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Discrete Affine Term Structure Models Applied to the Government Debt and Fiscal Imbalances

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Abstract

This paper discusses the relationship between government deficits and changes in total debt outstanding as well as the relationship between yields and real interest rates, inflation expectations and credit spreads using inflation linked swap data. This study also shows how affine term structure models (ATSMs) can be used to link the theory of the price level to term structure dynamics. When central banks are independent, increases in government deficits result in increases in the credit spreads and not necessarily in increases in the price level. Empirical results show that when modelling Spanish and Greek government yields fitted values improve significantly when credit spreads are included in the state vector. Most importantly, this study shows how affine term structure models can be used for the analysis of the time path of changes in government debt, government primary surplus and credit spreads. Finally, another novelty of this work is to apply the ATSM methodology to describe, for instance, how EU-governments' deficits deteriorate as a consequence of the time path of shocks in macroeconomic variables such as unemployment, as unemployment shocks can have widening effects on governments' fiscal imbalances and this will vary depending on the governments' risk profile.

Keywords:

Affine term structure model, Fiscal theory, Fiscal imbalances, Credit spreads.

JEL classification:

B22, C58, E27, G17.

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Modelos afines discretos de estructura temporal aplicados al **análisis de la dinámica de la deuda pública y los desequilibrios fiscales**

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Resumen

En este artículo se discute la relación existente entre el déficit público y los cambios en la deuda total en circulación, así como aquella entre los rendimientos y los tipos de interés reales, las expectativas inflacionarias y los spreads de crédito, a partir de la utilización de swaps de inflación. En él se muestra cómo los modelos afines de estructura temporal (ATSMs) pueden utilizarse para ligar la teoría del nivel de precios con la dinámica de la estructura temporal. Cuando los bancos centrales son independientes, los incrementos en los déficit públicos dan lugar a spreads de crédito y no se traducen necesariamente en un incremento del nivel de precios. Los resultados empíricos muestran que, cuando se modelan los rendimientos de los bonos griegos y españoles, los valores ajustados mejoran de forma significativa si los spreads de crédito se incluyen en el vector de estados. Más importante aún es el hecho de que este artículo revela cómo los ATSM pueden ser muy útiles en el análisis de la dinámica temporal de cambios en la deuda pública, el superávit público primario y los spreads de crédito. Finalmente, otra novedad de este trabajo es la aplicación de la metodología ATSM para describir, por ejemplo, cómo los déficit públicos europeos se deterioran como consecuencia de la dinámica temporal de los shocks en variables macroeconómicas como el desempleo, ya que los shocks en el desempleo pueden tener efectos amplificadores en los desequilibrios fiscales gubernamentales que variarán dependiendo del perfil de riesgo gubernamental.

Palabras clave:

Modelos afín de estructura temporal, teoría fiscal, desequilibrios fiscales, spreads de crédito.

1. Introduction

This essay explores the use of discrete time affine term structure models applied to the theory of the price level and debt management in order to study the optimal term structure, and hence contribute to fiscal stabilisation policies and the optimal taxation approach. This paper makes use of affine term structure models in a similar set up as seen in the celebrated papers from Backus, Foresi and Telmer (1998 and 1996) and Backus, Telmer and Wu (1999). The paper applies the single and multifactor cases under Vasicek (1977) taking into account some of the developments seen in the latest affine term structure research such as Duffie and Kan (1996), Piazzesi (2010) and Singleton (2006).

Starting point in this paper is the flow identity under the fiscal theory of the price level as seen in Cochrane (2001), Leeper (1995), Sims (1994), Woodford (1995, 1996) and Dupor (1997). With respect to the optimal taxation approach, this paper is a contribution to some of the developments achieved in Missale (1997), Faraglia *et al.* (2008), Angeletos (2002) and Buera and Nicolini (2004). The novelty of this paper is that uses affine term structure models to describe the path at which surplus change and hence affect the price level with the ultimately effect on yields. Hence, affine terms structure models can be used to link the theory of the price level, debt management and optimal taxation approach, in order to identify the optimal term structure.

This paper is organised as follows: Section 2 basic and new concepts are introduced and discussed; section 3 an affine terms structure model is introduced; section 4 a recursive solution is presented for two possible scenarios: when the theory of the price level is at work and when the theory is not at work; in section 5 and subsequent subsections the model is extended in order to show the path of surplus shocks on total notional debt outstanding; the path of shocks from total notional debt outstanding on the price level and credit spreads and; the path of shocks stemming from government revenues and government expenditure and its effect on primary surplus. Section 6 calibrates some of the most relevant models already discussed in previous section with real data and present results, section 7 discusses the policy implications of our findings and finally, section 8 outlines main conclusions and final remarks.

2. Recalling some basic concepts and introducing new ones

The flow identity depicts that surplus equals redemptions minus net new issuances.

$$S_t^N = P_t^N B_t^N - P_t^{N+1} B_t^{N+1}, \tag{1}$$

for $S_t^{(N)}$ being the net primary surplus in t cumulated in period N, $P_t^{(N)}$ being the redemption price in t for a zero coupon bond with maturity N and remaining time to maturity N= 0. $B_t^{(N)}$ depicts the notional amount of a bond in time t with maturity N and remaining time to maturity N=0. Analogously, $P_t^{(N+1)}$ being the price of a new zero bond issued in t with remaining time to maturity N+1 and $B_t^{(N+1)}$ depicts the new bond's notional amount with remaining maturity N+1 in t. Notice that it is assumed that at origination all bonds have same maturity profile, it is denoted with N and with N+1 in order to differentiate when a bond is maturing or when a bond is a new issue, as for N=0 implies that $P_t^{(N)}B_t^{(N)}$ are the maturing amounts and for N+1=1 implying that the new issue amount of $P_t^{(N+1)}B_t^{(N+1)}$ will matured in t+1 and, at that point, the maturity of the bond will be N=0. The analysis should not be limited to total notional debt outstanding and surplus, but also include all other assets in the economy for which the government acts as a guarantor. This is because as these assets deteriorate together with surplus, the government is also force to increase its issuance in order to support asset prices, particularly those from the banking system. For the sake of simplicity the analysis here is limited to surplus shocks, but the reader can also apply it to shocks to assets in the banking system for which the government acts as a guarantor.

Equation (1) shows that if surplus $S_t^{(N)}$> 0 implies that $P_t^{(N)}B_t^{(N)}>P_t^{(N+1)}B_t^{(N+1)}$, hence the government is reducing total debt outstanding, as redemptions $P_t^{(N)}B_t^{(N)}$are greater than the new bond issuances $P_t^{(N+1)}B_t^{(N+1)}$. Alternatively, if surplus $S_t^{(N)}$<0 (thus is a deficit) implies that $P_t^{(N)}B_t^{(N)}<P_t^{(N+1)}B_t^{(N+1)}$ which means the government is increasing its total debt outstanding, as the new bond issuances $P_t^{(N+1)}B_t^{(N+1)}$ required to be greater than redemptions $P_t^{(N)}B_t^{(N)}$ in order to have enough funding to cover deficits.

Thus, a deterioration of the net primary surplus – hence an increase in the government's deficit – would require an increase in net new issues. An increase in net new issues is necessary in order to roll over the maturing debt whilst still be able to cover the increase in current financing requirements. Should the new issue price $P_t^{(N+1)}$deteriorate, then the government will be forced to increase the new-issue notional amount $B_t^{(N+1)}$ in order to compensate for the fall in the price and thus be able to gather enough funds to pay back redemptions $P_t^{(N)}B_t^{(N)}$ and finance its deficit $S_t^{(N)}$.

It is assumed that the government only increases debt if it is strictly necessary, hence (1) would imply that should $S_t^{(N)}$ improve by exhibiting an increase in surplus, the government would reduce total debt outstanding as a consequence of a decrease in its funding requirements. If we think about investors' expectations for a 1 period forward at t+1, from equation (1) it is possible to intuit that the redemption price and redemption amount are known values at time t. However and, what the market participants do not know is the new issue cash equivalent of next debt roll-over in period t+1 of $P_{t+1}^{(N+1)}B_{t+1}^{(N+1)}$. In fact, for the case of governments under financial distress investors are wary about their ability to issue new debt in times of low consumption growth and thus might not believe that they would obtain access to funds enough to redeem the maturing debt of $P_t^{(N)}B_t^{(N)}$ and finance their deficits $S_t^{(N)}$. The idea is that new issuances need to be sufficient so that (1) equates without forcing the government to issue at unfavourable prices $P_t^{(N+1)}$ and hence at a higher yield. Notice that the government avoids default by accepting lower prices and increasing debt outstanding if necessary. In order to depict this more precisely, (1) can be re-arranged, by moving $S_t^{(N)}$ as an explanatory variable to the right hand side and $P_t^{(N+1)}B_t^{(N+1)}$ to the left as endogenous, which can be specified as follows:

$$P_t^{N+1}B_t^{N+1} = P_t^{N}B_t^{N} - S_t^{N}\ , \tag{2}$$

which means that if $\Delta S_t^{(N)} > 0$, then inevitably $\Delta\left(P_t^{N+1}B_t^{N+1}\right) < 0$, as a result of a fall in financing requirements.

The nominal price of a zero coupon bond will contain information about the price level as well as information about the real interest rates. Assuming that the real interest rates remain constant, it could be said that an increase in the price level will result in an increase in yields with the subsequent fall in the bond price. A possible specification could be:

$$y_t^{(N+1)} = -\frac{\ln E\left\lfloor P_t^{N+1}\right\rfloor}{N+1}\ . \tag{3}$$

Taking into account that yields contain information about real interests and expected inflation implies:

$$y_t^{(N+1)} = y_t^r + \ln\left(\frac{\Pi_{t+1}}{\Pi_t}\right), \tag{4}$$

for $y_t^{(N+1)}$ being the nominal yield of the zero coupon bond with maturity $N+1$ comprising the sum of the real interest rate y_t^r and the rate of growth of the price level or inflation being $\ln(\Pi_{t+1}/\Pi_t)$. Notice that by normalising current price level Π_t to 1 it makes no difference if the level or change in the level is used. The relationship resulting from (4) and (3) says that an increase in the price level would subsequently result in an increase in nominal yields with the subsequent fall in the present value of a new issue and hence increase government's costs of financing.

However, governments cannot always influence monetary policy which means they cannot determine the path of inflation, for instance, when central bank acts independently. In this case (4) would require a modification, whereby yields are obtained from a benchmark curve or short rate usually a risk free reference plus a credit spread. This could be specified as follows:

$$y_t^{(N+1)} = r_t^f + \theta_t^{(N+1)}\ , \tag{5}$$

for $r_t^f = y_t^{(r)} + E\left[\ln\frac{\Pi_t}{\Pi_{t-1}}\right]$

Equation (5) describes the relationship between yields $y_t^{(N+1)}$, the short rate r_t^f, real yields $y_t^{(r)}$ and the spreads $\theta_t^{(N+1)}$. Now, we know that the fiscal theory suggests that governments' choice of how to finance its debt play an important role on the determination of the time path of the inflation rate. However, if government debt is issued in a foreign currency or in a currency which governments' have no or little control, then the theory of the price level is less likely to be at work. Instead, governments' choice of how to finance its debt will have a subdued role on the determination of the time path of the inflation rate but rather an important role in the determination of the time path of the credit spread. This means that when the theory of the price level is at work, then (4) describes best yields behaviour as a function of log price level changes and when governments issue in a currency which they cannot control, equation (5) will best describe yields behaviour.

The ability of the government to issue new debt in t with maturity N+1 without incurring in a significant deterioration of its financing costs will depend largely on the market's view about government's ability to raise new funds in a future date, let us say t+1, which will also depend on the size of government's deficit at t+1. Why? Because the market's appetite to lend today will depend on their view about getting their investment redeemed in the future. To formulate this more precisely, I will adapt (2), and show that current issue price depends on the market's view about government's ability to issue new debt or to roll over the maturing one which will largely depend on the market's expected future government yields and thus government's surplus, hence:

$$E[P_t^{N+1}]B_t^{N+1} = E[m_{t+1}P_{t+1}^{N+1}]B_{t+1}^{N+1} - E[m_{t+1}S_{t+1}^{N+1}]. \qquad (6)$$

Equation (6) shows that net present value of total debt outstanding in t with maturity N+1 and hence maturing in time t+1 should equal the net present value of the new bonds to be issued in t+1 minus the present value of expected future surplus. This is because in this model governments issue new debt in order to repay maturing one. For m_{t+1} being the stochastic discount factor which is related to one-period bond yields (or the short rate) inversely as follows,

$$y_t^{(1)} = -\ln E\left[m_{t+1}\right] \qquad (7)$$

Between equations (2) and (6) there are important theoretical differences which are worth mentioning. Equation (2) says that the net present value of total debt outstanding depend on current surpluses and current redemptions. If surplus deteriorates, the government is required to rise more funding with the subsequent increase in the price level. However, (6) is instead saying that current bond prices depend not only on the expected future surplus, but also on markets' expectation on government's ability to issue new debt in the future, as the ability to issue new debt in the future will determine the size of redemptions of maturing bonds. This is an important difference, thus we are not saying anymore that the present value of total current debt outstanding depends on the present value of expected future surpluses but that also depends on expected future new bond prices. This means that in the hypothetical case that despite that surplus is expected to deteriorate in the future, current bond prices can still remain unaffected as long as the market continues to believe that the government will still be capable of issuing new debt, and hence be able to roll over maturing debt. I will show however, that this happens only if government's deficit remains within sustainable levels above which the bond becomes a risky asset and hence a Ponzi scheme.

Without loss of generality I will change (6) slightly as follows:

$$E\left[P_t^{N+1}\right]=E\left[m_{t+1}P_{t+1}^{N+1}\right]B_{t+1}^{N+1}-E\left[m_{t+1}S_{t+1}^{N+1}\right]. \tag{8}$$

In equation (8) above, $B_t^{(N+1)}$ equals 1 so that for simplicity's sake only $B_{t+1}^{(N+1)}$ is left in the expression. (8) shows that if $E\left[m_{t+1}S_{t+1}^{N+1}\right]=0$ then current present value $E\left[P_t^{N+1}\right]$ will equal the net present value of total new issuances rolling over in t+1, which has been specified as $E\left[m_{t+1}P_{t+1}^{N+1}\right]B_{t+1}^{N+1}$ and would imply that B_{t+1}^{N+1}=1 so that $E\left[P_t^{N+1}\right]=E\left[m_{t+1}P_{t+1}^{N+1}\right]$. Alternatively, if $E\left[m_{t+1}S_{t+1}^{N+1}\right]<0$, thus if the government incurs a deficit, then it would necessarily need to be that $B_{t+1}^{N+1}>1$.

Finally and recalling some basics, we know that the expected price at t with maturity N+1 of a bond that redeems at t+1 is usually specified as follows:

$$E\left[P_t^{N+1}\right]=E\left[m_{t+1}P_{t+1}^{N}\right] \tag{9}$$

And by applying natural logarithms yields:

$$\ln\left[P_t^{(N+1)}\right]=\ln\left[m_{t+1}\right]+\ln\left[P_{t+1}^{(N)}\right]. \tag{10}$$

Equations (8) and (9) show that there are two ways of solving this, as the right hand side of (8) also equals the right hand side of (9). This paper will start solution for equation (9) and workout (8) thereafter. Solution for (9) would be rather straight forward, as solution for (8) requires a definition of the time path for each of the components such as surplus, revenues, expenditure, total notional debt outstanding and the price level.

3. The theory of the price level and risk-free assets: the model

Equation (9) will be solved using an affine term structure model as in Backus, *et al.* (1998) and a Vasicek (1977) stochastic process for the state variables.

As seen in most recent affine term structure literature log prices can be specified as a linear function of a state vector x_{t+1} as follows:

$$-\ln\left[P_{t+1}^{(N)}\right] = A(N) + B(N)' x_{t+1} , \tag{11}$$

for $A(N)$ being a scalar, $B(N)'$ a $1{\times}k$ vector of coefficients and x_{t+1} a $k{\times}1$ vector of state variables. Note that the transpose of a vector or matrix is specified with a " ' ". Equation (11) is only a guess, as the functional form is not known. However, the literature appears to have generally accepted this as seen in Piazzesi (2010), Singleton (2006), Cochrane (2005), as well as in Backus *et al.* (1996) and (1998) and seminal papers of Duffie and Kan (1996).

From our guess shown in (11) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$. These parameters are obtained by linking observable yields to an observation equation describing the behaviour of a state space vector. This can be done by combining equations (3) at t+1 with (11) which boils down to:

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)'}{N} x_{t+1} . \tag{12}$$

Thus the short rate could be specified as follows:

$$y_{t+1}^{(1)} = A(N{=}1) + B(N{=}1)' x_{t+1} . \tag{13}$$

Empirically, equation (14) would look like:

$$y_{t+1}^{(1)} = \gamma_0 + \gamma_1' x_{t+1} \tag{14}$$

It is also needed to specify the stochastic process for x_{t+1} as well as for the stochastic discount factor shown in (7). A good starting point is to use the pricing kernel à la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977) process. A possible specification would be like:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1} \tag{15}$$

$$-\ln\left[m_{t+1}\right] = \delta + y_t^{(1)} + \lambda' \varepsilon_{t+1} \tag{16}$$

Equation (15) describes the stochastic process of the independent state variables. This is the usual mean reversing process whereby Δx_{t+1} is likely to be negative if x_t is above $\bar{x}$ and, is likely to be positive if x_t is below its mean $\bar{x}$. x_t and $\bar{x}$ are both k-dimensional vectors. Φ is a k x k matrix of diagonal elements Φ_i which represent the speed of adjustment at which each of $x_{i,t}$ elements reverse to their means. σ_x is a diagonal k x k matrix comprising the volatility of the state variables. ε_{t+1} is a k-vector of shocks moving x_t away from $\bar{x}$ and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance 1.

Equation (16) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (15) was originally the univariate Vasicek (1977) case. In this essay we transform this specification and adapt it for the multifactor case of a k-dimensional vector of state variables as in Jakas (2012). Same as in Backus *et al.* (1998) δ is specified as follows:

$$\delta = \frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 . \tag{17}$$

Clearly, specification (17) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that with (17), now (16) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 - y_t^{(1)} - \lambda' \varepsilon_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 - y_t^{(1)} \tag{18}$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{k} \lambda_i^2 - y_t^{(1)} - \lambda' \varepsilon_{t+1}\right] = \sum_{i=1}^{k} \lambda_i^2 . \tag{19}$$

And assuming $E[\ln x]=\mu(x)+ \frac{1}{2}\sigma^2(x)$, which yields:

$$E\left[\ln m_{t+1}\right] = -y_t^{(1)} , \tag{20}$$

it would be assumed that the price level Π_t is a function of total notional debt outstanding $B_t^{(N)}$ and total debt outstanding increases as $S_t^{(N)}$ deteriorates, hence as surplus turns into deficit. Finally, $S_t^{(N)}$ depends on the state vector x_{t+1}, so that macroeconomic shocks affecting government surplus will have an effect on the price level only if surplus shocks increase total debt outstanding and ultimately affecting the price level. These relationships could be specified as follows:

$$y_t^{(1)} = \alpha_0 + \alpha_1 y_t^r + \alpha_2 E\left[\ln\left(\frac{\Pi_t}{\Pi_{t-1}}\right)\right] \tag{21}$$

$$\Pi_t\left(B_t^N\right) = \eta_0 + \eta_1 B_t^N\left(S_t^N, S_t^{N*}\right) \tag{22}$$

$$B_t^N\left(S_t^N, S_t^{N*}\right) = \varphi_0 + \varphi_1\left[S_t^{N*} - S_t^N\left(x_t\right)\right] \tag{23}$$

$$S_t^N\left(x_t\right) = \beta_0 + \beta_1' x_t \tag{24}$$

Equation (21) is nothing but a way of estimating the unobservable parameters specified in (4), thus the coefficient for the real interest rate and the coefficient for the expected inflation are estimated via observed inflation-linked swap data. Equations (22) and (23) are our empirical interpretation of the theory of the price level linked to the short rate. Hence, here we describe that a deterioration of government's budget deficit beyond a certain unobservable limit results in a systematic increase in governments total debt outstanding and hence in an increase in the price level Π_t, as this theory is at work when central bank is not fully independent and governments are able to monetise their deficits.

Notice that for the case where the theory of the price level is not at work because the government does not have control over the monetary policy, it would imply that (21) and (22) need to be adapted to account for the credit spreads. In this case, (21) and (22) are transformed to (25) and (26) as specified below. Notice that the use of this numbering for the equations is fortuitous, as the intention is to call the reader's attention to the idea that (25) and (25) are a derivation of (21) and (22). If the theory of the price level is not at work it is because governments cannot decide over the path of inflation and instead any increase in deficits result in a deterioration of the credit spreads, instead of an increase in the price level:

$$\theta_t^{(1)}\left(B_t^N\right)=\eta_0+\eta_1 B_t^N\left(S_t^N,S_t^{N*}\right) \tag{25}$$

$$y_t^{(1)}=\alpha_0+\alpha_1 y_t^r+\alpha_2 E_{t-1}[\pi_t]+\alpha_2\theta_t^{(1)}\left(B_t^N\right), \tag{26}$$

for $E_{t-1}[\pi_t]=E\left[\ln(\Pi_t/\Pi_{t-1})\right]$ being the expected inflation obtained from inflation-linked (IL) swaps.

Notice that (26) proposes a possible way of estimating (5), under the assumption that the true risk free is observable from IL-Swap data. Substituting (22) in (21) (or (25) in (26) and then continue the complete chain of substitutions through (22) to (24) it is possible to show that yields are a function of a state space which I summarised below:

$$y_t^{(1)}=\psi_0+\psi_1' x_t\ . \tag{27}$$

Equations (21) to (27) will depend on the difference between the level of $S_t^{(N)}$ and an unobservable *sustainable* $S_t^{(N)*}$ shown in (23). Thus for any level of government deficit where $E_{t-1}\left[S_t^{(N)}\right]<0$ and $S_t^{(N)*}<0$ restricted to $E_{t-1}\left[S_t^{(N)}\right]<S_t^{(N)*}$ it would result in $\varphi_1<0$ and $\eta_1<0$, which implies that the fiscal theory of the price level is at work and thus any shocks in aggregate demand that results in a deterioration of government finances as a consequence of such surplus shocks will affect the price level and result in an increase in yields. Therefore, the short rate is governed by (27) and not by (14). If the price level is not at work and if surplus shocks have an effect in the total debt outstanding, the same logic applies but instead (21) and (22) are replaced by (25) and (26) because government's choice of how to finance its debt has an effect on the determination of the path of credit spreads and not on the determination of the path of price level.

Notice that for any $E_{t-1}\left[S_t^{(N)}\right] \geq S_t^{(N)*}$ would result in $\varphi_1=0$ and $\eta_1=0$, which means that any expected deficit within a given level of sustainability will have no effect on the price level and hence the fiscal theory of the price level would be subdued so that the short rate will be governed by (14) instead of (27). When the short rate is governed by (14) instead of (27) only then it could be said that the government bond acts as a hedge for times when aggregate marginal utility growth is high and government bond prices are negatively correlated to aggregate consumption growth. On the contrary, if the theory of the price level rules, government bond prices will be positively correlated to aggregate consumption growth mainly because a deficit that is perceived as unsustainable is expected to have an effect on the price level. If governments cannot monetise their deficits any deterioration of their surplus will only be financed with increases in total debt outstanding which will not generate inflation but increases on the credit spread. When the fiscal theory of the price level is at work, the government bond is considered a risky asset and hence there would be no difference between contingent and non-contingent bond payoffs, as they would all be contingent to the price level or the credit spreads. However, if the fiscal theory of the price level is not at work, the government bond is risk-free and the difference between contingent and non-contingent bonds does mater.

4. Solving when the theory of the price level is at work and when is not

When the theory of the price level is at work the signs of the coefficients depicted in (27) will rule. However, when the theory of the price level is subdued, the signs of the coefficients will be governed by (14). For which the solution assuming that (27) holds for a set of coefficients in specified in (12) boils down to:

$$A(N+1)=\psi_0+A(N)+B(N)'\Phi\bar{x}+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2-\left(\lambda'+B(N)'\sigma_x\right)^2\right) \quad (28)$$

$$B(N+1)'=\left(\psi_1'+B(N)'(I-\Phi)\right) . \quad (29)$$

If the reader is interested in the algebra on how to obtain equations (28) and (29) refer please to Appendix, Section I.

The solution is obtained by computing the present value recursively using (10) for some guess of coefficients from (11). Since $P_{t+1}^{(N)}=1$ and $A(N=0)=B(N=0)'=0$, which means this can be solved recursively, as for 1 period would imply $A(N=1)=\psi_0$ and

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$B(N{=}1)' = \psi_1'$ which means that equals the short rate as described in (27). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (10) to compute the present value of an N+1 maturity bond.

All is needed is to line up (28) and (29) into (12) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. Parameters ψ_0 and ψ_{1i} are free and obtained empirically via OLS and the signs for parameters $B(N)_i$ in (12) depend on ψ_{1i} .

When the theory of the price level is not at work, equations (28) and (29) change, and this is because equation (14) is at work instead of equation (27) and by applying the same algebra discussed in Appendix Section I it would yield:

$$A(N+1) = \gamma_0 + A(N) + B(N)'\,\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left(\lambda' + B(N)'\sigma_x\right)^2\right) \quad (30)$$

$$B(N+1)' = \left(\gamma_1' + B(N)'(I-\Phi)\right) . \quad (31)$$

When governments issue in a foreign currency or in a currency in which they have no or little control the theory of the price level is less likely to occur, however, the same forces which have an effect on surplus shocks will result instead in an increase in the credit spreads. So that in a similar fashion to equations (21) to (27) to obtain (28) and (29) replacing the price level for the credit spread as endogenous variable would give us similar analytical results. All is needed is to account for (25) and (26).

Thus, macroeconomic shocks that result in a deterioration of government's net primary surplus beyond a sustainable level can result in increases in total notional debt outstanding which ultimately either result in increases in yields due to increases in the price level or results in increases in yields due to deterioration of the credit spreads. Either ways result in higher yields and hence in a further deterioration of government finances.

5. Solving by analysing fiscal shocks on surplus debt rollover risk

In order to solve equation (8) it is required to have a model describing the behaviour of the effects of: 1) surplus shocks on total notional debt outstanding; 2) the effects of innovations in total notional debt outstanding on the price level or credit spreads and; 3) modelling the path of government surplus by analysing macroeconomic innovations on government revenues and government expenditure. So this section will be organised in three subsections, outlining the above mentioned points.

5.1. The effects of surplus shocks on total notional debt outstanding

This chapter would make use of affine terms structure models to describe the path of surplus shocks and its effect on total notional debt growth:

$$E\left[B_t^{(N)} g_{t+1}\right] = E\left[B_{t+1}^{(N)}\right], \tag{32}$$

where $B_t^{(N)}$ being the current notional debt outstanding in time t which is expected to grow at g_{t+1} resulting in a total nominal debt at t+1 equal to $E\left[B_{t+1}^{(N)}\right]$. Equation (32) describes the path at which $B_t^{(N)}$ would need to grow from t to t+1, for which a *growth kernel* will be used similar to a pricing kernel as used in the affine term structure literature.

Thus g_{t+1} will be assumed that is a sort of growth kernel at time t+1 for period $N = t+1$ so that if $g_{t+1}>1$ it would imply that nominal debt increases during period t to t+1, if $g_{t+1}<1$ it would imply that nominal debt decreases between t to t+1. Rearranging (32) as a function of current total notional debt outstanding would yield:

$$E\left[B_t^{(N)}\right] = E\left[\left(g_{t+1}\right)^{-1} B_{t+1}^{N}\right] . \tag{33}$$

Using the natural logarithm notation (33) would look like:

$$\ln\left[B_t^N\right] = \ln\left[B_{t+1}^N\right] - \ln\left[g_{t+1}\right]. \tag{34}$$

Now we need to specify $\ln\left[B_{t+1}^{(N)}\right]$ and $-\ln\left[g_{t+1}\right]$ for which a similar expression will be used as in Backus *et al.* (1998) and which for simplicity's sake and without loss of generality has been specified as follows:

$$-\ln\left[g_{t+1}\right] = \frac{1}{2}\lambda^2 + \Delta b_t^{(1)} + \lambda \varepsilon_{t+1} \tag{35}$$

$$\Delta b_t^{(1)} = \varphi_0 + \varphi_1 \Delta s_t^{(1)}, \tag{36}$$

for λ depicting the sensitivities at which $-\ln\left[g_{t+1}\right]$ changes due to shocks in ε_{t+1}. Equation (35) is fortuitous, the notation is on purpose so that under normality $E\left[\ln g_{t+1}\right]$ equals $-\Delta b_t^{(1)}$. In equation (36) $-\Delta b_t^{(1)}$ represents the debt growth rate in 1 year as a consequence of surplus shocks, φ_0 and φ_1 are coefficients and $\Delta s_t^{(1)}$ depicts the one period government's cumulated total primary surplus in time t.

Under these assumptions a government's total notional debt outstanding is expected to grow as follows:

$$B_{t+1}^{(N)} = B_t^{(N)} \exp\left[\Delta b_{t+1}^{(1)} \times N\right] , \quad (37)$$

For simplicity's sake and without loss of generality it is assumed $B_{t+1}^{(N)} = 1$. Applying natural logarithms to (37) and rearranging yields:

$$\Delta b_{t+1}^{(N)} = \frac{1}{N} \ln\left[B_{t+1}^{N}\right] . \quad (38)$$

It is also assumed that:

$$-\ln\left[B_{t+1}^{(N)}\right] = A(N) + B(N)\Delta s_{t+1}^{(1)} . \quad (39)$$

Notice that the functional form shown in (39) is only a guess that works quite well when plugging (39) in (40), which results in:

$$\Delta b_{t+1}^{(N)} = -\frac{A(N)}{N} - \frac{B(N)}{N} \Delta s_{t+1}^{(1)} . \quad (40)$$

The final assumption here is that surplus $\Delta s_{t+1}^{(1)}$ follows a Vasicek (1977) process as follows:

$$\Delta s_{t+1}^{(1)} = \Delta s_t^{(1)} + \phi\left(\overline{s}_{\Delta s}^{(1)} - \Delta s_t^{(1)}\right) + \sigma_s \, \varepsilon_{t+1} . \quad (41)$$

Equation (41) says that surplus will have a mean reversing AR(1) behaviour. Equation (41) describes the stochastic process of the government primary surplus. This is the usual mean reversing process whereby $\Delta s_{t+1}^{(1)}$ is likely to be negative if $\Delta s_t^{(1)}$ is above its mean and, is likely to be positive if $\Delta s_t^{(1)}$ is below its mean. If on average surplus is zero, the adjustment will depend fully on $\phi \Delta s_t^{(1)}$. For which the solution for the coefficients in (40) would look like:

$$A(N+1) = -\varphi_0 - A(N) - B(N)\phi \overline{s}_{\Delta s}^{(1)} + \frac{1}{2}\left[B(N)\sigma_{\Delta s}\right]^2 \quad (42)$$

$$B(N+1) = \left[B(N)(1-\phi) - \varphi_1\right] . \quad (43)$$

For a detail algebra on how to get to (42) and (43) the reader should refer to Appendix Section II.

5.2. The effect of changes in total notional debt outstanding on the price level and credit spread

In a similar fashion to previous chapter, this part of the paper will make use of affine terms structure models to describe the path of shocks on notional debt outstanding and its effects on the price level. The path of the price level can be specified as follows:

$$E\left[\Pi_t^{(N)}\,\pi_{t+1}^{(N)}\right]=E\left[\Pi_{t+1}^{(N)}\right], \tag{44}$$

where $E\left[\Pi_t^{(N)}\pi_{t+1}^{(N)}\right]$ being the price level in time t which is expected to grow at $\pi_{t+1}^{(N)}$ resulting in an expected future price level at t+1 equal to $E\left[\Pi_{t+1}^{(N)}\right]$. Equation (44) describes the path at which $\Pi_t^{(N)}$ would need to grow from t to t+1. For $\pi_{t+1}^{(N)}$ being the growth kernel at t+1 for $N = t+1$ so that if $\pi_{t+1}^{(N)}>1$ it would imply that the price level increases during period t to t+1, if $\pi_{t+1}^{(N)}<1$ it would imply that the price level decreases between t to t+1.

Rearranging (44) as a function of current price level yields:

$$E\left[\Pi_t^{N}\right]=E\left\lfloor\left(\pi_{t+1}^{N}\right)^{-1}\Pi_{t+1}^{N}\right\rfloor . \tag{45}$$

Using the natural logarithm notation (45) would look like:

$$\ln\left[\Pi_t^{(N)}\right]=\ln\left[\Pi_{t+1}^{(N)}\right]-\ln\left[\pi_{t+1}^{(N)}\right]. \tag{46}$$

Now we need to specify $\ln\left[\Pi_{t+1}^{(N)}\right]$ and $-\ln\left[\pi_{t+1}^{(N)}\right]$ for which a similar expression will be used as in previous chapter which for this case could be specified as follows:

$$-\ln\left[\pi_{t+1}^{N}\right]=\frac{1}{2}\lambda^2+\Delta\pi_t^{(1)}+\lambda\varepsilon_{t+1} \tag{47}$$

$$\Delta\pi_t^{(1)}=\gamma_0+\gamma_1\,\Delta b_t^{(1)}, \tag{48}$$

for λ depicting the sensitivities at which changes $-\ln\left[\pi_{t+1}^{(N)}\right]$ due to shocks in ε_{t+1}. $\Delta\pi_t^{(1)}$ represents the price level growth rate in 1 year as a consequence of changes in total notional debt outstanding, γ_0 and γ_1 are coefficients and $\Delta b_t^{(1)}$ depicts the government's one year growth on total notional debt at time t. And again, as in previous chapter, the notation is on purpose so that under normality $E\left[-\ln\pi_{t+1}^{(N)}\right]$ equals $-\Delta\pi_t^{(1)}$.

$$\Pi_{t+1}^{(N)}=\Pi_t^{(N)}\exp\left[\Delta\pi_{t+1}^{(N)}\times N\right] \tag{49}$$

For simplicity's sake it is assumed $\Pi_{t+1}^{(N)}=1$. Applying natural logarithms to (49) and rearranging yields:

$$\Delta\pi_{t+1}^{(1)}=-\frac{1}{N}\ln\left[\Pi_t^{(N)}\right]. \tag{50}$$

If $\Pi_{t+1}^{(N)}>\Pi_t^{(N)}$ implies that the price level for period N is expected to increase which means that $\Pi_t^{(N)}<1$ and $\Delta\pi_{t+1}^{(1)}>0$. If on the contrary $\Pi_{t+1}^{(N)}<\Pi_t^{(N)}$ implies that the price level for period N is expected to decrease which means that $\Pi_t^{(N)}>1$ and $\Delta\pi_{t+1}^{(1)}<0$.

It is also assumed as in similar fashion to previous sections:

$$\ln\left[\Pi_{t+1}^{(N)}\right]=A(N)+B(N)\Delta b_{t+1}^{(1)}. \tag{51}$$

Plugging (51) in (50) results in the inflation curve as a function of short term (e.g. one year) growth in total debt outstanding:

$$\Delta\pi_{t+1}^{(N)}=\frac{A(N)}{N}+\frac{B(N)}{N}\Delta b_{t+1}^{(1)}. \tag{52}$$

The final assumption here is that total notional debt outstanding $\Delta b_t^{(1)}$ follows a Vasicek (1977) process as follows:

$$\Delta b_{t+1}^{(1)}=\Delta b_t^{(1)}+\phi\left(\overline{b}_{\Delta b}^{(1)}-\Delta b_t^{(1)}\right)+\sigma_{\Delta b}\varepsilon_{t+1}. \tag{53}$$

Equation (53) says that changes in total notional debt outstanding will have a mean reversing AR(1) behaviour. This is a Vasicek (1977) stochastic process whereby the term $\left(\overline{b}_{\Delta b}^{(1)}-\Delta b_t^{(1)}\right)$ is likely to be negative if $\Delta b_t^{(1)}$ is above its mean $\overline{b}_{\Delta b}^{(1)}$ and, is likely to be positive if $\Delta b_t^{(1)}$ is below $\overline{b}_{\Delta b}^{(1)}$. If on average the change in total notional debt outstanding is zero, the adjustment will depend fully on $\phi\Delta b_t^{(1)}$. For which the solution for the coefficients in (52) would look like:

$$A(N+1)=\gamma_0+A(N)+B(N)\phi\overline{b}_{\Delta b}+\frac{1}{2}\left[B(N)\sigma_{\Delta b}\right]^2 \tag{54}$$

$$(N+1)=\left[B(N)(1-\phi)+\gamma_1\right]. \tag{55}$$

Details refer to Appendix Section III.

5.3. Modelling Government's net primary surplus as a function of state variables

In a similar fashion to previous chapters, this section deals with the use of affine terms structure models to describe the path of macroeconomic shocks on government revenues and government expenditure. This is crucial, as macroeconomic shocks can have different paths for revenues as well as for expenditure and need to be studied separately. Structural deficits can result from problems arising from these differences:

$$S_t^{(N)}=\tau_t^{(N)}(x_t)-G_t^{(N)}(x_t). \tag{56}$$

Equation (56) says that total cumulated surplus during holding period N at time t denoted $S_t^{(N)}$ equals total government revenues cumulated during N at time t which is denoted as $\tau_t^{(N)}(x_t)$ less government expenditure $G_t^{(N)}(x_t)$. Notice that we denote that government revenues as well as government expenditure are both a function of $k\times 1$

a state space vector x_t of macroeconomic variables. For simplicity's sake and to save some effort in the notation we will omit this in the following equations however, the reader will see that these variables will continue to be a function of vector x_t.

Cumulated government revenues and government expenditures path from t to t+1 can be specified as follows:

$$E\left[\tau_t^{(N)}\delta_{t+1}^{(N)}\right] = E\left[\tau_{t+1}^{(N)}\right] \tag{57}$$

$$E\left[G_t^{(N)}g_{t+1}^{N}\right] = E\left[G_{t+1}^{(N)}\right]. \tag{58}$$

Equation (57) describes $E\left[\tau_t^{(N)}\delta_{t+1}^{(N)}\right]$ as being the government revenue in time t which is expected to grow at $\delta_{t+1}^{(N)}$ resulting in an expected future revenue level at t+1 equal to $E\left[\tau_{t+1}^{(N)}\right]$. Equation (57) describes the path at which $E\left[\tau_t^{(N)}\right]$ would need to grow from t to t+1. In similar fashion, equation (58) describes the growth path for government expenditure at which G_t^N would need to grow from t to t+1. For $\delta_{t+1}^{(N)}$ and $g_{t+1}^{(N)}$ being the growth kernels at t+1 for $N = t+1$ so that if e.g. $\delta_{t+1}^{(N)} > 1$ it would imply that the revenue level increases during period t to t+1, if $\delta_{t+1}^{(N)} < 1$ it would imply that the revenue growth decreases between t to t+1. The same would apply for the growth in government expenditure $g_{t+1}^{(N)}$. Rearranging (57) and (58)

$$E\left[\tau_t^{(N)}\right] = E\left[\left(\delta_{t+1}^{(N)}\right)^{-1}\tau_{t+1}^{(N)}\right] \tag{59}$$

$$E\left[G_t^{(N)}\right] = E\left[\left(g_{t+1}^{(N)}\right)^{-1}G_{t+1}^{(N)}\right]. \tag{60}$$

Using the natural logarithm for (59) and (60) yields:

$$\ln\left[\tau_t^{(N)}\right] = \ln\left[\tau_{t+1}^{(N)}\right] - \ln\left[\delta_{t+1}^{(N)}\right] \tag{61}$$

$$\ln\left[G_t^{(N)}\right] = \ln\left[G_{t+1}^{(N)}\right] - \ln\left[g_{t+1}^{(N)}\right]: \tag{62}$$

Now we need to specify $\ln\left[\tau_{t+1}^{(N)}\right], -\ln\left[\delta_{t+1}^{(N)}\right]$, $\ln\left[G_{t+1}^{(N)}\right]$ and $-\ln\left[g_{t+1}^{(N)}\right]$ for which same notation will be used yielding:

$$\ln\left[\delta_{t+1}^{(N)}\right] = \frac{1}{2}\lambda_\delta^2 + \Delta\delta_t^{(1)} + \lambda_\delta\varepsilon_{t+1} \tag{63}$$

$$\Delta\delta_t^{(1)} = \alpha_0 + \alpha_1 x_t \tag{64}$$

$$-\ln\left[g_{t+1}^{(N)}\right] = \frac{1}{2}\lambda_g^2 + \Delta g_t^{(1)} + \lambda_g\varepsilon_{t+1} \tag{65}$$

$$\Delta g_t^{(1)} = \beta_0 + \beta_1 x_t\,. \tag{66}$$

For λ_δ and λ_g depicting the sensitivities at which $\delta_{t+1}^{(N)}$ or $g_{t+1}^{(N)}$ change due to shocks in ε_{t+1}. $\Delta\delta_{t+1}^{(1)}$ represents the revenue growth rate in 1 year as a explained by innovations in macroeconomic state vector x_t and the same for $\Delta g_t^{(1)}$ which in this case describes the government expenditure growth in 1 year which is also a function of macroeconomic state vector x_t. And for α_0, α_1, β_0 and β_1 being coefficients obtained empirically, e.g. via OLS.

Under these assumptions the revenues and expenditure are expected to grow as follows:

$$\tau_{t+1}^{(N)} = \tau_t^{(N)} \exp\left[\Delta\delta_{t+1}^{(1)} \times N\right] \quad (67)$$

$$G_{t+1}^{(N)} = G_t^{(N)} \exp\left[\Delta g_{t+1}^{(1)} \times N\right]. \quad (68)$$

For simplicity's sake it is assumed $\tau_{t+1}^{(N)} = 1$ and $G_{t+1}^{(N)} = 1$. Applying natural logarithms to (67) and (68), and rearranging yields:

$$\Delta\delta_{t+1}^{(N)} = -\frac{1}{N}\ln\left[\tau_t^{(N)}\right]. \quad (69)$$

$$\Delta g_{t+1}^{(N)} = -\frac{1}{N}\ln\left[G_t^{(N)}\right]. \quad (70)$$

It is also assumed that:

$$\ln\left[\tau_{t+1}^{(N)}\right] = A(N)_\tau + B(N)'_\tau x_{t+1} \quad (71)$$

$$\ln\left[G_{t+1}^{(N)}\right] = A(N)_G + B(N)'_G x_{t+1}, \quad (72)$$

for $A(N)_\tau$ and $A(N)_g$ being scalars, $B(N)'_\tau$ and $B(N)'_g$ being $1\times k$ vectors of coefficients and x_{t+1} a $k \times 1$ vector of state variables.

From our guess shown in (71) and (72) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$, and hence by plugging (71) in (69) and (72) in (70) results in:

$$\Delta\delta_{t+1}^{(N)} = \frac{A(N)_\tau}{N} + \frac{B(N)'_\tau}{N} x_{t+1} \quad (73)$$

$$\Delta g_{t+1}^{(N)} = \frac{A(N)_G}{N} + \frac{B(N)'_G}{N} x_{t+1}. \quad (74)$$

From (73) and (74), and recalling 56) boils down to:

$$\Delta s_{t+1}^{(N)} = \Delta\delta_{t+1}^{(N)} - \Delta g_{t+1}^{(N)} \quad (75)$$

$$y_{t+1}^{(N)} = \kappa_0 + \kappa_1 \Delta s_{t+1}^{(N)}. \quad (76)$$

The final assumption here is that the state space vector x_t follows a Vasicek (1977) process already discussed in (15) and now reproduced below for convenience as follows:

$$x_{t+1} = x_t + \Phi\left(\overline{x} - x_t\right) + \sigma_x \varepsilon_{t+1} . \tag{77}$$

Equation (77) says that the state space vector with elements $x_{i,t+1}$ will have a mean reversing AR(1) behaviour. For which the solution for the coefficients in (73) and (74) would look like:

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)'_\tau \Phi\overline{x} + \tfrac{1}{2}\left[B(N)'_\tau \sigma_x\right]^2 \tag{78}$$

$$B(N+1)'_\tau = \left[B(N)'_\tau \left(I - \Phi\right) + \alpha'_1\right]. \tag{79}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)'_G \Phi\overline{x} + \tfrac{1}{2}\left[B(N)'_G \sigma_x\right]^2 \tag{80}$$

$$B(N+1)'_G = \left[B(N)'_G \left(I - \Phi\right) + \beta'_1\right]. \tag{81}$$

For details the reader should refer to Appendix, Section IV.

In order to depict how this works let us assume an economy where the only source of income is from employment and if the individuals are not employed they receive a transfer from the government. For simplicity's sake, we will assume that unemployment is a valid state variable which describes well shocks on government revenues and on government expenditure. As unemployment increases, government revenues increase, but also expenditure decreases, as there are less unemployed people and hence less payments from the government and therefore, surplus improves. However, if unemployment deteriorates, government revenues decrease, as there is less taxable income. At the same time expenses increase as there are more unemployed people and hence more transfer payments.

If, let's say the constant terms in (73) and (74) are $A(N)_G / N = A(N)_\tau / N = 0$, $B(N)_G / N > 0$ $B(N)_\tau / N < 0$ for all N, and x_t being the unemployment rate, this will imply that an increase in unemployment has results in a deterioration of government surplus and that this might have an effect for various periods. The size and the sign (positive or negative) of the 1-period shock are dependent on the coefficients estimated as described in (61) and (66) However, because the term is being divided by N, this means that shocks dissipate as N becomes greater. With this model it is possible to study persistence of these shocks along N which is largely dependent on Φ which is nothing else but the speed of adjustment of a state variable such as unemployment towards its mean from x_t to $\overline{x}$. The closer Φ is to 1 the faster the adjustment, but the closer Φ is to zero the greater the persistence of these shocks in affecting government surplus for various periods. Here, it is possible to have state variables which have offsetting effects in period 1. However, if they differ in persistence, hence on the size of Φ it could result in cumulating deficits mainly because on the long run one variable has persisting negative effects on surplus.

6. Empirical Analysis

We analyse Bloomberg and Eurostat monthly data mostly from the period August 2008 to September 2012. As we are trying to link macroeconomic data with financial markets data such as inflation-linked swap yields, we are very much dependent on its availability. For the case of European inflation-linked swaps we only observe a complete inflation curve for all maturities in the Euro-Zone since August 2008 which leaves us only with 50 data points.

Before the affine term structure models are calibrated, this section will start first by plotting the EONIA, the inflation as well as the real interest rates from 1 year inflation-linked swaps. The intention is to study how movements in inflation and real interest rates explain movements in the short rate.

In a second stage, we will regress via OLS, the EONIA rate to the inflation as well as the real interest rates from 1 year inflation-linked swaps following discussions on (4). Subsequently we will repeat this exercise for European and German government yields of various maturities. We will show that results are consistent across maturities.

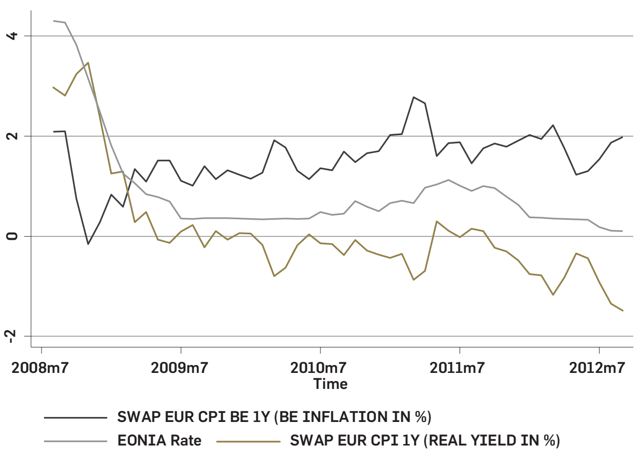
In a third stage, we will repeat this exercise for Greek and Spanish government yields and show that though the fitted values appear to follow some of the movements in the yields, results are not as encouraging as for the European and German yields. However, we will show that we can remediate this by regressing (26), hence by incorporating the spread of these yields to the German benchmark in the regression as a third state variable. By doing so we can show the reader that inflation and real interest rates do not suffice as explanatory variables for non-benchmark yields, as there is a credit risk component which appears that needs to be taken into account when modelling Spanish and Greek yields.

In a fourth stage, we attempt to replicate the behaviour of governments total debt outstanding by using the price level, which for simplicity's sake we use as proxy the production price index (PPI) for the respective countries. Here, we regress (22) as the intention is to analyse how the stochastic path of governments' total debt outstanding can be used as a state variable when modelling the time path of the price level. In this section we also show an example of fitting for the two year Spanish government yield using Spanish cumulated government deficit and Spanish unemployment rate as state variables.

The final piece of the analysis introduces the use of affine term structure models for: 1) to generate an inflation curve (or so-called the "breakeven inflation" curve) using as state variable government's total debt outstanding; 2) likewise we generate the average spread curves for Spanish and Greek government yields and analyse the coefficients $B(N_i)N_i$ for various maturities using as state variables Spanish and Greek governments' total debt outstanding and; 3) we present and example where we will calibrate an affine model as specified in (73) and (74) using as a state variable the country's unemployment rate and analyse the coefficients $B(N_i)N_i$ to show how affine term structure models can help understand how innovations in the unemployment rate are useful for determining the time path of governments' revenues and expenses and hence ultimately contribute to the understanding of the effects that innovations in macroeconomic variables have on governments' deficits.

Figure 1 below shows the EONIA and the inflation as well as the real interest rates from a 1 year inflation-linked swap. Only by looking at the data it is possible to see the effects of expected inflation and real interest rate movements on the short rate (EONIA). These components can be used as proxies, in order to replicate the identity described in equation (4).

Figure 1. **Development of the EONIA, breakeven inflation and real interest rate observed from Euro 1 year inflation swap.**



Note: This is a possible observation of the various variables discussed in equation (4).

Figure 2, replicates the EONIA which has been obtained by regressing equation (4). The EONIA is the proxy for the nominal short rate and proxies for the real interest rate as well as for the inflation rate are the 1-year real yield and the breakeven inflation observed from the 1-year Euro inflation-linked (CPI) swaps. The reader can appreciate that the fitted values fit remarkably well the observed data.

Figure 2. Replication of EONIA using as state variables the real interest rates and the breakeven inflation rates from observed inflations swaps data.

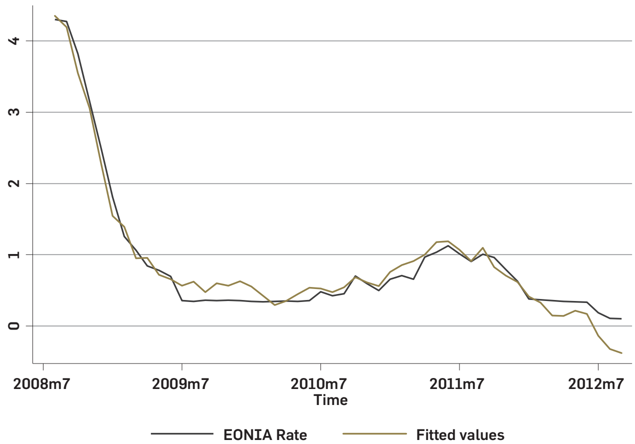


Figure 3 replicates the European government benchmark curve which has been obtained by regressing equation (4) for different maturities. The European government benchmark curve is here used as proxy for the nominal risk-free rates for various maturities. In addition, the proxies for the real interest rates as well as for the inflation rates are the corresponding 2, 5, 10, 15, 20 and 30-year real inflation-linked swap yields and the breakeven inflation rates which are observed from these same swaps. The reader can appreciate that the fitted values replicate remarkably well the observed data. This is the beauty of the availability of inflation-linked financial data, as it gives the possibility to segregate inflation expectations and expectations on real yields.

Similar to Figure 3, Figure 4 replicates the same regression for the German government benchmark yields and results are just as encouraging. These results are notably better than those seen in Jakas (2012), as these empirical results show that inflation swap data performs better than the usual monthly survey data.

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AESTIMATIO, THE IEB INTERNATIONAL JOURNAL OF FINANCE, 2013. **7**: 48-93

■ Figure 3. **Bloomberg EU Government Yield Curves (Bloomberg Index), observed vs. fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.**

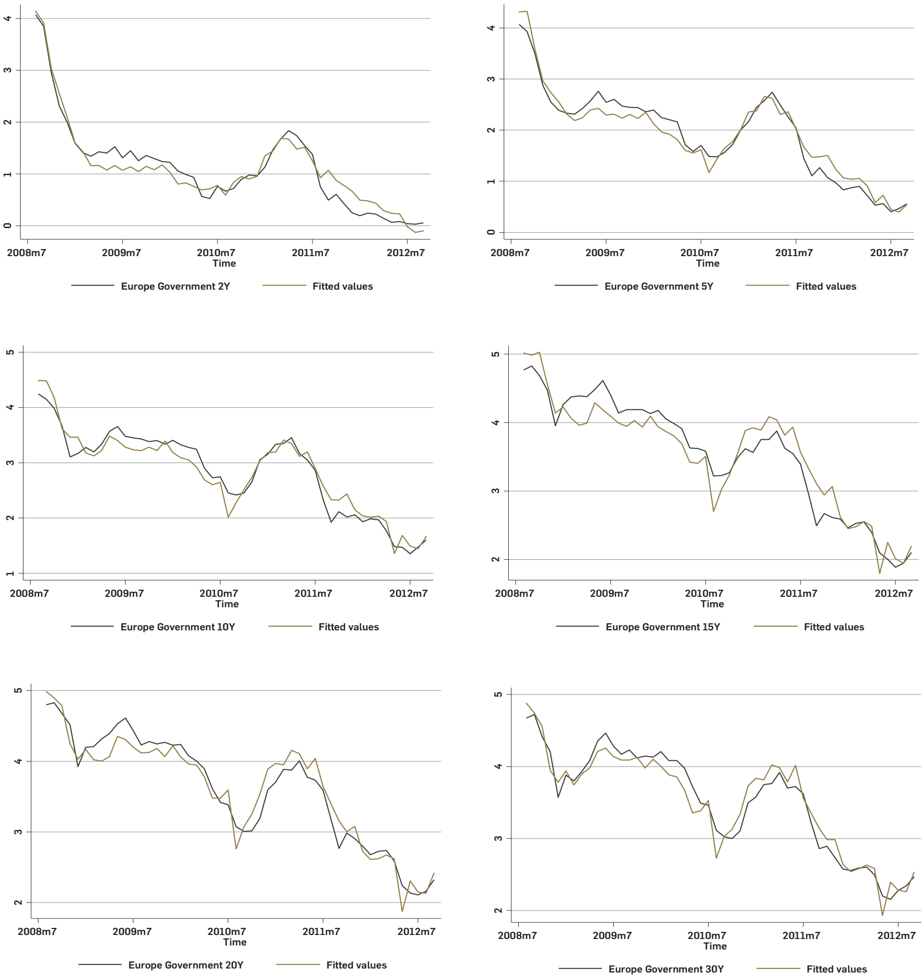
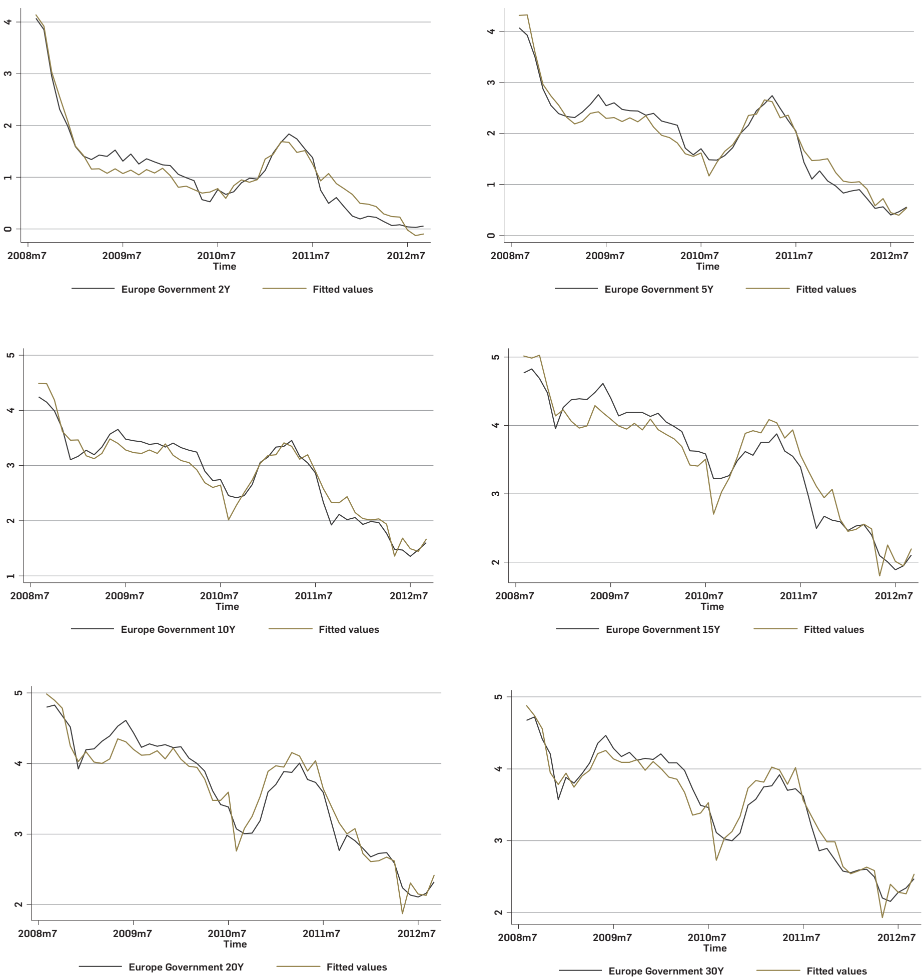
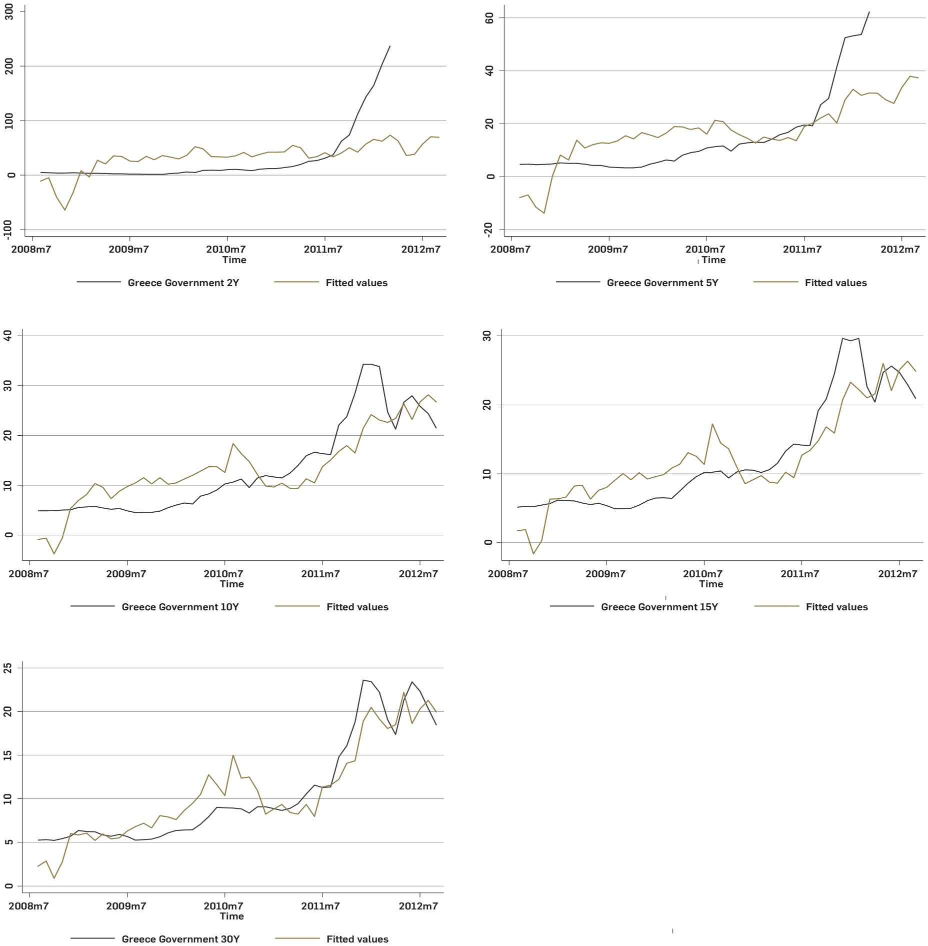


Figure 4. Bloomberg German Government Yield Curves (Bloomberg Index), observed vs. fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.



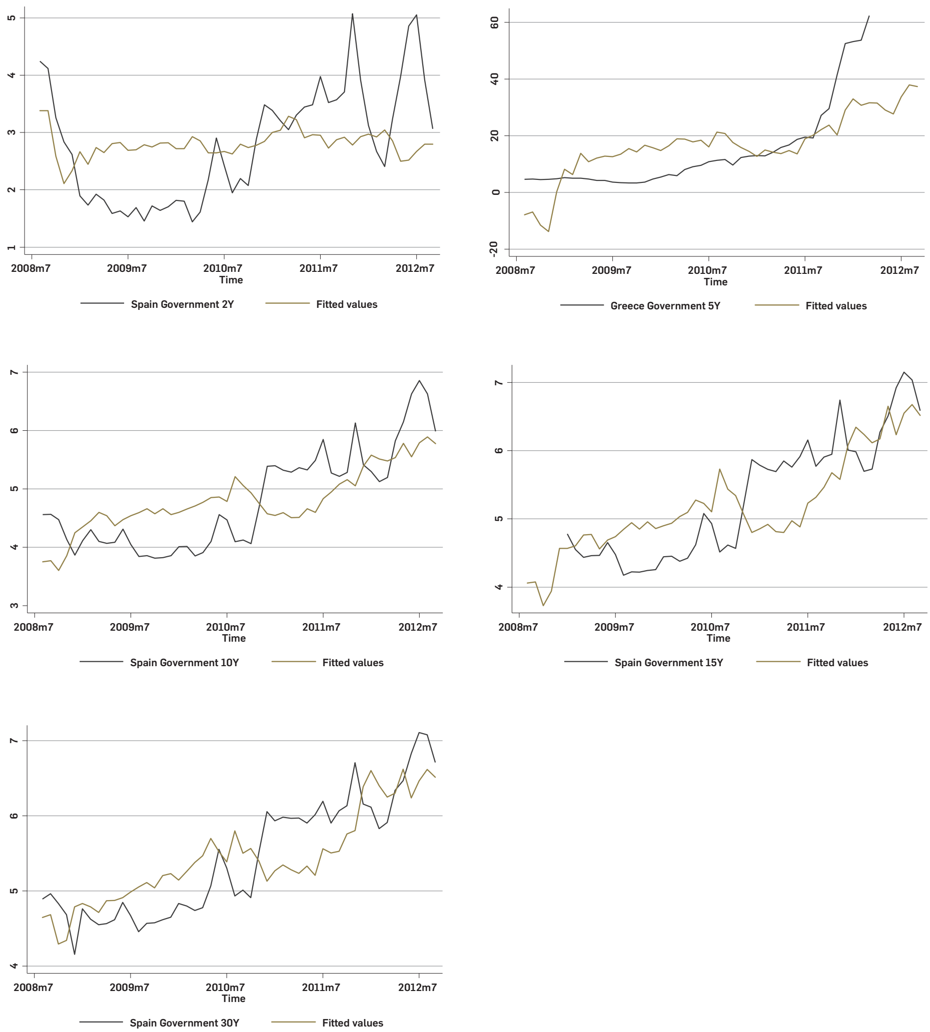
Figures 5 and 6 replicate the Greek and Spanish government benchmark curves which have been obtained by regressing equation (4) in a same fashion as we did for Figures 3 and 4. Not surprisingly, real yields and inflation have less predictive power on Greek and Spanish benchmark yields compared to the European or the German government benchmark curves. The reason is the existence of a credit spread as these are risky assets.

Figure 5. Bloomberg Greek Government Yield Curves, Observed vs. Fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.



Note: The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis.

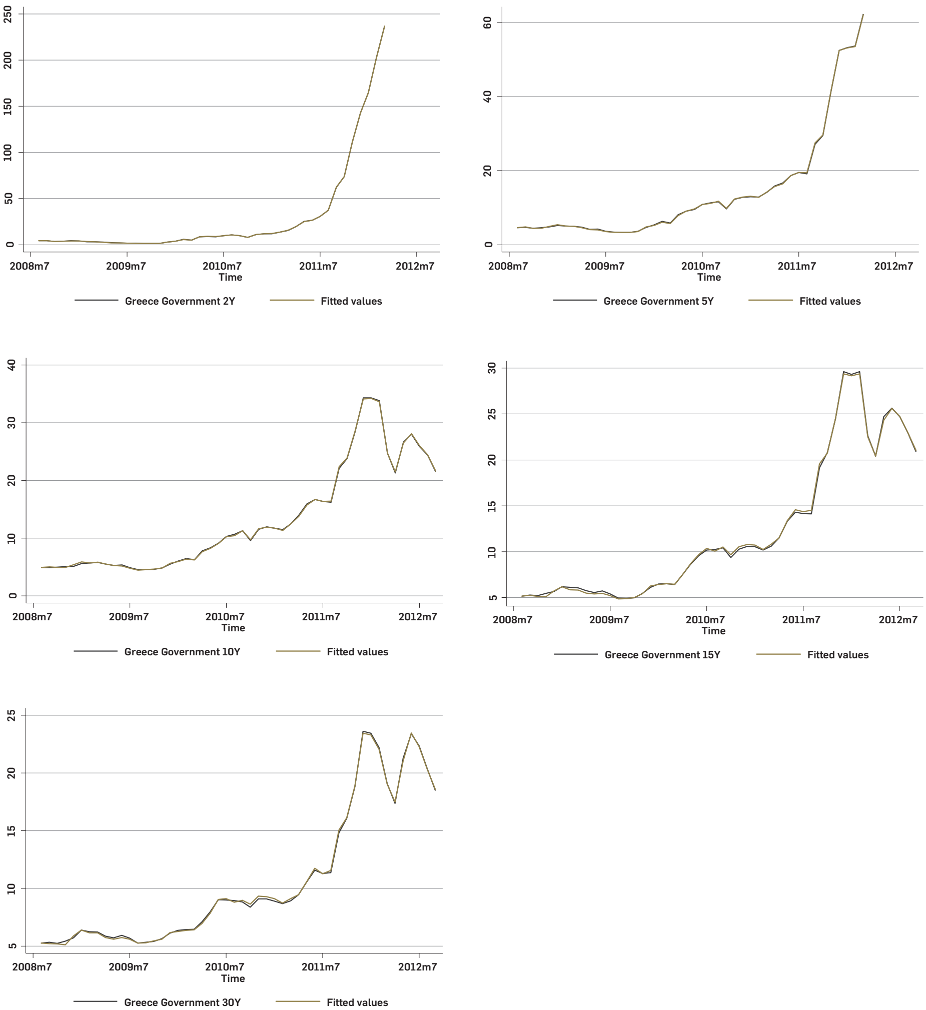
Figure 6. Bloomberg Spanish Government Yield Curves, Observed vs. Fitted using observed breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.



Note: The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis.

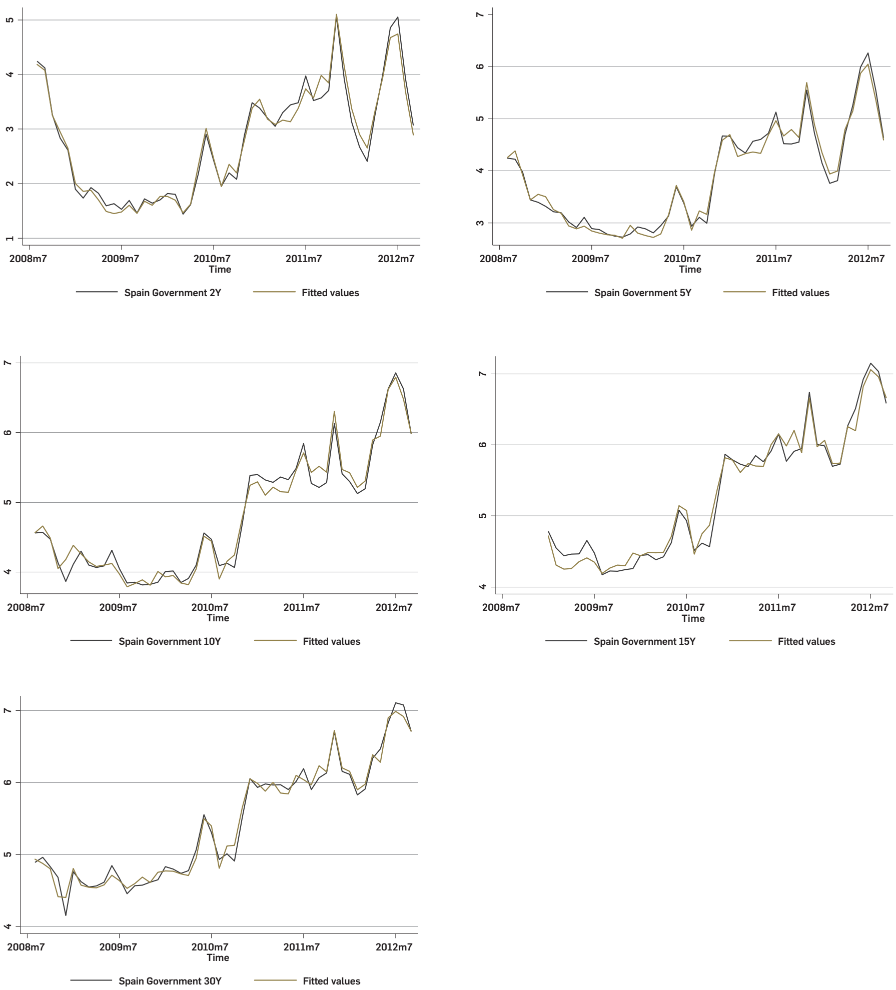
We will include the credit spread to equation (4) as discussed in (5) and as specified in (26), by doing so the fitted values show remarkable improvements as shown in Figures 7 and 8. Figures 7 and 8 replicate the Greek and Spanish government benchmark curves which have been obtained by regressing nominal the yields to a state space vector comprising real yields, breakeven inflation and the spreads to the German benchmark.

Figure 7. Fitting by incorporating the spread as in equation (26). Bloomberg Greek Government Yield Curves, Observed vs. Fitted using the observed credit spreads, breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.



Note: The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis. Credit spreads are calculated as the difference to the German Benchmark.

Figure 8. Fitting equation (26). Bloomberg Spanish Government Yield Curves, Observed vs. Fitted using the observed credit spreads, breakeven inflation and real interest rates from the Euro Inflation (CPI) Swap curve.



Note: The 20 years is not available in Bloomberg, so we have omitted this piece in our analysis. Credit spreads are calculated as the difference to the German Benchmark.

The theory of the price level suggests that an increase in total debt outstanding can lead to increases in the price level. Figure 9 shows PPI from EU, Germany, Spain and Greece, observed versus fitted values obtained by regressing equation (22). These time series show that levels in governments' total debt outstanding can be used as explanatory variables in order to explain changes in the price level.

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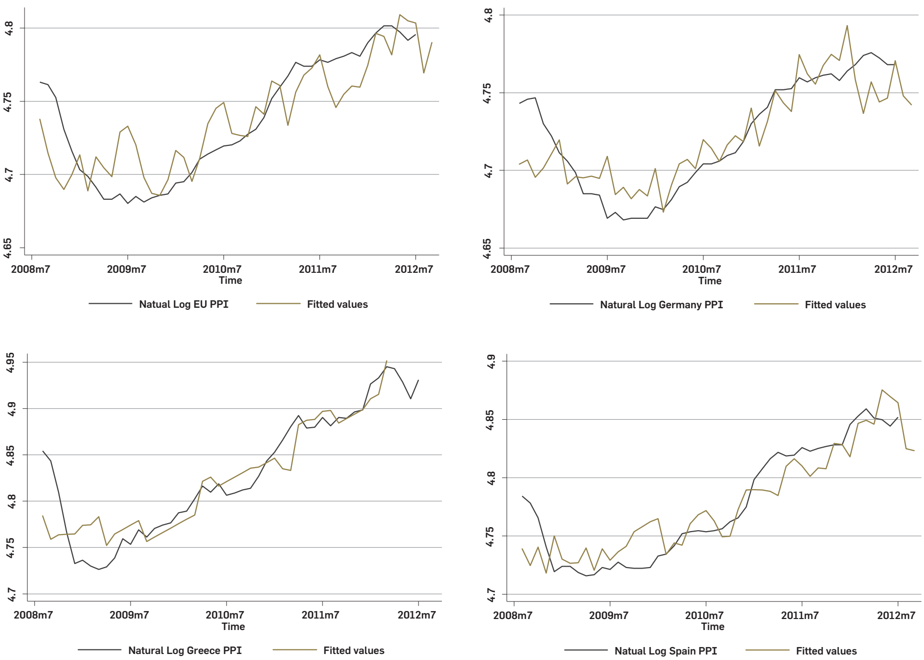
Figure 9. Fitting equation (22). Replication of Production Price Indices (PPI) for the Euro-Zone, Germany, Spain and Greece. Observed vs. Fitted values using the observed total government debt outstanding available in Bloomberg.



Deficits and unemployment can result in a deterioration in government yields. The theory of the price level suggests that the net present value of a bond or the yield must reflect the net present value of expected future surpluses. If unemployment and surplus (deficits) deteriorate, then this should be observed in the short term yields. Figure 10 tries to replicate this relationship. Though it is possible to replicate the trend, we are not capable of replicating the volatility observed which appears to increase as yields become greater.

Figure 10. Replication of 2-year Spanish Government yield applying two factors, Spanish cumulated government deficit and unemployment rates: Observed vs. Fitted values.

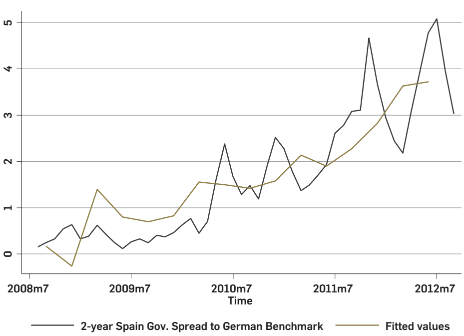


Figure 11 replicates the average Euro Breakeven Inflation curve by calibrating equation (52). It is possible to produce the average inflation curve (left quadrant) but it is not possible to replicate a reasonable time series that fits well the observed data (right quadrant). The fitted data is much less volatile which shows this requires further research. However, and despite this deficiency, it should be mentioned that the model is still capable of fitting the underlying upward trend.

Figure 12 replicates the spreads for Spanish and Greek government bonds. On average, we observe that the Spanish spreads show a "normal" upward-sloping, except the front end (2 years) exhibiting lower spreads. Spreads are constant between 5-years and 30-year maturities. The average Greek government spread curve gives a completely different view. The downward-sloping shows that the yields in the front end are more risky than the rest of the curve. This is because the market's view is that the Greek government will not be capable or raising new funds in order to pay the 2-year maturing bonds. The issuance is a Ponzi scheme and speculation is focused solely on the ability of the Greek government to convince core European partners on a bail out. In this case the theory of the price level is not at work, as aggregated European inflation is influencing very little the deterioration in the underlying yields and is rather a credit risk element.

Figure 11. **(Left) Replication of the Euro Breakeven Inflation observed vs. affine fitted values, using equation (52). (Right) 1-year Euro Breakeven Inflation observed vs. fitted.**

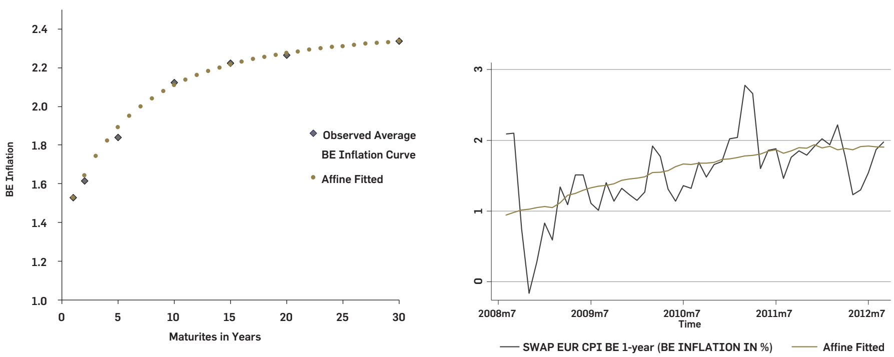


Figure 12. Replication of the Spanish and Greek Government Spreads for each maturity bucket vs. Affine Fitted values, using equation (52) and replacing $\Delta\pi_{t+1}^{(N)}$ (inflation) for $\theta_{t+1}^{(N)}$ (credit spread).

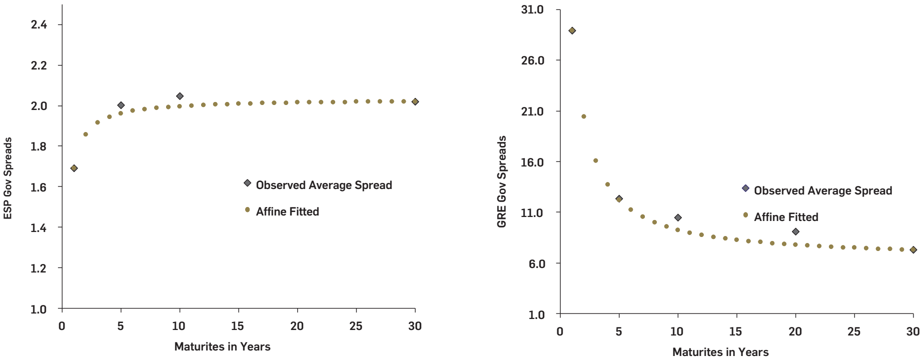
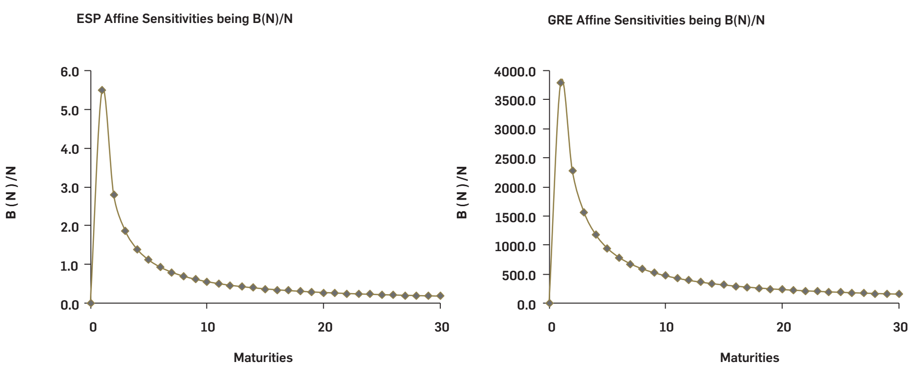


Figure 13 shows how the $B(N)/N$ coefficients which depict the sensitivity of changes in the credit spread $\theta_{t+1}^{(N)}$ as a consequence of increases in total debt outstanding and the effect is diluted as the maturity increases, which makes sense because in times when default risk is high, the markets believe that the issuer will default first on the issuance that is just about to mature, thus where debt-roll-over risk is high. Interestingly, comparing the parameters $B(N)/N$ estimated for Spain with those estimated for Greece it is possible to observe the magnitude of how sensitive is Greece to changes in total debt outstanding.

Figure 13. Affine term structure sensitivities of the Spanish and Greek Government Spreads for each maturity bucket vs. Affine Fitted values, using equation (52) and replacing $\Delta\pi_{t+1}^{(N)}$ (inflation) for $\theta_{t+1}^{(N)}$ (credit spread).



In Figures 14 and 15 we show a cumulated expense versus income ratio for German, Spanish and Greek governments and the cumulated expense and income levels. Here, we try to capture discussion seen in equations (75) and (76). This idea is taken from the popular *cost-income ratio* used in the corporate finance literature, however we are particularly interested in its cumulated value as an indication of debt growth. For Figures 14 and 15 we cumulate for values starting since 2001 and 2008. The cumulated expense to cumulated income ratio shown in Figure 14 can be specified as follows

$$ratio_{t+i} = \sum_{t=1}^{t+i} Exp_t \Big/ \sum_{t=1}^{t+i} Inc_t$$

For the case of Spain and Greece, there is a clear deterioration of this ratio particularly since 2008, as GDP falls, with the subsequent fall in tax revenues, increase in unemployment and thus further increase in Government expenses. We can see here we move from a relatively stable ratio and then moving towards higher levels since 2008. The case of Spain is interesting as the ratio has been in better shape compared to Germany throughout the decade (chart on the left) however, the data also show that if we instead cumulated from 2008 onwards (chart on the right), the ratio exhibits similar levels as those seen in the Greek case. No doubt we see that in case of macroeconomic shocks influencing surplus result in a deterioration of this ratio and unless this trend is reversed the gap remains thereafter for several periods. Figure 15 is not less impressive, as the gap between cumulated expenses to cumulated government revenues or income increases at a faster pace for Greece and Spain compared to the German case.

Figure 14. **Cumulated government expenses to cumulated government revenues ratios (cost-income-ratios) for Germany, Spain and Greece.**

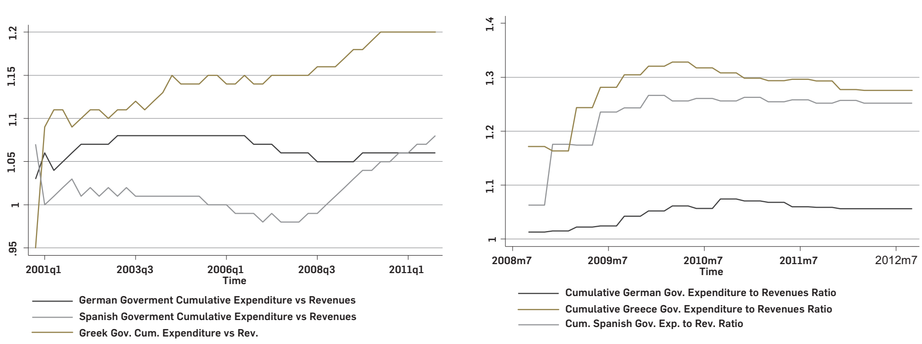
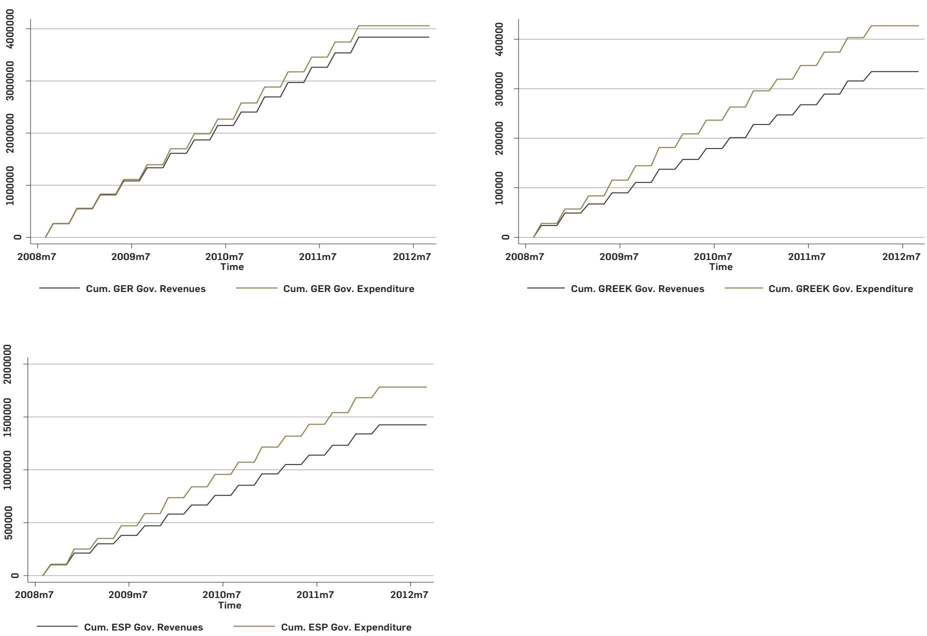


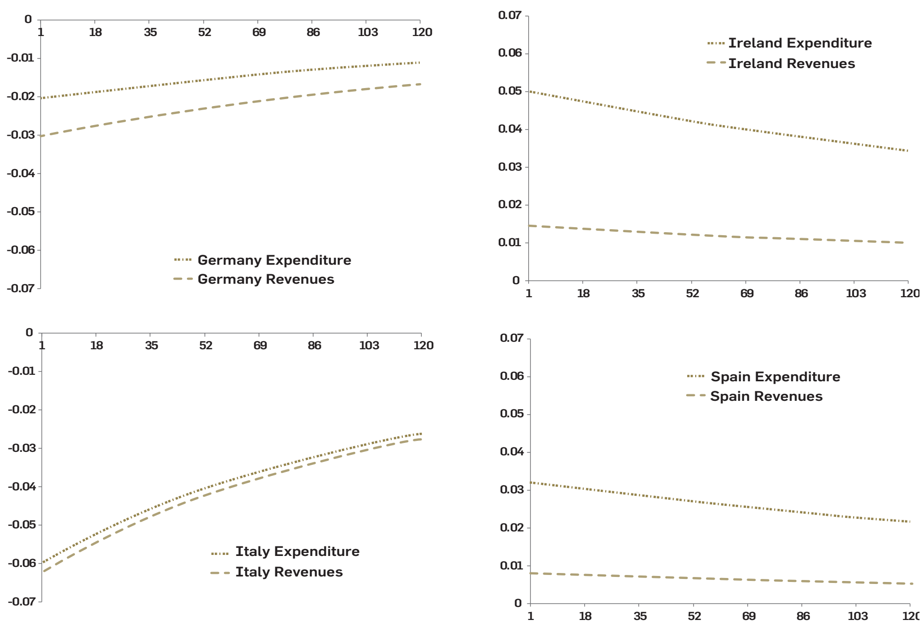
Figure 15. Cumulated government expense and cumulated government revenues for Germany, Spain and Greece. IMF Quarterly data, cumulative since 2008.

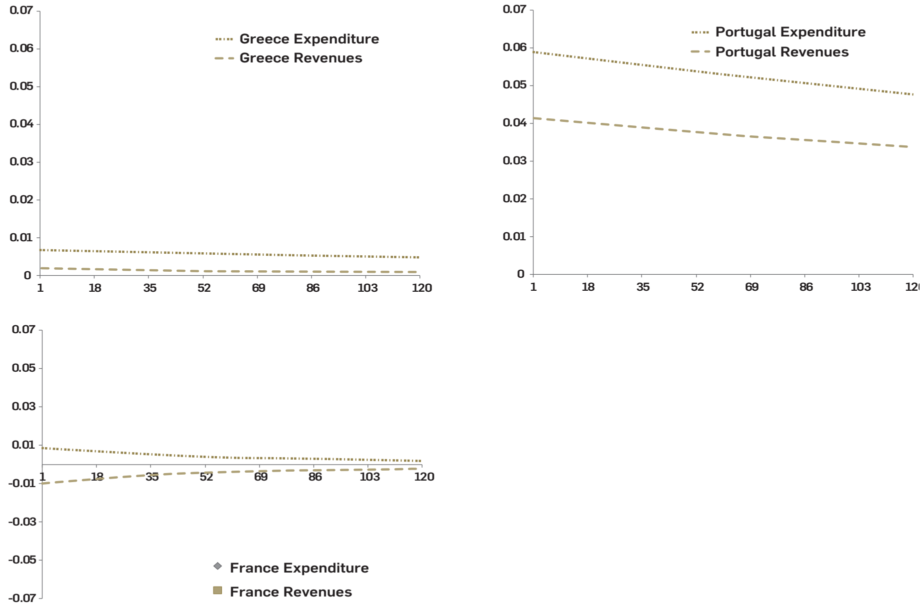


Note: IMF Quarterly data, cumulative since 2001 (left) and 2008 (right) respectively.

We calibrate the model discussed in section 5.3 by using unemployment rate as the state variable x_t influencing government revenue and expenditure for Germany, Ireland, Greece, Spain, Italy, Portugal and France. Hence, using one factor version of equations (78) to (81) and restricted to (64), (66), (73) and (74). Figure 15 summarises the results obtained for coefficients $B(N)_\tau/N$ and $B(N)_G/N$ which describes the growth path in government's tax revenues and government's expenditure for various quarters as a consequence of shocks in the unemployment rate. In all countries we observe that an increase in unemployment results in a deterioration of government's surplus. For all cases in Figure 15, unemployment exhibits persistence, thus shocks in unemployment remain for long periods of time and therefore has long lasting effects on governments' revenues and expenses. This is attributed to the autocorrelation coefficients of the lagged term in equation (77) specified as Φ which are very close to zero. There are only subtle differences between countries mostly when it comes to the size of the sensitivity of government's deficits to innovations in unemployment. For the cases of Ireland, Spain, Portugal and Greece, and increase in unemployment results in government expenditure increasing at a faster pace than revenues. Thus we can see that increases in unemployment result in fiscal imbalances that last various periods. It is possible to see that this gap is significantly large for Ireland, Spain and Portugal. Unemployment appears to have a lesser role for Greece's budgetary constraints however, still exhibits a deterioration of fiscal imbalances as a consequence of increases in unemployment. Alternatively, in the case of Germany, increases in unemployment result in government revenues falling at a faster pace than government expenditure thus, unemployment also has a negative effect on its surplus but mainly because the decrease in revenues has greater damaging effect on its fiscal imbalances than the increase in expenditure due to higher transfer payments to those unemployed. It is also possible to observe that the government reduces on expenditure when unemployment grows, presumably, in order to adjust fiscal imbalances however this adjustment appears only to partially offset the decrease in tax revenues, hence resulting in an overall deterioration of the German government surplus. Italy exhibits a similar result to Germany however the difference here is that the gap is tighter and that the unemployment rate despite exhibiting less persisting effects on Italy's government surplus the initial shock is larger than in for the Germany government. France exhibits a so-called "normal" case, thus an increase in unemployment results in a decrease in revenues and an increase in expenditure, closer to discussions in section 5.3.

Figure 16. $B(N)_\tau/N$ **and** $B(N)_G/N$ **coefficients calibrated with unemployment rate figures for Germany, Ireland, Greece, Spain, Italy, Portugal and France using one factor version of equations (78) to (81) and restricted to (64), (66), (73) and (74) using quarterly data for tax revenues and unemployment from Eurostat and ECB Data warehouse.**





7. Policy Implications, Conclusions and Final Remarks

This paper is an attempt to link term structure models, the theory of the price level to debt dynamics and fiscal imbalances. This paper starts calibrating a state space vector for breakeven inflation and real yields observed from inflation linked swap data and results were encouraging. We can observe that they do a poorer job for Spanish and Greek bonds however we remediated this by incorporating the credit spreads. It shows that inflation has lower predictive ability on these yields and most of the movement is captured in the credit risk component, as fitted values improved significantly when incorporating the spread to German benchmark yields in the model.

We understand that if governments need to provide counter-cyclical policies in times when aggregate marginal utility growth is high and hence, when consumption growth is low, this will largely depend on their ability to issue new debt without exhibiting a deterioration of their financing costs. We have seen that increases in total debt outstanding can lead to higher inflation and higher yields, but for the governments where this appears not to be significant, such as for instance, Spain or Greece, the increases in higher levels of total debt outstanding translate in higher spreads rather in higher inflation. We see that either ways financing becomes more expensive. The German case is different, because greater levels of total debt outstanding do not translate in a deterioration of their cumulated surplus. This, we believe, can be attributed to the idea that in times of low consumption growth German yields are indeed low so that the Government can issue more debt without incurring higher financing costs and undertake counter-cyclical policies. This assertion has testable implications and hence could be part of future research or follow-ups.

From our analysis using an affine term structure model, we observed that increases in unemployment, for instance, generated a gap between revenues and expenses that was significantly tighter for the case of Germany compared to the cases of Ireland, Spain and Portugal. Therefore, it appears that the German government budget is less sensitive to innovations in unemployment and hence cumulated government surplus remains relatively unaffected when controlling for this state variable compared to Ireland, Spain and Portugal. We understand that for these countries is more difficult to undertake countercyclical policies in the same as Germany can, mostly because we observe that during periods of financial distress issuing new debt becomes more expensive.

In addition, we have observed that the front end is very sensitive to innovations in increases on government's total debt outstanding, possibly because investors perceive that that is the first tranche that the issuer is likely to default. It should be mentioned that few governments exhibit low financing costs in times when aggregate marginal utility growth is high. In fact this demonstrates that governments should run stress tests to their budgets and secure liquidity reserves by running orthodox fiscal policies during economic booms and issuing long term debt enough to cover eventualities in case of a stressed scenario that would have extraordinary deteriorating effects on their budgetary deficits. This implies ensuring long periods of enough liquidity reserves for the case when several short term tranches mature and hence avoid debt roll-over risk, thus the risk of not being able to issue enough new debt at low financing costs in order to redeem the maturing one.

Acknowledgements

Special thanks to Prof. Dr. Ashok Kaul for his valuable guidance.

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Appendix

I. Solving when the theory of the price level is at work and when is not

Here it is shown how to get to the solution. Starting first with equation (10) and substituting the right hand term for (16) and (11) which boils down to:

$$\ln\left[P_t^{(N+1)}\right]=-\delta-y_t^{(1)}-\lambda'\varepsilon_{t+1}-A(N)-B(N)'x_{t+1}\ . \qquad (I.1)$$

In order to solve recursively δ is replaced by (17), and to be able to account for the theory of the price level $y_t^{(1)}$ is replaced by (27). In addition, x_{t+1} is also replaced for (15) to account for the Vasicek (1977) process, which all boils down to:

$$\ln\left[P_t^{(N+1)}\right]=-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2-\psi_0-\psi_1'x_t-\lambda'\varepsilon_{t+1}-A(N)-B(N)'\left[x_t+\Phi(\overline{x}-x_t)+\sigma_x\varepsilon_{t+1}\right]. \qquad (I.2)$$

The constant terms and the terms multiplying x_t and ε_{t+1} are grouped, so that at the end it would look something like this:

$$\ln\left[P_t^{(N+1)}\right]=-\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2+\psi_0+A(N)+B(N)'\Phi\overline{x}\right)$$
$$-\left(\psi_1'+B(N)'(I-\Phi)\right)x_t-\left(\lambda'+B(N)'\sigma_x\right)\varepsilon_{t+1} \qquad (I.3)$$

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The right hand side of equation (10) which has now developed to (I.3) has the following conditional moments,

$$E\left[\ln m_{t+1}+\ln P_{t+1}^{(N)}\right]=-\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2+\psi_0+A(N)+B(N)'\Phi\bar{x}\right)-\left(\psi_1'+B(N)'(I-\Phi)\right)x_t \qquad (I.4)$$

and

$$Var\left[\ln m_{t+1}+\ln P_{t+1}^{(N)}\right]=\left(\lambda'+B(N)'\sigma_x\right) \qquad (I.5)$$

Recalling that the implied present value of a fixed income security yields:

$$-E\left[\ln P_t^{(N+1)}\right]=-E\left[\ln m_{t+1}+\ln P_{t+1}^{(N)}\right]-\frac{1}{2}Var\left[\ln m_{t+1}+\ln P_{t+1}^{(N)}\right] \qquad (I.6)$$

Substituting (I.4) and (I.5) into (I.6) yields:

$$-E\left[\ln P_t^{(N+1)}\right]=\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2+\psi_0+A(N)+B(N)'\Phi\bar{x}$$
$$+\left(\psi_1'+B(N)'(I-\Phi)\right)x_t-\frac{1}{2}\left(\lambda'+B(N)'\sigma_x\right)^2 \qquad (I.7)$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (11) yields:

$$A(N+1)=\psi_0+A(N)+B(N)'\Phi\bar{x}+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2-\left(\lambda'+B(N)'\sigma_x\right)^2\right) \qquad (I.8)$$

$$B(N+1)'=\left(\psi_1'+B(N)'(I-\Phi)\right). \qquad (I.9)$$

The solution is obtained by computing the present value recursively using (10) for some guess of coefficients from (11). Since $P_{t+1}^{(N)}=1$ and $A(N=0)$ = $B(N=0)$ ' = 0, which means this can be solved recursively, as for 1 period would imply $A(N=1)$ = ψ_0 and $B(N=1)$ ' = ψ_1' which means that equals the short rate as described in (27). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (10) to compute the present value of an N+1 maturity bond.

All is needed is to line up (I.8) and (I.9) into (12) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. Parameters ψ_0 and ψ_{1i} are free and obtained empirically and the signs for parameters $B(N)_i$ in (12) depend on ψ_{1i}.

When the theory of the price level is not at work, equations (I.8) and (I.9) change, and this is because equation (14) is at work instead of equation (27) and by applying the same algebra discussed in (I.1) to (I.7) it would now yield:

$$A(N+1)=\gamma_0+A(N)+B(N)'\Phi\overline{x}+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2-\left(\lambda'+B(N)'\sigma_x\right)^2\right) \quad (I.10)$$

$$B(N+1)'=\left(\gamma_1'+B(N)'(I-\Phi)\right) \quad (I.11)$$

II. Solving the coefficients *A*(*N*+1) and *B*(*N*+1) assuming innovations in government surplus influence changes in total notional debt outstanding

I will solve (33) recursively starting with equation (34) and substituting terms for (35), (36), (39) and (41), and operating results in:

$$\begin{aligned}\ln\left[B_t^N\right]= -A(N)-B(N)\Delta s_t^{(1)}-B(N)\phi\overline{s}_{\Delta s}^{(1)}+B(N)\phi\Delta s_t^{(1)}-B(N)\sigma_{\Delta s}\varepsilon_{t+1}\\ -\frac{1}{2}\lambda^2-\varphi_0-\varphi_1\Delta s_t^{(1)}-\lambda\varepsilon_{t+1}\end{aligned} \quad (II.1)$$

Rearranging the constant terms and the terms multiplying $\Delta s_t^{(1)}$ and those multiplying ε_{t+1} boils down to:

$$\begin{aligned}\ln\left[B_t^N\right]= -\varphi_0-A(N)-B(N)\phi\overline{s}_{\Delta s}^{(1)}-\frac{1}{2}\lambda^2+\left[B(N)(1-\phi)-\varphi_1\right]\Delta s_t^{(1)}\\ -\left[B(N)\sigma_{\Delta s}+\lambda\right]\varepsilon_{t+1}\end{aligned} \quad (II.2)$$

Equation (II.2) has the following conditional mean and variance:

$$E\left[\ln B_t^N\right]= -\varphi_0-A(N)-B(N)\phi\overline{s}_{\Delta s}^{(1)}-\frac{1}{2}\lambda^2+\left[B(N)(1-\phi)-\varphi_1\right]\Delta s_t^{(1)}$$

$$Var\left[\ln B_t^N\right]=\left[B(N)\sigma_{\Delta s}+\lambda\right]^2 .$$

Recalling normality as in (37) and substituting:

$$E\left[\ln B_t^N\right]=E\left[\ln B_{t+1}^N-\ln g_{t+1}\right]+\frac{1}{2}Var\left[\ln B_{t+1}^N-\ln g_{t+1}^N\right]. \quad (II.3)$$

Substituting the above conditional moments in (II.3) yields,

$$\begin{aligned}E\left[\ln B_t^N\right]=-\varphi_0-A(N)-B(N)\phi\overline{s}_{\Delta s}^{(1)}-\frac{1}{2}\lambda^2\\ +\left[B(N)(1-\phi)-\varphi_1\right]\Delta s_t^{(1)}+\frac{1}{2}\left[B(N)\sigma_{\Delta s}+\lambda\right]^2\end{aligned} \quad (II.4)$$

Rearranging and grouping the constant terms and the terms multiplying $\Delta s_t^{(1)}$ as well as lining up with (39) yields,

$$A(N+1)=-\varphi_0-A(N)-B(N)\phi\overline{s}_{\Delta s}^{(1)}+\frac{1}{2}\left(\left[B(N)\sigma_{\Delta s}+\lambda\right]^2-\lambda^2\right) \quad (II.5)$$

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$$B(N+1) = \left[B(N)(1-\phi)-\varphi_1\right]. \tag{II.6}$$

Equations (II.5) and (II.6) are resolved recursively using what is known from (34) restricted to (39) and (40). Since it has been assumed that $B_{t+1}^{(N)}=1$, and $A(N=0)$ = $B(N=0) = 0$, which means this can be solve recursively, as for 1 period would imply $A(N=1) = -\varphi_0$ and $B(N=1) = -\varphi_1$ which means that equals the 1 year debt growth rate $\Delta b_t^{(1)}$ described in (36). Now for any set of surplus shocks the resulting nominal debt outstanding can be computed.

Equations (II.5) and (II.6) also contain the parameter λ which is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ to zero for which (II.5) and (II.6) would be:

$$A(N+1)=-\varphi_0-A(N)-B(N)\phi\overline{s}_{\Delta s}^{(1)}+\tfrac{1}{2}\left[B(N)\sigma_{\Delta s}\right]^2 \tag{II.7}$$

$$B(N+1) = \left[B(N)(1-\phi)-\varphi_1\right]. \tag{II.8}$$

III. Solving the coefficients *A*(*N*+1) and *B*(*N*+1) assuming that changes in total notional debt outstanding influence the price level or credit spreads

I will solve (44) recursively starting with equation (46) and substituting terms for (47), (48), (51) and (53), and operating results in:

$$\begin{aligned}\ln\left[\Pi_t^{(N)}\right] = {} & A(N)+B(N)\Delta b_t^{(1)}+B(N)\phi\overline{b}_{\Delta b}-B(N)\phi\Delta b_t^{(1)}+B(N)\sigma_{\Delta b}\varepsilon_{t+1} \\ & +\tfrac{1}{2}\lambda^2+\gamma_0+\gamma_1+\Delta b_t^{(1)}+\lambda\varepsilon_{t+1}\end{aligned} \tag{III.1}$$

Rearranging the constant terms and the terms multiplying $\Delta b_t^{(1)}$ and those multiplying ε_{t+1} boils down to:

$$\ln\left[\Pi_t^{(N)}\right]=\gamma_0+A(N)+B(N)\phi\overline{b}_{\Delta b}+\tfrac{1}{2}\lambda^2+\left[B(N)(1-\phi)+\gamma_1\right]\Delta b_t^{(1)}+\left[B(N)\sigma_{\Delta b}+\lambda\right]\varepsilon_{t+1} \tag{III.2}$$

Equation (III.2) has the following conditional mean and variance:

$$E\left[\ln \Pi_t^{(N)}\right]=\gamma_0+A(N)+B(N)\phi\overline{b}_{\Delta b}+\tfrac{1}{2}\lambda^2+\left[B(N)(1-\phi)+\gamma_1\right]\Delta b_t^{(1)} \tag{III.3}$$

$$Var\left[\ln \Pi_t^{(N)}\right]=\left[B(N)\sigma_{\Delta b}+\lambda\right]^2 . \tag{III.4}$$

Recalling normality for (46):

$$E\left[\ln\Pi_t^{(N)}\right]= E\left[\ln\Pi_{t+1}^{(N)}-\ln\pi_{t+1}^{(N)}\right]+\tfrac{1}{2}Var\left[\ln \Pi_{t+1}^{(N)}-\ln\pi_{t+1}^{(N)}\right]. \tag{III.5}$$

Substituting (III.3) and (III.4) in (III.5) yields,

$$E\left[\ln\Pi_t^{(N)}\right]=\gamma_0+A(N)+B(N)\phi\overline{b}_{\Delta b}+\frac{1}{2}\lambda^2+\left[B(N)(1-\phi)+\gamma_1\right]\Delta b_t^{(1)}$$
$$+\frac{1}{2}\left[B(N)\sigma_{\Delta b}+\lambda\right]^2 \qquad \text{(III.6)}$$

Rearranging and grouping the constant terms and the terms multiplying as well as lining up with (51) yields,

$$A(N+1)=\gamma_0+A(N)+B(N)\phi\Delta b_t^{(1)}+\frac{1}{2}\left(\left[B(N)\sigma_{\Delta b}+\lambda\right]^2+\lambda^2\right) \qquad \text{(III.7)}$$

$$B(N+1)=\left[B(N)(1-\phi)+\gamma_1\right]. \qquad \text{(III.8)}$$

Equations (III.7) and (III.8) are resolved recursively. Since it has been assumed that $\Pi_{t+1}^{(N)}=1.$ and $A(N=0) = B(N=0) = 0$, which means this can be solve recursively, as for 1 period would imply $A(N=1) = \gamma_0$ and $B(N=1) = \gamma_1$ which means that equals the 1 year price level growth rate $\Delta\pi_t^{(1)}$ described in (48). Now for any set of notional debt outstanding the resulting price level can be computed and by doing so it is possible to study inflationary shocks or credit spreads – for the case where the government does not control monetary policy.

Equations (III.7) and (III.8) also contain the parameter λ which, as already mentioned, is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ to zero for which (III.7) and (III.8) would be:

$$A(N+1)=\gamma_0+A(N)+B(N)\varphi\overline{b}_{\Delta b}+\frac{1}{2}\left[B(N)\sigma_{\Delta b}\right]^2 \qquad \text{(III.9)}$$

$$(N+1)=\left[B(N)(1-\phi)+\gamma_1\right]. \qquad \text{(III.10)}$$

IV. Solving the coefficients *A*(*N*+1) and *B*(*N*+1) assuming net primary surplus is a function of a state vector of macroeconomic variables

I will solve (59) and (60) recursively starting with equation (61) and (62), and substituting terms for (63) to (66) as well as (71), (72) and (77) and which results in:

$$\ln\left[\tau_t^{(N)}\right]=A(N)+B(N)_\tau' x_t+B(N)_\tau'\Phi\overline{x}-B(N)_\tau'\Phi x_t+B(N)_\tau'\sigma_x\varepsilon_{t+1}$$
$$+\frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2+\alpha_0+\alpha_1' x_t+\lambda_\delta'\varepsilon_{t+1} \qquad \text{(IV.1)}$$

$$\ln\left[G_t^{(N)}\right] = A(N) + B(N)'_G x_t + B(N)'_G \Phi \bar{x} - B(N)'_G \Phi x_t + B(N)'_G \sigma_x \varepsilon_{t+1}$$

$$+\frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \beta_0 + \beta'_1 x_t + \lambda'_g \varepsilon_{t+1} \qquad \text{(IV.2)}$$

Rearranging the constant terms and the terms multiplying x_t and those multiplying ε_{t+1} boils down to:

$$\ln\left[\tau_t^{(N)}\right] = \alpha_0 + A(N) + B(N)'_\tau \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2 + \left[B(N)'_\tau (I-\Phi) + \alpha'_1\right] x_t$$

$$+\left[B(N)'_\tau \sigma_x + \lambda'_\delta\right]\varepsilon_{t+1} \qquad \text{(IV.3)}$$

$$\ln\left[G_t^{(N)}\right] = \beta_0 + A(N) + B(N)'_G \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \left[B(N)'_G (I-\Phi) + \beta'_1\right] x_t$$

$$+\left[B(N)'_G \sigma_x + \lambda'_g\right]\varepsilon_{t+1} \qquad \text{(IV.4)}$$

Equations (IV.3) and (IV.4) have the following conditional mean and variance:

$$E\left[\ln \tau_t^{(N)}\right] = \alpha_0 + A(N) + B(N)'_\tau \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2 + \left[B(N)'_\tau (I-\Phi) + \alpha'_1\right] x_t \qquad \text{(IV.5)}$$

$$E\left[\ln G_t^{(N)}\right] = \beta_0 + A(N) + B(N)'_G \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2 + \left[B(N)'_G (I-\Phi) + \beta'_1\right] x_t . \qquad \text{(IV.6)}$$

$$Var\left[\ln \tau_t^{(N)}\right] = \left[B(N)'_\tau \sigma_x + \lambda'_\delta\right]^2 \qquad \text{(IV.7)}$$

$$Var\left[\ln G_t^{(N)}\right] = \left[B(N)'_G \sigma_x + \lambda'_g\right]^2 . \qquad \text{(IV.8)}$$

Recalling normality for (61) and (62) yields:

$$E\left[\ln \tau_t^{(N)}\right] = E\left[\ln \tau_{t+1}^{(N)} - \ln \delta_{t+1}^{(N)}\right] + \frac{1}{2} Var\left[\ln \tau_{t+1}^{(N)} - \ln \delta_{t+1}^{(N)}\right] \qquad \text{(IV.9)}$$

$$E\left[\ln G_t^{(N)}\right] = E\left[\ln G_{t+1}^{(N)} - \ln g_{t+1}^{(N)}\right] + \frac{1}{2} Var\left[\ln G_{t+1}^{(N)} - \ln g_{t+1}^{(N)}\right] \qquad \text{(IV.10)}$$

Substituting (IV.5), (IV.6), (IV.7) and (IV.8) in (IV.9) and (IV.10) respectively yields,

$$\ln\left[\tau_t^{(N)}\right] = \alpha_0 + A(N)_\tau + B(N)'_\tau \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{i\delta}^2$$

$$+\left[B(N)'_\tau (I-\Phi) + \alpha'_1\right] x_t + \frac{1}{2}\left[B(N)'_\tau \sigma_x + \lambda'_\delta\right]^2 \qquad \text{(IV.11)}$$

$$E\left[\ln G_t^{(N)}\right] = \beta_0 + A(N)_G + B(N)'_G \Phi \bar{x} + \frac{1}{2}\sum_{i=1}^{k}\lambda_{ig}^2$$

$$+\left[B(N)'_G (I-\Phi) + \beta'_1\right] x_t + \frac{1}{2}\left[B(N)'_\tau \sigma_x + \lambda'_\delta\right]^2 \qquad \text{(IV.12)}$$

Rearranging and grouping the constant terms and the terms multiplying x_t as well as lining up with (71) and (72) yields,

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)'_\tau \Phi\overline{x} + \frac{1}{2}\left(\left[B(N)'_\tau \sigma_x + \lambda'_\delta\right]^2 + \lambda_{i\delta}^2\right) \quad \text{(IV.13)}$$

$$B(N+1)_\tau = \left[B(N)'_\tau \left(I - \Phi\right) + \alpha'_1\right] \quad \text{(IV.14)}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)'_G \Phi\overline{x} + \frac{1}{2}\left(\left[B(N)'_G \sigma_x + \lambda'_g\right]^2 + \lambda_{ig}^2\right) \quad \text{(IV.15)}$$

$$B(N+1)_G = \left[B(N)'_G \left(I - \Phi\right) + \beta'_1\right] \quad \text{(IV.16)}$$

Equations (IV.13) to (IV.16) are resolved recursively. Since it has been assumed that $\tau_{t+1}^{(N)} = 1$ and $G_{t+1}^{(N)} = 1$, and $A(N=0)_{\tau,G} = 0$ and for $B(N=0)'_{\tau,G}$ would result in a vector with all elements being equal to 0, which means this can be solve recursively, as for 1 period would imply $A(N=1)_{\tau,G} = [\alpha_0, \beta_0]$ and $B(N=1)'_{\tau,G} = [\alpha_1, \beta_1']$ which means that equals the 1-year revenue growth rate $\Delta\delta_t^{(1)}$ and 1-year expenditure growth rate $\Delta g_t^{(1)}$, restricted to (73) and (74). Now for any set of macroeconomic state variables the resulting surplus can be computed and by doing so it is possible to study surplus shocks resulting from innovations in macroeconomic variables.

Equations (IV.13) to (IV.16) also contain the parameters λ_δ and λ_g which, as already mentioned, is quite handy for adjusting to observable data. If this is not desired, still the researcher can set λ_δ and λ_g to zero for which (IV.13) to (IV.16) would be

$$A(N+1)_\tau = \alpha_0 + A(N)_\tau + B(N)'_\tau \Phi\overline{x} + \frac{1}{2}\left[B(N)'_\tau \sigma_x\right]^2 \quad \text{(IV.17)}$$

$$B(N+1)'_\tau = \left[B(N)'_\tau \left(I - \Phi\right) + \alpha'_1\right]. \quad \text{(IV.18)}$$

$$A(N+1)_G = \beta_0 + A(N)_G + B(N)'_G \Phi\overline{x} + \frac{1}{2}\left[B(N)'_G \sigma_x\right]^2 \quad \text{(IV.19)}$$

$$B(N+1)'_G = \left[B(N)'_G \left(I - \Phi\right) + \beta'_1\right]. \quad \text{(IV.20)}$$