friend gives her 13 M&Ms. Now she has 91 M&Ms. How many did she give her friend?	5.2
Missing mathematical content	$45 - n = 21$	Kevin has some pigs. He gave away a certain amount. Now Kevin has 21 pigs. How many pigs did Kevin give away?	5.5
Adds mathematical content	$6 \times 13 = n$	Alayna has some M&Ms. She has 6 of them, but she buys 13 more bags that hold 6 each. How many does she have now?	5.6
No story action	$6 \times 13 = n$	Ian has 6×13 CDs. How many CDs is that?	8.5
Wrong question	$63 + 41 - 13 = n$	Ian had 63 CDs and got 41 new ones. 13 of the new CDs didn't work. How many new CDs did work?	3.0

Table 1 (continued)
Errors types and examples

Type of Error	Equation	Student Example	Percent of Problems
No end statement	$6 \times 13 = n$	Tara is saving for a bicycle. She is making 13 dollars an hour for watching her younger brother. She watches him for 6 hours.	3.5
Convert two-operator to one-operator equation	$21 \times 4 - 17 = n$	Mike is baking cookies. He has 84 cookies made. Then the dog eats 17. How many cookies does Mike have left?	1.7
Convert start-unknown to result-unknown	$45 - n = 21$	Sara has 45 pencils. She broke 21 pencils. How many are left?	3.4

To assess reliability of the coding procedures, a second trained coder recoded 10% of the story-writing data. Agreement was 84% for identifying errors and 83% for classifying errors into categories.

Results

We focus first on how structural characteristics of the equations (position of unknown, number of operations, and operation type) influenced students' performance on the two tasks. We then examine the most common types of student errors on the story-writing task, with an eye towards investigating what such errors imply about students' conceptual understanding of algebraic equations.

Equation Solving Performance

To evaluate students' performance on equation solving, we used mixed effects logistic regression in the *lme4* package in the R statistics software (Bates, Maechler, Bolker, & Walker, 2014). We fit a model that included the manipulated factors (unknown position and number of operations), their interaction, and grade level (sixth or seventh) as fixed effects, and that used a maximal random effects structure (Barr, Levy, Scheepers, & Tily, 2013). We evaluated all fixed effects using likelihood-ratio tests in which we compared the full model containing the fixed effect of interest to an identical model in which only that effect was removed (i.e., Type 3-like tests; Barr et al., 2013).

On average, students succeeded on 9.8 out of the 12 equation-solving items. The percent of participants who succeeded for each equation type is presented in Figure 1.
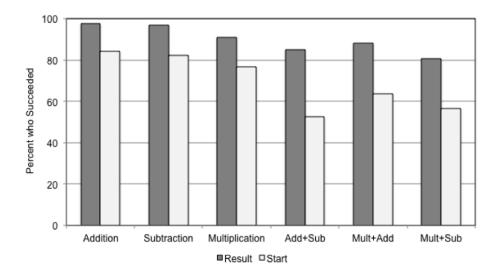
Figure 1. Percent of participants who succeeded on the equation-solving task for each operation or operation combination and each position of the unknown.

The data pattern suggests that both number of operations and unknown position influenced students' performance on equation solving. Indeed, a model with number of operations yielded a substantially better fit to the data than a model without number of operations, χ^2 (1) = 23.21, $p < .001$, and a model with unknown position yielded a substantially better fit to the data than a model without unknown position, χ^2 (1) = 24.50, $p < .001$. Not surprisingly, participants were more successful on one-operator equations than on two-operator equations, and they were more successful on result-unknown equations than on start-unknown equations. The odds of correctly solving a one-operator equation were estimated to be 5.42 times the odds of correctly solving a two-operator equation, 95% CI [3.63, 8.08], and the odds of correctly solving a result-unknown equation were estimated to be 6.08 times the odds of correctly solving a result-unknown equation, 95% CI [4.02, 9.21]. The interaction of unknown position and number of operations did not improve model fit. A model that included grade level yielded a somewhat better fit to the data than a model without grade level, χ^2 (1) = 2.96, $p = .085$. Surprisingly, sixth-grade students performed slightly better

than seventh-grade students, (M = 9.64, SE = 0.16 vs. M = 9.02, SE = 0.37, out of 12). The odds of sixth-grade students successfully solving an equation were estimated to be 1.65 times those of seventh-grade students, 95% CI [0.96, 2.84]. This may be due to the fact that the sixth-grade sample included some students in accelerated classes, whereas the seventh-grade sample did not.

We also wished to examine whether there were variations in equation-solving performance across the specific pairs of operations and across the specific individual operations that we tested. To do so, we examined one-operator and two-operator equations separately. For two-operator equations, a model that included equation type (addition-subtraction, addition-multiplication, or subtraction-multiplication) fit the data better than a model without equation type, χ^2 (2) = 10.16, p = .006. Participants performed best on multiplication-addition equations (M = 1.52 correct, SE = 0.04, out of 2), and similarly, but slightly less well on addition-subtraction equations (M = 1.37 correct, SE = 0.04, out of 2) and multiplication-subtraction items (M = 1.37 correct, SE = 0.05, out of 2). The odds of succeeding on multiplication-addition stories were estimated to be 1.97 times the odds of succeeding on addition-subtraction stories, 95% CI [1.39, 2.78]. The odds of succeeding on multiplication-subtraction stories and addition-subtraction stories did not differ significantly.

For one-operator equations, the main effect of equation type was not significant. Performance was similar and high for all three types of one-operator equations (addition: M = 1.82 correct, SE = 0.03, subtraction: M = 1.79 correct, SE = 0.03 correct, multiplication: M = 1.68, SE = .04 correct, all out of 2).

Story Writing Performance

We also used mixed effects logistic regression to evaluate students' performance on story writing. Recall that there were two versions of the story writing assessment, each of which included six of the twelve equation types (see Appendix). Participants' total scores were comparable across versions (version A, M = 3.73 correct, SE = 0.15, version B, M = 3.61 correct, SE = 0.15, t(252) = 0.58, ns). The percent of participants who succeeded in writing stories for each equation type is presented in Figure 2.
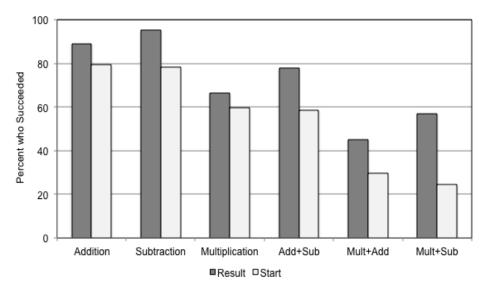
Figure 2. Percent of participants who succeeded on the story-writing task for each operation or operation combination and each position of the unknown.

The main findings for story writing were similar to those for equation solving. A model with number of operations yielded a substantially better fit to the data than a model without number of operations, χ^2 (1) = 17.42, $p <$ .001, and a model with unknown position yielded a substantially better fit to the data than a model without unknown position, χ^2 (1) = 7.54, $p <$ .006. Participants were more successful generating correct stories for one-operator equations than for two-operator equations, and they were more successful generating correct stories for result-unknown equations than for start-unknown equations (see Figure 2). The odds of correctly writing a one-operator story were estimated to be 7.85 times the odds of correctly writing a two-operator story, 95% CI [3.53, 17.42], and the odds of correctly writing a result-unknown story were estimated to be 3.40 times the odds of correctly writing a start-unknown story, 95% CI [1.53, 7.56]. The interaction of unknown position and number of operations did not improve model fit. A model that included grade level yielded a significantly better fit to the data than a model without grade level, χ^2 (1) = 6.02, $p =$ .01. As for

equation solving, sixth-grade students outperformed seventh-grade students, though the margin was small (sixth $M = 3.90$ correct, $SE = 0.11$, vs. seventh $M = 3.29$ correct, $SE = 0.23$, out of six). The odds of sixth-grade students successfully writing stories were estimated to be 2.12 times those of seventh-grade students, 95% CI [1.19, 3.79].

We also wished to examine whether there were variations in story writing performance across the specific pairs of operations and across the specific individual operations that we tested. For two-operator equations, a model with equation type fit the data better than a model without equation type, $\chi^2 (2) = 28.28$, $p <.001$. A majority of participants were successful at writing addition-subtraction stories (68% of participants); fewer participants succeeded at writing multiplication-addition stories (37% of participants) and multiplication-subtraction stories (41% of participants). The odds of succeeding on addition-subtraction stories were 6.72 times the odds of succeeding on multiplication-addition stories, 95% CI [4.34, 10.40], and 5.43 times the odds of succeeding on multiplication-subtraction stories, 95% CI [3.50, 8.41].

For one-operator equations, there was also a main effect of equation type, $\chi^2 (2) = 19.74$, $p < .001$. A comparable percentage of participants succeeded on writing addition stories (84% of participants) and subtraction stories (87% of participants), whereas fewer participants succeeded on writing multiplication stories (63% of participants). The odds of successfully writing addition stories were estimated to be 4.24 times the odds of successfully writing multiplication stories, 95% CI [2.66, 6.76]. The odds of successfully writing addition stories and subtraction stories did not differ significantly.

To investigate the possible existence of a "composition effect" (Heffernan & Koedinger, 1997) in story generation, we next examined whether writing stories for each type of two-operator equation was more difficult than would be expected on the basis of performance writing stories for the corresponding one-operator equations. We estimated the probability of success at writing stories for each of the six types of two-operator equations (i.e., addition-subtraction, addition-multiplication, and subtraction-multiplication for start- and result-unknown equations) by multiplying the rates of success in writing stories for the relevant one-operator equations. We then compared these estimated probabilities of

success with the actual probabilities of success observed in the data. This analysis revealed that writing stories for two-operator equations was indeed more difficult than would be expected on the basis of performance writing stories for the corresponding one-operator equations, $t(5) = 4.03$, $p < .01$. Thus, combining operations in stories presented a substantial challenge for students.

Performance on the equation-solving task and the story-writing task was significantly correlated, $r(240) = .44$, $t(239) = 7.53$, $p < .001$. This finding is consistent with reports in the literature from other domains indicating that students' conceptual understanding and procedural skill are positively associated (e.g., Baroody & Gannon, 1984; Dixon & Moore, 1996; Hiebert & Wearne, 1996; Knuth, Stephens, McNeil & Alibali, 2006; Rittle-Johnson & Alibali, 1999).

Analysis of Story-Writing Errors
We turn next to an analysis of the errors students produced in story writing. Here we present a detailed analysis of those error categories that were assigned on more than 5% of all items (with the exception of the Other category, which was a heterogeneous category): (1) Wrong operation, (2) No story action, (3) Missing mathematical content, and (4) Added mathematical content.

Wrong-operation errors are errors in which some aspect of the student's story reflected an operation different from the one in the given equation. For example, given the equation $6 \times 13 = n$, one student wrote, "Kevin lives on a farm. He has 6 cows and he buys 13. How many does he have?" In this story, the student used a story action that reflects addition rather than multiplication. Table 2 presents the distribution of different types of *Wrong-operation* errors in stories generated for one-operator ($N = 31$) and two-operator ($N = 48$) items. As seen in the table, in the large majority of cases, wrong-operation errors involved converting multiplication to addition.

Table 2

Proportion of Wrong-operation errors of each type

Operation	1-operator items	2-operator items
Addition		
To multiplication	0.06	0.00
To subtraction	0.10	0.06
Addition total	*0.16*	*0.06*
Subtraction		
To addition	0.03	0.10
Subtraction total	*0.03*	*0.10*
Multiplication		
To addition	0.68	0.63
To subtraction	0.03	0.00
To division	0.06	0.00
Multiplication total	*0.77*	*0.63*
N	31	48

Note: Totals do not sum to 1.0 because in some cases the specific change of operation (either which operation was changed, or what it was changed to) could not be precisely identified. This often occurred when other errors were also present.

No-story-action errors are errors in which the student did not provide a story context for some element of the given equation. For example, given the equation $4 \times 13 + 25 = n$, one student wrote, "Kevin lives on a farm. He has 4×13 pigs. The next day he gets 25 more. How many does he have now?" In this story, the student did not provide a story context for the multiplication operation. Table 3 presents the distribution of equation elements that were not described in story form for one-operator ($N = 29$) and two-operator ($N = 101$) items. As seen in the table, when students omitted an element from their stories, it was most often the element that corresponded with multiplication in the given equation.

Table 3
Proportion of No-story-action errors of each type

Content element	1-operator items	2-operator items
Addition operation	0.00	0.36
Subtraction operation	0.00	0.27
Multiplication operation	1.00	0.80
Result quantity	0.00	0.02
N	29	101

Note: Total for 2-operator items does not sum to 1.0 because some stories included multiple No-story-action errors.

Missing-mathematical-content errors are errors in which students failed to include some of the mathematical content from the given equation in their stories. For example, given the equation $6 \times n = 78$, one student wrote, "Alayna has some M&Ms. A bag has 6 M&Ms in a bag. How many more bags does she need?" In this story, the student described a multiplicative relationship involving 6, but did not include the result quantity, 78. Table 4 presents the distribution of elements that were missing for one-operator ($N = 23$) and two-operator ($N = 61$) items. As seen in the table, when an element was missing, it was most often either the start or result quantity. However, in cases where a mathematical operation was missing, it was most often multiplication.

Table 4
Proportion of Missing-mathematical-content errors of each type

Content element	1-operator items	2-operator items
Addition operation	0.09	0.12
Subtraction operation	0.00	0.13
Multiplication operation	0.04	0.26
Start quantity	0.48	0.39
Result quantity	0.44	0.38
N	23	61

Note: Totals do not sum to 1.0 because some stories included multiple Missing-mathematical-content errors.

Added-mathematical-content errors are errors in which students included mathematical content in their stories that was not present in the given equation. Such errors were coded only when the added content was integral to the solution of the story problem, and not when it was simply "distractor" information that was not needed for solving the problem. In coding the data, it became apparent that students often made *Added-mathematical-content* errors of a particular type when the given operation was multiplication. Specifically, given the expression $n \times m$, students often expressed the initial quantity on its own before describing the multiplication operation. Combining these statements, the mathematical relationship described was $n + n \times m$ rather than $n \times m$. For example, given the equation $4 \times 21 = n$, one student wrote, "Mike is making cookies for a school bake sale. He has made 21, but now needs to make 4 times that amount. How many cookies will he have made altogether?" Inspection of the *Added-mathematical-content* errors indicated that fully 79% were of this type (including 74% of the *Added-mathematical-content* errors made on one-operator items, and 81% of such errors made on two-operator items).

The analyses of these most-frequent errors—*Wrong operation*, *No story action*, *Missing mathematical content*, and *Added mathematical content*—converge to suggest that students lack a full-fledged conceptual understanding of the operation of multiplication and its symbolic representation.

Distribution of Story-Writing Errors on One- and Two-operator Equations

We next examined whether particular story-writing errors were especially likely to occur for two-operator items. To address this issue, we examined whether particular error codes were assigned more frequently on stories generated for two-operator equations than would be expected on the basis of their frequency in stories generated for the corresponding one-operator equations. We performed this analysis on each of the error categories that occurred on more than 5% of all items: (1) *Wrong operation*, (2) *No story action*, (3) *Missing mathematical content*, (4) *Added mathematical content*, and (5) *Other*. We also performed a comparable analysis on the *Convert start-unknown to result-unknown* error category, which was only applicable

to stories generated for start-unknown equations, and which occurred on 6.7% of start-unknown items.

We estimated the probability of each type of error on stories generated for each of the two-operator equations (e.g., start- and result-unknown versions for addition-subtraction, addition-multiplication and subtraction-multiplication) by adding the probabilities of that type of error on stories generated for the relevant one-operator equations and then subtracting their joint probability. For example, to estimate the probability of a *Wrong-operation* error on a story generated for a result-unknown addition-multiplication equation, we added the probabilities of *Wrong-operation* errors on stories generated for result-unknown addition equations (3.1%) and result-unknown multiplication equations (8.6%) and then subtracted their joint probability (0.27%). We then compared these estimated probabilities with the actual probabilities for that error category.

The frequency of *Wrong-operation*, *Missing-mathematical-content*, and *Added-mathematical-content* errors on stories generated for two-operator equations did not differ from what would be expected on the basis of their frequency on stories generated for the corresponding one-operator equations. However, No-story-action errors occurred more frequently on stories generated for two-operator equations than would be expected on the basis of their frequency on stories generated for the corresponding one-operator equations, $t(5) = 4.86$, $p = .002$, one-tailed. This finding suggests that, for two-operator equations, students often avoided generating a story action, rather than face the challenge of generating a coherent two-operator story.

Convert-start-unknown-to-result-unknown errors also occurred more frequently on stories generated for two-operator equations than would be expected on the basis of their frequency on stories generated for the corresponding one-operator equations, $t(2) = 3.99$, $p = .03$. Thus, for start-unknown two-operator items, students sometimes "simplified" their task by writing stories that reflected result-unknown scenarios.

Errors in the *Other* category also occurred more frequently on stories generated for two-operator equations than would be expected on the basis of their frequency on stories generated for the corresponding one-operator equations, $t(5) = 2.25$, $p = .04$, one-tailed. Because the *Other* category is a heterogeneous category, it is not clear how this finding should be

interpreted. Nevertheless, some of the errors observed in the *Other* category are of interest because they belie difficulties integrating multiple operations into a coherent story. In some cases, students generated stories that were incoherent because different units applied to each operation. For example, given the equation $14 \times 7 - 23 = n$, one student wrote, "Nicole wants to buy some necklaces for her[self] and her friends. They come in packs of 14 for $7. She wants to have a few leftovers for her[self], so if she has 23 friends, how many will she keep for herself?" In this example, the multiplication element of the story focuses on the *cost* of the necklaces, but the subtraction element of the story focuses on the *number* of necklaces. In other cases, students appeared to have difficulties assigning meaning to the quantities involved in operations. For example, given the equation $63 + 41 - 13 = n$, one student wrote, "Kevin lives on a farm. He has 63 cows, 41 ducks, and 13 pigs. The pigs are on a sale, though. How [many] animals will he have after the pigs are sold?" In this example, the student incorporated story actions that reflect addition (finding the total number of animals) and subtraction (selling the pigs) but treated the value $63 + 41$ as indicating the number of animals *including* the pigs, rather than only the number of cows and ducks. In both of these examples, students displayed some understanding of the operations involved in the equations but had difficulty integrating multiple operations into coherent stories.

Discussion

Our primary aim in this study was to investigate middle school students' understanding of algebraic equations. In past work, such understanding has often been assessed by asking students to solve equations. We too asked students to complete an equation-solving task; however, we also employed a novel story-writing task in an attempt to gain further insight into students' conceptual understanding of the meanings of the algebraic equations, by making it impossible for them to rely on rote or memorized procedures. Our findings suggest that the story-writing task did indeed reveal much about students' thinking.

Although students in our study were fairly successful at solving algebraic equations, they experienced difficulties with equations that

involved two operations and equations with unknown starting quantities. Students' performance on the story-writing task showed a similar pattern, with two-operator items being more difficult than one-operator items, and start-unknown items being more difficult than result-unknown items. These results are consistent with reports of middle and high school students' difficulties in interpreting word problems (Kenney & Silver, 1997; Koedinger & Nathan, 2004; Sowder, 1988) and symbolic equations (Stephens, 2003).

The nature of these errors revealed two broad areas of concern in students' conceptual understanding. First, students' errors indicated that their conceptual understanding of some arithmetic operations—in particular, multiplication—was weak or incomplete. This finding is compatible with past research identifying middle school students' difficulties in identifying which operations need to be performed to *solve* story problems (Sowder, 1988) and reports that 8th-grade students' intuitive understanding of multiplication is weaker than their understanding of addition (Dixon, Deets, & Bangert, 2001). Second, students' errors indicated that they had difficulties combining multiple operations into coherent stories. This finding is reminiscent of findings that students have difficulties solving and symbolizing story problems that involve multiple operations (Heffernan & Koedinger, 1997; 1998; Koedinger, Alibali, & Nathan, 2008). We consider each of these issues in turn.

A closer analysis of student work falling into four common error categories indicated that, for many students, their conceptual understanding of multiplication was weak or incomplete. When students made *Wrong-operation* errors, in the overwhelming majority of cases, the operation that they represented incorrectly was multiplication. In most of these cases, students wrote stories reflecting the operation of addition instead. When students made *Missing-mathematical-content* errors, they often neglected the equation's starting or resulting quantity; however, in cases where the omitted portion of the equation was an operation, that omitted operation was usually multiplication. Students who made *No-story-action* errors were most likely to have had difficulty generating a story situation that could be represented by a given multiplication operation. Finally, students' *Added-mathematical-content* errors again indicated difficulty generating a story that appropriately corresponded to a given multiplication operation. The

vast majority of *Added-mathematical-content* errors occurred when students composed a story reflective of the expression $n + n \times m$ rather than the given $n \times m$.

Carpenter and colleagues (Carpenter, Fennema, Franke, Levi, & Empson, 1999) have noted that even very young children can solve multiplication word problems such as the following one: "Megan has 5 bags of cookies. There are 3 cookies in each bag. How many cookies does Megan have all together?" (p. 34). Students' success on such problems indicates that they do have some grasp of the operation of multiplication. We suggest, however, that the link between such a story situation and its symbolic representation (i.e., 5×3) may be tenuous for many students. Whereas students often successfully model and subsequently solve multiplication word problems using repeated addition of groups (Carpenter et al., 1999), students who are provided a multiplication operation in symbolic form do not necessarily connect these symbols to a repeated addition scenario (Koehler, 2004). This interpretation points to the importance of spending ample instructional time on the symbolization process, so that students can make stronger connections between symbolic representations and their referents and develop facility both *looking through* and *looking at* symbols (Kaput et al., 2008).

A second area of concern raised by students' performance on the story-writing task has to do with their abilities to combine multiple operations into coherent stories. Our data point to the existence of a "composition effect" in story writing, as has been shown in past work on symbolization. Students often simply avoided generating story actions for two-operator equations –and did so much more frequently than would have been expected given the frequency of such errors on stories generated for one-operator equations. In addition, on the challenging two-operator start-unknown items, students frequently simplified their task by generating stories that reflected simpler, result-unknown situations—again, more frequently than would have been expected given the frequency of such errors on one-operator items. Taken together, these findings suggest that students found it difficult to integrate multiple mathematical operations. Consistent with this view, students sometimes generated stories that included all the relevant numbers, but not in ways that fit together conceptually. For example, students sometimes generated stories in which different units applied to

each operation, rendering the stories as a whole incoherent. The present findings are reminiscent of past research indicating that students have difficulties symbolizing story problems that involve multiple operations (Heffernan & Koedinger, 1997) as well as solving equations that involve multiple operations (Koedinger et al., 2008).

The story-writing task was designed to assess students' conceptual understanding of symbolic expressions. We believe that it did in fact provide insight into such understanding—particularly concerning multiplication and operation composition issues—that the equation-solving task on its own did not reveal. Although performing multiplication operations was not necessarily difficult for students (as was evident in their good performance on the equation-solving task), the story-writing task revealed difficulty with the underlying meaning of multiplication. Likewise, students' abilities to generate stories to correspond with two-operator equations were poorer than their abilities to solve comparable equations. The nature of students' errors suggests that integrating operations poses a special challenge.

Finally, our findings are consistent with past research that has documented associations between knowledge of concepts and knowledge of procedures. Although we do not wish to argue that the equation-solving task is a purely procedural one, we believe that students who have extensive practice with equation solving can be successful without possessing or activating deep conceptual understanding of algebraic equations. We believe that the novelty of the story-writing task, on the other hand, encourages students to rely more heavily on their conceptual understanding, and thus story writing can provide greater insight into their conceptual understandings of algebraic equations.

Our findings have implications for the mathematics instruction of students in the elementary and middle grades. First, our findings support Russell, Schifter, and Bastable's (2011; Schifter, 1999) call for an increased focus on generalized arithmetic in the elementary grades, especially regarding articulating generalizations about the behavior of the operations. The *Common Core State Standards for Mathematics* also call for opportunities to develop such understanding (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Both the Standards for Mathematical Practice and the

middle school content standards emphasize the need to describe real-world relationships mathematically Students at all grade levels are expected to "make sense of problems and persevere in solving them," which includes "explain[ing] correspondences between equations, verbal descriptions, tables, and graphs...." (p. 9). Asking students to write story problem scenarios to represent different mathematical expressions and equations (including ones that involve multiplicative relationships) is one way to address this standard.

Our findings further suggest that students could benefit from instructional activities that focus on multiplicative relationships and on combining multiple mathematical relationships. One such activity might involve interpreting various components of equations in relation to their referents, including not only isolated numbers and operations, but also expressions such as 14×7, $14 \times n$, or $5 + 14 \times n$. Another activity might involve working with verbally presented problems, which present fewer challenges for meaning making than do symbolic problems (Koedinger & Nathan, 2004). Once students successfully solve verbally presented problems, they could then be guided to apply their solution processes to corresponding symbolic problems, or to symbolize those verbally presented problems.

In brief, our findings document gaps in middle school students' conceptual understanding of algebraic equations, and they highlight the importance of fostering connections between symbols and their referents among middle school students. More broadly, our findings support Kaput et al.'s (2008) argument that although algebraic symbols are powerful tools that can foster students' algebraic reasoning, we should not cut short the process of symbolization if our aim is to promote meaning-making and conceptual understanding.

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Appendix

Equations Used in the Equation-Solving Task
$17 + 54 = n$
$67 - 41 = n$
$5 \times 19 = n$
$28 + n = 74$
$84 - n = 53$
$7 \times n = 91$
$42 + 26 - 13 = n$
$4 \times 12 + 21 = n$
$16 \times 5 - 27 = n$
$35 + n - 18 = 46$
$5 \times n + 23 = 93$
$13 \times n - 22 = 56$

Equations Used in the Story-Writing Task
Version A

Number set 1	Number set 2
$19 + 33 = n$	$43 + 18 = n$
$63 + n - 13 = 91$	$37 + n - 15 = 46$
$45 - n = 21$	$93 - n = 61$
$21 \times 4 - 17 = n$	$14 \times 7 - 23 = n$
$6 \times 13 = n$	$4 \times 21 = n$
$6 \times n + 23 = 89$	$4 \times n + 25 = 77$

Version B

Number set 1	Number set 2
$93 - 32 = n$	$45 - 24 = n$
$37 + 24 - 15 = n$	$63 + 41 - 13 = n$
$43 + n = 61$	$19 + n = 52$
$4 \times 13 + 25 = n$	$6 \times 11 + 23 = n$
$4 \times n = 84$	$6 \times n = 78$
$14 \times n - 23 = 75$	$21 \times n - 17 = 67$