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# Approximate solutions for the model of evolution of cocaine consumption in Spain using HPM and BPEs methods

# Soluciones aproximadas para el modelo de la evolución del consumo de la cocaína en España utilizando el Método de Perturbación Homotópica y el Método de Expansión Polinomial de Boubaker

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# Resumen

En este trabajo, dos métodos son aplicados a un sistema de ecuaciones diferenciales no lineales que modela la evolución del consumo de cocaína en España. Consideraciones teóricas han sido detalladas como guías para demostrar la potencia y la confiabilidad de ambos métodos. Al comparar los resultados obtenidos empleando éstas técnicas se revela que son muy eficientes y convenientes.

***Palabras clave:*** Consumo de cocaína, ecuaciones diferenciales no lineales, modelos matemáticos

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# Abstract

In this paper, two methods are applied to a system of nonlinear differential equations that models the evolution of consumption of cocaine in Spain. Theoretical considerations have been detailed as guides to demonstrate the ability and reliability of both methods. Comparing results obtained by employing these techniques revealed that they are effective and convenient.

***Keywords:*** Cocaine consumption, nonlinear differential equations, mathematical models

## 1. Introduction

Many physical phenomena are modelled, commonly, using nonlinear differential equations, which is a straightforward method to describe the behaviour of their dynamics. Several methods are focused to find approximate solutions to nonlinear differential equations like: Homotopy perturbation method (HPM) [1-4,7-10,12], variational iteration method (VIM) [13-17], Boubaker Polynomials Expansion Scheme (BPES) [18-37,46], Rational Homotopy Perturbation Method[5,6,46], nonliearities distribution homotopy perturbation method [11], among many others.

Epidemic models are an important area of research, due to the need of professionals to predict the behaviour of epidemics over the population; this kind of information can help governments to take important decisions related to the public health policy. In particular, addictions are dangerous epidemics that can cause severe damage to the economy of the countries. Most models which describe cocaine consumption evolution in limited spatial ranges are mainly based on either rational addiction or classical lifetime-utility functions approaches. The first models correlate current, past, and future consumption to the raw demand for cocaine, while second one quantify needs and consumption in terms of unmeasured life cycle variables, time discount factor, and lagged consumption marginal utility. In Spain, many early numerical studies have outlined the particularities of consumption dynamics. Barrio *et al.* [47] proposed epidemic models, while De la Fuente *et al.* [48] and Torrens *et al.* [49] introduced treatment variables and human behaviour patterns, respectively. Therefore, we propose to obtain approximate solutions for the model of evolution of cocaine consumption in Spain reported in [50] using HPM, HPM coupled with Padé [54] approximant [5,38] and BPES methods.

This paper is structured as follows. In Section 2, we present the model of evolution of cocaine consumption in Spain. Sections 3 and 4 present the fundamentals about HPM and BPES methods, respectively. The solutions obtained using both methods are explained in Section 5. Comparisons between the two methods and some other results presented in recent literature are provided in Section 6. Section 7 provides the conclusions about this work.

## 2. Model for evolution of cocaine consumption in Spain

The following equations describe the evolution of the system [50] (see Figure 1 for model synopsis)

$$\begin{cases} \dot{y}_1(t) = \mu P - d y_1(t) - \dfrac{\beta y_1(t)(y_2(t)+y_3(t)+y_4(t))}{P} + \varepsilon y_4(t) \\ \dot{y}_2(t) = \dfrac{\beta y_1(t)(y_2(t)+y_3(t)+y_4(t))}{P} - d_c y_2(t) - \gamma y_2(t) \\ \dot{y}_3(t) = \gamma y_2(t) - d_c y_3(t) - \sigma y_3(t) \\ \dot{y}_4(t) = \sigma y_3(t) - d_c y_4(t) - \varepsilon y_4(t) \\ P = \sum_{i=1}^{4} y_i(t) \\ \text{with initial conditions:} \begin{cases} y_1(0) = y_1^{(0)} \\ y_2(0) = y_2^{(0)} \\ y_3(0) = y_3^{(0)} \\ y_4(0) = y_4^{(0)} \end{cases} \end{cases} \tag{1}$$

where the variables are defined as follows:

I. $y_1(t)$ - No consumers- The population of individuals who have never consumed cocaine.
II. $y_2(t)$ - Occasional consumers - The population of individuals who have consumed cocaine sometimes in their lives.
III. $y_3(t)$ - Regular consumers - The population of individuals who have consumed cocaine sometimes during last year.
IV. $y_4(t)$ - Habitual consumers - The population of individuals who have consumed cocaine sometimes during last month.
V. We assume that the population size is normalized and constant, then $P = 1$.

As reported in Ref. [50], the definition of parameters is:

I. $\mu = 0.01\text{years}^{-1}$ - Represents the average birth rate in Spain.
II. $\beta = 0.09614$ - Represents the transmission rate due to social pressure to consume cocaine.
III. $\gamma = 0.0596$ - Shows the rate at which an occasional consumer becomes a regular consumer.
IV. $\sigma = 0.0579$ - Provides the rate at which a regular consumer becomes a habitual consumer.
V. $\varepsilon = 0.0000456\,\text{years}^{-1}$ - Represents the rate at which a habitual consumer leaves cocaine consumption due to therapy programs.

VI. $d = 0.008388\text{years}^{-1}$ - The average death rate in Spain.

VII. $d_c = 0.01636\text{years}^{-1}$ - The augmented death rate due to drug consumption.

VIII. As reported in Ref. [50], initial conditions deduced from statistics of population from Spain are: $y_1(0) = r_1 = 0.944$, $y_2(0) = r_1 = 0.034$, $y_3(0) = r_3 = 0.018$, and $y_4(0) = r_4 = 0.004$.

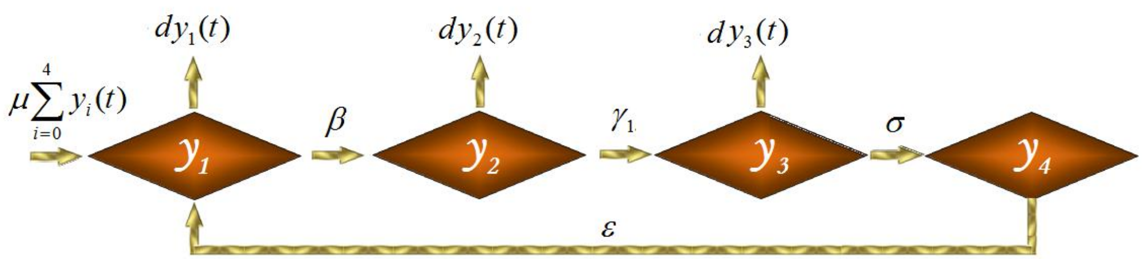


**Figure 1:** Model synopsis for model (1) divided into four classes of consumers

The Ref. [47] and [48] stated that interaction within the four classes of population has different patterns, particularly when the population size is supposed to be constant. Example, if we mix regular and habitual consumers, a big amount of information is lost.

In Ref. [50] the authors report the model (1) and its qualitative characteristics. Nonetheless, in this work we propose some approximate solutions for Eq. (1) based in HPM, HPM-Padé and BPES.

## 3. Fundaments of the homotopy perturbation method

The homotopy perturbation method HPM [1-13] can be considered as a combination of the classical perturbation technique [38,39] and the homotopy (whose origin is in the topology) [40-45], but not restricted to a small parameter like traditional perturbation methods. For example, HPM requires neither small parameter nor linearization, but only few iterations to obtain accurate solutions.

To figure out how HPM method works, consider a general nonlinear equation in the form

$$A(u)-f(r)=0, \qquad r\in\Omega, \tag{2}$$

with the following boundary conditions

$$B(u,\frac{\partial u}{\partial \eta})=0, \qquad r\in\Gamma, \tag{3}$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ a known analytical function, $\Gamma$ is the boundary of domain $\Omega$ and $\partial u/\partial\eta$ denotes differentiation along the normal drawn outwards from $\Omega$ [3,4]. $A$ can be divided into two operators, $L$ and $N$, where $L$ is linear and $N$ nonlinear; from this last statement, Eq. (2) can be rewritten as

$$L(u)+N(u)-f(r)=0. \tag{4}$$

Generally, a homotopy can be constructed in the form [1-3]

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0, \quad p\in[0,1], \; r\in\Omega \tag{5}$$

where $p$ is a homotopy a parameter whose values are within range of 0 and 1, $u_0$ is the first approximation for the solution of Eq. (2) that satisfies the boundary conditions.

When $p\to 0$, (5) is reduced to

$$L(v)-L(u_0)=0, \tag{6}$$

here, operator $L$ possesses trivial solution $v=u_0$.

When $p\to 1$, Eq. (5) is reduced to the original problem

$$N(v)+L(v)-f(r)=0. \tag{7}$$

Assuming that solution for Eq. (5) can be written as a power series of $p$.

$$v=v_0+v_1p+v_2p^2+... \tag{8}$$

Substituting Eq. (8) into Eq. (5) and equating identical powers of $p$ terms it is possible to find values for the sequence $v_0, v_1, v_2,...$ ; where $v_0$ fulfil the boundary conditions of Eq. (2), and the following terms $v_1, v_2,...$ are set to zero at the boundary conditions.

When $p\to 1$ in (8), it yields to the approximate solution for Eq. (2) in the form

$$u=\lim_{p\to 1}(v)=v_0+v_1+v_2+\cdots \tag{9}$$

## 4. Fundaments of the Boubaker Polynomials Expansion Scheme BPES

The resolution of system (2) along with boundary conditions has been achieved using the Boubaker Polynomials Expansion Scheme BPES [18-37]. This scheme is a resolution protocol which has been successfully applied to several applied-physics and mathematics problems. The BPES protocol ensures the validity of the related boundary conditions regardless of main equation features. The Boubaker Polynomials expansion scheme BPES is based on the Boubaker polynomials first derivatives properties

$$\begin{cases} \left.\sum_{q=1}^{N} B_{4q}(x)\right|_{x=0} = -2N \neq 0; \\ \left.\sum_{q=1}^{N} B_{4q}(x)\right|_{x=r_q} = 0; \end{cases} \tag{10}$$

and

$$\begin{cases} \left.\sum_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=0} = 0 \\ \left.\sum_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=r_q} = \sum_{q=1}^{N} H_q \\ \text{with}: H_n = B'_{4n}(r_n) = \left( \frac{4r_n[2-r_n^2] \times \sum_{q=1}^{n} B_{4q}^2(r_n)}{B_{4(n+1)}(r_n)} + 4r_n^3 \right) \end{cases} \tag{11}$$

Several solutions have been proposed through the BPES in many fields such as numerical analysis, theoretical physics, mathematical algorithms, heat transfer, homodynamic, material characterization, fuzzy systems modelling, and biology [18-37].

## 5. Approximations of case study

### 5.1 Solution using HPM method

Using Eq. (5), we establish the following HPM formulation

$$
\begin{aligned}
&(1-p)(\dot{v}_1-\dot{y}_1(0))+p\left(\dot{v}_1(t)-\mu P+dv_1(t)+\frac{\beta v_1(t)(v_2(t)+v_3(t)+v_4(t))}{P}-\varepsilon v_4(t)\right)=0,\\
&(1-p)(\dot{v}_2-\dot{y}_2(0))+p\left(\dot{v}_2(t)-\frac{\beta v_1(t)(v_2(t)+v_3(t)+v_4(t))}{P}+d_c v_2(t)+\gamma v_2(t)\right)=0,\\
&(1-p)(\dot{v}_3-\dot{y}_3(0))+p\left(\dot{v}_3(t)-\gamma v_2(t)+d_c v_3(t)+\sigma v_3(t)\right)=0,\\
&(1-p)(\dot{v}_4-\dot{y}_4(0))+p\left(\dot{v}_4(t)-\sigma v_3(t)+d_c v_4(t)+\varepsilon v_4(t)\right)=0.
\end{aligned} \tag{12}
$$

Where prime denotes differentiation with respect to time $t$, and the initial approximations are

$$
\begin{aligned}
v_{1,0}&=y_1(0)=r_1,\\
v_{2,0}&=y_2(0)=r_2,\\
v_{3,0}&=y_3(0)=r_3,\\
v_{4,0}&=y_4(0)=r_4.
\end{aligned} \tag{13}
$$

From Eq. (8), we assume that the solution for Eq. (12) can be written as a power series of $p$ as follows

$$
\begin{aligned}
v_1&=v_{1,0}+pv_{1,1}+p^2v_{1,2}+p^3v_{1,3}\cdots,\\
v_2&=v_{2,0}+pv_{2,1}+p^2v_{2,2}+p^3v_{2,3}\cdots,\\
v_3&=v_{3,0}+pv_{3,1}+p^2v_{3,2}+p^3v_{3,3}\cdots,\\
v_4&=v_{4,0}+pv_{4,1}+p^2v_{4,2}+p^3v_{4,3}\cdots.
\end{aligned} \tag{14}
$$

We substitute Eq. (14) into Eq. (12), regrouping terms, and equating those with identical powers of $p$ it is possible to fulfil boundary condition for Eq. (17); it follows that $v_{j,k}(0)=0$ ( $j=1,2,3,4.$ and $k=1,2,3,\ldots$) for the homotopy map. The results are recast in the following systems of differential equations

$$
\begin{array}{lll}
p^0: & v'_{1,0}=0, & v_{1,0}(0)=r_1, \\
p^1: & v'_{1,1}+dv_{1,0}-\mu+\beta v_{1,0}v_{3,0}+\beta v_{1,0}v_{4,0}-\varepsilon v_{4,0}+\beta v_{1,0}v_{2,0}=0, & v_{1,1}(0)=0, \\
\vdots & \vdots & \vdots \\
p^0: & v'_{2,0}=0, & v_{2,0}(0)=r_2, \\
p^1: & v'_{2,1}-\beta v_{1,0}v_{4,0}+d_c v_{2,0}-\beta v_{1,0}v_{2,0}-\beta v_{1,0}v_{3,0}+\gamma v_{2,0}=0, & v_{2,1}(0)=0, \\
\vdots & \vdots & \vdots \\
p^0: & v'_{3,0}=0, & v_{3,0}(0)=r_3, \\
p^1: & v'_{3,1}+d_c v_{3,0}-\gamma v_{2,0}+\sigma v_{3,0}=0, & v_{3,1}(0)=0, \\
\vdots & \vdots & \vdots \\
p^0: & v'_{4,0}=0, & v_{4,0}(0)=r_4, \\
p^1: & v'_{4,1}+d_c v_{4,0}+\varepsilon v_{4,0}-\sigma v_{3,0}=0, & v_{4,1}(0)=0, \\
\vdots & \vdots & \vdots
\end{array}
\qquad (15)
$$

Solving Eq. (15) yields

$$
\begin{aligned}
&v_{1,0}=r_1,\\
&v_{1,1}=\left(-dr_1+\mu+\varepsilon r_4+\beta(-r_1r_3-r_1r_4-r_1r_2)\right)t,\\
&\vdots\\
&v_{2,0}=r_2,\\
&v_{2,1}=\left(\beta(r_1r_3+r_1r_2+r_1r_3)-r_1r_2-d_c r_2\right)t,\\
&\vdots\\
&v_{3,0}=r_3,\\
&v_{3,1}=\left(-d_c r_3-\sigma r_3+\gamma r_2\right)t,\\
&\vdots\\
&v_{4,0}=r_4,\\
&v_{4,1}=\left(-d_c r_4+\sigma r_3-\varepsilon r_4\right)t,\\
&\vdots
\end{aligned}
\qquad (16)
$$

Substituting Eq. (16) into Eq. (14) and calculating the limit when $p \to 1$, we obtain the 20th-order approximation

$$
y_k(t)=\lim_{p\to 1}\left(\sum_{i=0}^{20} v_i p^i\right), \qquad k=1,2,3,4 \qquad (17)
$$

We apply parameter values and initial conditions presented in Section 2 to Eq. (17). Next, we apply the resummation method denominated Padé approximation [5,38,54], to obtain the approximations $y_1(t)_{[12/12]}$, $y_2(t)_{[12/12]}$, $y_3(t)_{[8/14]}$, and $y_4(t)_{[12/12]}$, which possesses larger domain of convergence than Eq. (17) as we will see in the discussion section. We will denominate to such coupling of methods as HPM-Padé. From experimentation, we notice that at least a 20th-order approximation (see Eq. (17)) was required to give enough information to the Padé approximant to recast and predict the behaviour of (1) for a larger domain than the power series (17) as depicted in figure 2 in Section 6.

### 5.2 Solution using the Boubaker Polynomials Expansion Scheme BPES

The resolution protocol is based on setting $\tilde{y}_i\big|_{i=1..4}$ as estimators to the *t*-dependent variables $y_i\big|_{i=1..4}$

$$\begin{cases} \tilde{y}_1(t) = \dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0} \xi_k^{(1)} \times B_{4k}(t \times r_k) \\ \tilde{y}_2(t) = \dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0} \xi_k^{(2)} \times B_{4k}(t \times r_k) \\ \tilde{y}_3(t) = \dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0} \xi_k^{(3)} \times B_{4k}(t \times r_k) \\ \tilde{y}_4(t) = \dfrac{1}{2N_0}\displaystyle\sum_{k=1}^{N_0} \xi_k^{(4)} \times B_{4k}(t \times r_k) \end{cases} \tag{18}$$

where $B_{4k}$ are the 4k-order Boubaker polynomials [23-33], $r_k$ are $B_{4k}$ minimal positive roots, $N_0$ is a prefixed integer, and $\xi_k^{(m)}\big|_{k=1..N_0, m=1..4}$ are unknown pondering real coefficients.

The main advantage of this formulation is the verification of initial conditions with respect to time, expressed in Eq. (1), in advance to the resolution process. In fact, thanks to the properties expressed in Eq. (10) and Eq. (11), these conditions are reduced to the inherently verified linear equations

$$\sum_{k=1}^{N_0} \xi_k^{(m)}\Big|_{m=1..4} = -N_0 y_m^{(0)} \tag{19}$$

The BPES solution for Eq. (1) is obtained, according to the principles of the BPES, by determining the non-null set of coefficients $\xi_k^{(m)}\big|_{m=1..4}$ that minimizes the absolute difference between left and right sides of the following equations, which follow a majoring of the sum $P=\sum_{i=0}^{4} y_i(t)$

$$\begin{cases} \frac{1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(1)} r_k \times \frac{dB_{4k}(\hat{t})}{dt} + \frac{d}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(1)} B_{4k}(\hat{t}) - \frac{\varepsilon}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(4)} B_{4k}(\hat{t}) = 0 \\ \frac{1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(2)} r_k \times \frac{dB_{4k}(\hat{t})}{dt} + \frac{d_c}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(2)} B_{4k}(\hat{t}) + \frac{\gamma_1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(2)} B_{4k}(\hat{t}) = 0 \\ \frac{1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(2)} r_k \times \frac{dB_{4k}(\hat{t})}{dt} - \frac{\gamma_1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(2)} B_{4k}(\hat{t}) + \frac{d_c}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(3)} B_{4k}(\hat{t}) \\ \qquad + \frac{\sigma}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(3)} B_{4k}(\hat{t}) = 0 \\ \frac{1}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(4)} r_k \times \frac{dB_{4k}(\hat{t})}{dt} + \frac{\sigma}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(3)} B_{4k}(\hat{t}) + \frac{\varepsilon}{2N_0}\sum_{k=1}^{N_0}\xi_k^{(4)} B_{4k}(\hat{t}) = 0 \\ \hat{t} = t \times r_k \end{cases} \tag{20}$$

Where $N_0 = 241$ in order to maintain a high accuracy and the size constrained.

The final solution is obtained by substituting the obtained values of the coefficients $\left.\xi_k^{(m)}\right|_{m=1..4}$ in Eq. (18).

## 6. Results plots and discussion

Figures 2 and 3 shows a comparison between the Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant (RKF45) [52, 53] solution (built-in function of Maple software for Eq. (1), HPM, HPM-Padé, and BPES approximations. In order to obtain a good numerical reference the accuracy of RKF45 was set to an absolute error of $10^{-7}$ and relative error of $10^{-6}$. Moreover, Figure 2 (a) shows, for all solutions, a non-uniform decreasing profile for the population of individuals who have never consumed cocaine. This feature is a master key for understanding transmission dynamics. In fact, for the given value of transmission rate ($\beta = 0.09614$), it was expected that a short period of constancy (0< $t$ <8) is followed by an avalanche of contamination. Divergence between numerical and analytical solutions is recorded for the period $t$ >40 for BPES , $t$ >40 for HPM and $t$ >80 for HPM-Padé. Therefore, HPM-Padé exhibited a wider domain of convergence. This is due to the known characteristic of the Padé resummation method [54] to recast and predict the behaviour of power series solutions;

increasing notoriously the domain of convergence. The BPES and HPM approximations exhibit a poor convergence in contrast to HPM-Padé because equations (17) and (18) are pure polynomial solutions while HPM-Padé produces rational expressions. BPES method is merely based on strict respect of initial conditions; consequently BPES protocol is less sensitive to long term dynamics than HPM. For perusal, references [25-29] evoke this item for "avalanche of contamination"-like long term perturbation. Since the difference concerns only the population of occasional consumers or individuals who have never consumed cocaine, it can be formulated that long-term prediction among safe population cannot be subjected to analytical modelling, oppositely to that of regular and habitual consumers groups. This phenomenon has been already recorded by Sánchez *et al.* [51].

## 7. Conclusion

In this paper, powerful analytical methods Homotopy Perturbation Method (HPM) and Boubaker Polynomials Expansion Scheme (BPES) are presented to construct analytical solutions for the model of evolution of cocaine consumption in Spain. The numerical experiments are presented to support the theoretical results. In order to enlarge the domain of convergence of the HPM polynomials, we apply the Padé resummation method. Therefore, the HPM-Padé solution exhibited a wider domain of convergence than HPM and BPES, reaching a good agreement with exact solution for the range $t \in [0,80]$. Further research is required in order to obtain solution with larger domain of convergence that can lead to a better understanding of the dynamics of the cocaine consumption in Spain and the relationship with its parameters.

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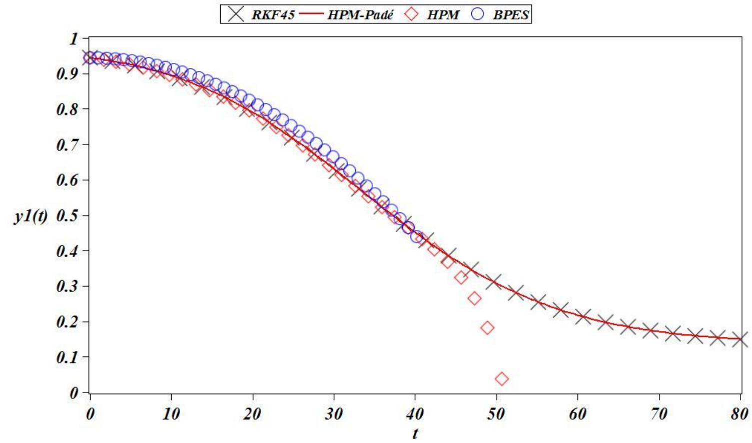
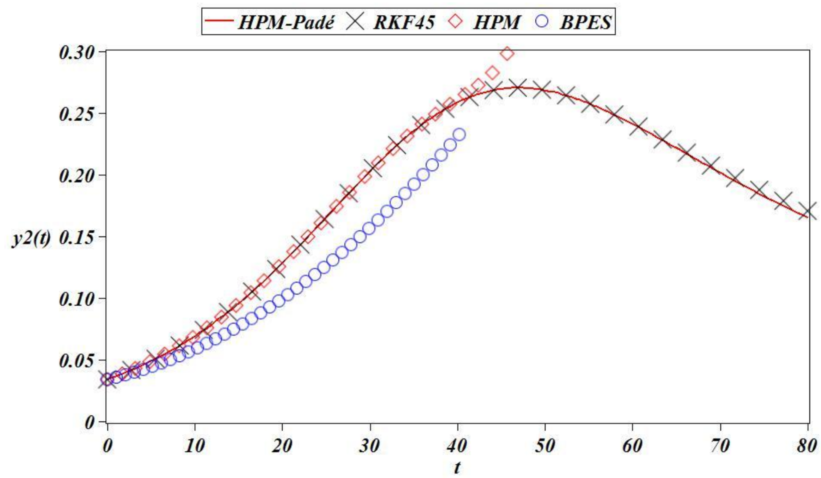
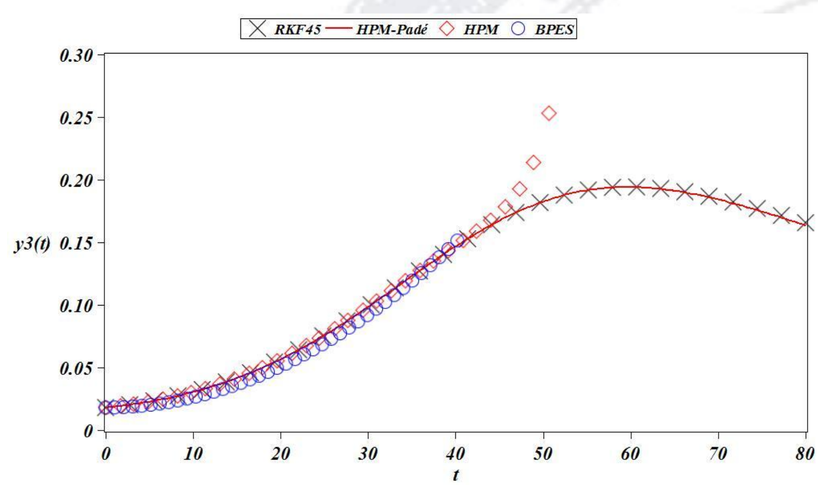



(a)

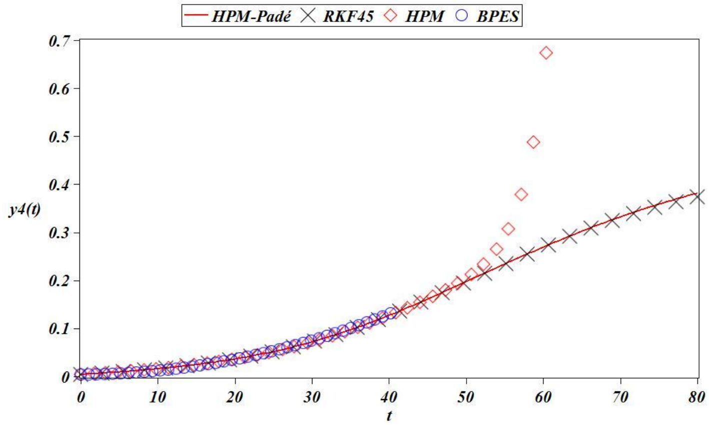



(b)

**Figure 2:** HPM, HPM-Padé, and BPES approximations ( $y_1(t), y_2(t)$ ) for RKF45 solution of (1).



(c)




(d)

**Figure 3:** HPM, HPM-Padé, and BPES approximations ( $y_3(t), y_4(t)$ ) for RKF45 solution of (1).

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