teorema
Vol. XXVIII/3, 2009, pp. 173-190
[BIBLID 0210-1602 (2009) 28:3; pp. 173-190]

Strong Soundness-Completeness Theorem: A Semantic Approach*

José Alfredo Amor

RESUMEN

El objetivo de este artículo es exponer dos cosas. En primer lugar, lo que entendemos por una prueba de tipo semántico del teorema de corrección–completud fuerte de la lógica clásica. En segundo lugar presentar una prueba de ese estilo, usando el teorema de compacidad y otros resultados semánticos. Los elementos usados son conocidos, sin embargo la prueba es original, elegante y sencilla porque evita la manipulación sintáctica en un sistema axiomático dado. Usaré la lógica de primer orden clásica y el concepto de sistema axiomático en una concepción ligeramente diferente a la tradicional, donde la definición de derivación formal incluye la posibilidad de establecer restricciones a la aplicación de las reglas de inferencia.

PALABRAS CLAVE: *semántica, compacidad, corrección, completud y metateorema de la deducción.*

ABSTRACT

This paper has a twofold aim. On one side, it explains what is understood by a *semantic approach proof* of the strong soundness–completeness theorem. On the other, it introduces such a proof for Gödel's famous theorem, using the compactness theorem and other semantic results. The ingredients of the proof are well–known, but the proof itself is original, elegant and clear, because it avoids the unintuitive syntactic manipulations derived from the use of a given deductive calculus. We will be working within first order classical logic and a slightly modified version of the formal system concept, where the definition of formal derivation includes the possibility to establish restrictions to the applications of inference rules.

KEYWORDS: *semantics, compactness, soundness, completeness, and deduction metatheorem*.

I. BASIC NOTIONS

Let us assume that the notions of first order language with equality and the interpretation for it —as they are presented in most mathematical logic books— are well known. In this section we will go over some basic notions such as "logical consequence", "formal derivation in an axiomatic system" and the related theorems of strong soundness–completeness and compactness, in order to clarify concepts or simply refresh the reader's memory. We will also define the meaning of "a system satisfies Modus Ponens" and "a system satisfies the Deduction Metatheorem".

In Section II the main ideas about the correctness–completeness theorem will be shown. In Section III, we introduce the semantic theorems of compactness, Skolem and Herbrand; we will explain what is understood by a semantic approach proof, with a special emphasis on the difference between semantic and syntactic heuristics for the completeness theorem. Then we will generalize a proof due to J. Malitz [Malitz (1979)], that works as a semantic approach proof of the *weak* (or restricted) soundness–completeness theorem, but it does not for the strong one. With the outline of Malitz proof we will then use two metalogical results previously introduced to define —in a semantic approach— an axiomatic system in order to get the *strong* version of soundness and completeness. Our system will be named *MA*, for it is a modification of that of Malitz, and it will be formally defined in Section IV. It is in our notion of derivability of *MA* the most interesting contribution, since it was not obvious how to adapt the notion of derivability so as to get the strong soundness proof. In Section V we will introduce the related heuristic ideas and some conclusions.

In what follows we will refer only to first order formal languages with equality and to first order classical logic. We will be working with Hilbert type axiomatic systems, though with a version slightly different from the usual one, which is really just an improvement, as we will see later.

The basic semantic relational concept of being forced to be true by other truths is known as "logical consequence".

DEFINITION 1.1 φ *is a logical consequence of* Σ or Σ *logically implies* φ ($\Sigma \vDash \varphi$) if and only if in every interpretation A every variable–assignment s that satisfies α for every $\alpha \in \Sigma$, also satisfies φ.

Notice that the definition of "φ is a logical consequence of Σ" means that it is impossible to have an interpretation and an assignment where all the formulas of the set Σ are satisfied but the formula φ is not.

The particular case $\Sigma = \varnothing$ denoted $\vDash \varphi$ instead of $\varnothing \vDash \varphi$, means according to the above definition, that in every interpretation A every variable assignment s satisfies φ. In this case we say that φ is *logically valid.*

The basic syntactic concept of getting a formula from a set of formulas in an axiomatic system S is known as formal derivability in the system S. We say that a formula α is derivable from a set of formulas Σ in an axiomatic system S, if there is a formal derivation of α from Σ in S.

DEFINITION 1.2

a) An *axiomatic system S* is defined by the following two conditions:
 1) A decidable set Δ of formulas.[1] The formulas of Δ are called *the axioms* of *S*.
 2) A finite set of decidable inference rules[2] of *S*.
b) A *formal derivation of a formula α from a set of formulas* Σ *in an axiomatic system S* is a finite list of n formulas $\alpha_1,\ldots,\alpha_n$, with $n \geq 1$, such that $\alpha_n = \alpha$ and for all i

($1 \leq i \leq n$) α_i is either an axiom of S, or α_i is a formula of Σ[3] or α_i is obtained from previous formulas in the list by means of some inference rule of S. Some restriction to the application of a rule can be established in advance, but it must always be effectively decidable.[4] If such a derivation exists, this is denoted by $\Sigma \vdash_S \alpha$ and it is read as "α is derivable from Σ in the system S".

The particular case $\Sigma = \varnothing$ denoted $\vdash_S \varphi$, instead of $\varnothing \vdash_S \varphi$, is defined as "$\alpha$ is a formal theorem in the system S". The only difference is that there are no hypotheses.

DEFINITION 1.3

a) An axiomatic system S *satisfies strong soundness* if and only if every formula obtained with the process of derivation from a set of formulas Σ is a logical consequence of the same set Σ. In symbols:

$$\text{if } \Sigma \vdash_S \alpha \text{ then } \Sigma \vDash \alpha.$$

b) An axiomatic system S *satisfies weak soundness* if and only if every formal theorem is a logically valid formula. This is just the particular case $\Sigma = \varnothing$. In symbols:

$$\text{if } \vdash_S \alpha \text{ then } \vDash \alpha.$$

c) An axiomatic system S *satisfies strong completeness* if and only if all logical consequences from Σ can be obtained with the process of derivation from Σ in S. In symbols:

$$\text{if } \Sigma \vDash \alpha \text{ then } \Sigma \vdash_S \alpha$$

d) An axiomatic system S *satisfies weak completeness* if and only if all logically valid formulas are formal theorems in S. This is also the particular case $\Sigma = \varnothing$. In symbols:

$$\text{if } \vDash \alpha \text{ then } \vdash_S \alpha$$

DEFINITION 1.4 Given an axiomatic system S, we define the following:

a) The system S *satisfies Modus Ponens* (*MP*) if and only if for any set of formulas $\Sigma \cup \{\alpha, \beta\}$,

$$\Sigma, \alpha, (\alpha \to \beta) \vdash_S \beta$$

b) The system S *satisfies the Deduction Metatheorem* (*DMT*) if and only if for any set of formulas $\Sigma \cup \{\alpha, \beta\}$:

$$\text{if } \Sigma, \alpha \vdash_S \beta \text{ then } \Sigma \vdash_S (\alpha \to \beta)$$

c) Let Σ be a set of formulas, then: *Σ is S–consistent* if and only if there is no formula α such that $\Sigma \vdash_S \alpha$ and $\Sigma \vdash_S \neg\alpha$. Cf. [Amor (2003)].

II. THE STRONG SOUNDNESS–COMPLETENESS THEOREM

To figure out whether or not there is a semantic approach proof for the strong soundness–completeness theorem, using the compactness theorem, was the origin of this paper. Now we can gratefully say it is possible to define an axiomatic system from a semantic perspective, such that all its logical consequences are just the formal derivations. In other words, we can say that the proof we were searching for exists and, as we will show later, it can be carried out without working within the given system, but instead by using the semantic properties of the system and some other semantic results such as the compactness theorem, a semantic version of a theorem due to Skolem and a semantic version of Herbrand theorem.[5] Of course we need to define a few syntactic concepts because the theorem itself involves them and we need to use some elementary syntactic results that follow trivially from the definitions and are fulfilled *in any* axiomatic system. However the arguments that we will give throughout the proof are purely semantic.

This paper introduces not only the semantic approach proof mentioned above but also a general way of understanding and formulating the strong soundness–completeness theorem, as a result about the *existence* of certain axiomatic systems and not as a theorem that refers to particular properties of an alleged axiomatic system.

From now on we will say "*soundness–completeness theorem*" instead of "*strong soundness–completeness theorem*" and maintain "*weak soundness–completeness theorem*" for the weak form of the theorem. It must be noticed that within the formulation of the soundness–completeness theorem, the axiomatic system mentioned plays a fundamental role (that is usually not recognized). So from a metalogical point of view, our aim is that of emphasizing *the existence* of an axiomatic system in the theorem's assertion, relative to which the properties of soundness and completeness are fulfilled. Let us consider the following two equivalent forms of the theorem:

THEOREM 2.1 (*Soundness–Completeness Theorem. First form*) There exists a formal axiomatic system S such that for any set of formulas $\Sigma\cup\{\alpha\}$,

$$\Sigma\vDash\alpha \text{ if and only if } \Sigma\vdash_S\alpha$$

THEOREM 2.2 (*Soundness–Completeness Theorem. Second form*) There exists a formal axiomatic system S such that:

a) S satisfies *MP*
b) For all formulas β and γ, $\vdash_S(\beta\rightarrow(\neg\beta\rightarrow\gamma))$.
c) For any formula α, $\vdash_S(\neg\alpha\rightarrow\alpha)\rightarrow\alpha$.
d) S satisfies *DMT*
e) For any set of formulas Γ,

$$\Gamma \text{ is } S\text{–consistent if and only if } \Gamma \text{ is satisfiable.}$$

In this paper we will give a semantic approach proof of the first form of the theorem. Notice that property e) of the second form includes Henkin theorem.[6] It must be said that in order to have the equivalence between these two forms, properties a) to d) of the second form are necessary.

THEOREM 2.3 The two forms of the Soundness–Completeness Theorem are equivalent.

Proof (sketch): Suppose the first form and let S be the given system. It is clear that for any set of formulas $\Sigma\cup\{\alpha,\beta\}$: $\Sigma,\alpha,(\alpha\rightarrow\beta)\vDash\beta$; for all formulas β and γ: $\vDash(\beta\rightarrow(\neg\beta\rightarrow\gamma))$; for any formula α: $\vDash(\neg\alpha\rightarrow\alpha)\rightarrow\alpha$; and for any set of formulas $\Sigma\cup\{\alpha,\beta\}$: if $\Sigma,\alpha\vDash\beta$ then $\Sigma\vDash(\alpha\rightarrow\beta)$. Now by using the first form we have a), b), c), d). For e) let Γ be any set of formulas. Suppose Γ is unsatisfiable, then $\Gamma\vDash\alpha$ for any formula α, then (by first form) $\Gamma\vdash_S\alpha$ for any formula α, then Γ is S–inconsistent. On the other hand, suppose Γ is S–inconsistent, then there is a formula β such that $\Gamma\vdash_S\beta$ and $\Gamma\vdash_S\neg\beta$. Then (by first form) we have $\Gamma\vDash\beta$ and $\Gamma\vDash\neg\beta$, therefore Γ is unsatisfiable.

Now suppose the second form and let S be the given system. Let $\Sigma\cup\{\alpha\}$ be any set of formulas. For the "only if" part, suppose that $\Sigma\nvdash_S\alpha$. Then $\Sigma\cup\{\neg\alpha\}$ is S–consistent because if not, there would be a formula β such that $\Sigma\cup\{\neg\alpha\}\vdash_S\beta$ and $\Sigma\cup\{\neg\alpha\}\vdash_S\neg\beta$, but then by b) and a) of the second form we have $\Sigma\cup\{\neg\alpha\}\vdash_S\gamma$ for any formula γ; in particular $\Sigma\cup\{\neg\alpha\}\vdash_S\alpha$. Now by d) we have $\Sigma\vdash_S(\neg\alpha\rightarrow\alpha)$ and by c) and a) we have $\Sigma\vdash_S\alpha$. So $\Sigma\cup\{\neg\alpha\}$ is S–consistent. Finally by e) we have that $\Sigma\cup\{\neg\alpha\}$ is satisfiable and therefore $\Sigma\nvDash\alpha$.

For the "if" part, suppose that $\Sigma\nvDash\alpha$. Then $\Sigma\cup\{\neg\alpha\}$ is satisfiable and by e) $\Sigma\cup\{\neg\alpha\}$ is S–consistent. Then $\Sigma\nvdash_S\alpha$, because if $\Sigma\vdash_S\alpha$ then $\Sigma\cup\{\neg\alpha\}\vdash_S\alpha$ by monotony of any formal system and $\Sigma\cup\{\neg\alpha\}\vdash_S\neg\alpha$ against the S–consistency of $\Sigma\cup\{\neg\alpha\}$.

For the proof of the equivalence of this two forms, cf. [Amor (2006), pp.140–141]. The first form, with the particular case $\Sigma=\varnothing$, will be called *weak* soundness–completeness.

THEOREM 2.4 (*Weak Soundness–Completeness Theorem*) There exists a formal axiomatic system S such that for any formula α,

$$\vDash\alpha \text{ if and only if } \vdash_S\alpha$$

It is quite clear that soundness–completeness trivially implies the weak soundness–completeness, because the last one is just the particular case $\Sigma=\varnothing$ of the first. On the other side, weak soundness–completeness together with Modus Ponens (*MP*) and the Deduction Metatheorem in the system (*DMT*) along with the compactness theorem, imply soundness–completeness.

III. A SEMANTIC APPROACH TO SOUNDNESS–COMPLETENESS

There is a direct proof of the soundness–completeness theorem, which consists on defining an axiomatic system, and proving that it satisfies soundness and completeness. This proof is usually done by working inside the axiomatic system and making many formal derivations until both properties are shown to be fulfilled by the system.

There is an alternative way: a *semantic approach proof*, which consists on defining, in an ad hoc way and with a semantic perspective, an ad hoc system whose axioms and rules have semantic grounds and that satisfies soundness and completeness. In this case the justification is semantic and avoids working inside the defined system except for using some general properties of all systems**.** This proof is an extension and essential modification of the proof of Malitz, who introduced a system and proved that it satisfies *weak* soundness–completeness, but satisfies neither soundness nor completeness.

In the proof we use the Compactness theorem, a Skolem semantic theorem and Herbrand semantic theorem which by the way, is proved by using the Compactness theorem. The proofs of these theorems are semantic and appear in Malitz's book. Cf. [Malitz (1979)].

We introduce now the compactness theorem, which is a purely semantic theorem proved also by Gödel in 1930. We give here two equivalent versions of that theorem.

THEOREM 3.1 (*Compactness Theorem. First form*) Given a set of formulas Σ of a first order language with equality, if for each finite subset $\Gamma\subseteq\Sigma$, Γ is satisfiable, then Σ is satisfiable.

Proof (*sketch*): First let us take an infinite set of formulas Σ such that each finite subset of it is satisfiable (call this property "finitely satisfiable"). Two results need to be proven, the first one consists in getting a set Γ such that $\Sigma\subseteq\Gamma$ and Γ is a maximal finitely satisfiable set of formulas; for that the Zorn Lemma is used. The second result consists in getting a set Ω such that $\Sigma\subseteq\Omega$, Ω is closed under existential "witnesses" and is a finitely satisfiable set of formulas. Then these two results are iterated by recursion and the union of all those sets is a set Σ^* such that $\Sigma\subseteq\Sigma^*$ and Σ^* is a maximal closed under witness and finitely satisfiable set of formulas. Then we build a structure interpretation for the language and an assignment for variables that satisfy all formulas of Σ^*. That interpretation and assignment obviously satisfy all formulas of Σ and so we can conclude that Σ is satisfiable. Actually Σ^* equals the set of all formulas satisfied by that assignment in that structure.

THEOREM 3.2 (*Compactness Theorem. Second form*) If $\Sigma\cup\{\varphi\}$ is a set of formulas of a first order language with equality and $\Sigma\vDash\varphi$, then there is a finite subset $\Gamma\subseteq\Sigma$, such that $\Gamma\vDash\varphi$.

Proof: Assume that $\Sigma\vDash\varphi$. Thus $\Sigma\cup\{\neg\varphi\}$ is unsatisfiable, and hence by first form, there is a finite $\Delta\subseteq\Sigma\cup\{\neg\varphi\}$ which is unsatisfiable. But then $(\Delta\backslash\{\neg\varphi\})\cup\{\neg\varphi\}$ is also unsatisfiable and therefore there is $\Gamma=\Delta\backslash\{\neg\varphi\}$ finite, $\Delta\backslash\{\neg\varphi\}\subseteq\Sigma$ and $\Delta\backslash\{\neg\varphi\}\vDash\varphi$.

This second form is which we will use in our main result. Notice that the compactness theorem is a purely semantic assertion; it does not involve deductions at all. It refers only to semantic notions and it must be pointed out here that there exist several purely semantic proofs for it; that is, proofs based on semantic properties.

What we have called "Skolem theorem" should not be confused with Skolem famous theorem about the existence of countable models for sets of formulas. We are rather referring to a theorem about the special form of formula that can be associated to each formula, known as *its Skolem Form of Validity* (*SFV*).

DEFINITION 3.1 Let φ, ψ be formulas. We say that φ and ψ are *equivalid formulas* if and only if it happens that: $\vDash\varphi$ if and only if $\vDash\psi$.

The relationship between a formula and its associated Skolem form of validity is which we have called *equivalidity* and is strictly weaker than that of logical equivalence, but it is still quite useful, since it establishes "equivalence" in terms of logical validity, as we will see later with Skolem's theorem.

A sentence can be transformed by means of an algorithm to its "Skolem Form of Validity", $SFV(\varphi)$. This algorithm gives out always a "pure" existential quantification (without universal quantifiers) that affects a matrix formula without quantifiers. In other words, for any sentence φ, $SFV(\varphi)=(\exists x_1\ldots\exists x_k\psi)$, where $\psi=\psi(x_1,\ldots,x_k)$ is a formula without quantifiers.

We will next show how to get these Skolem forms. Given any sentence φ, the $SFV(\varphi)$ is obtained through the following algorithmic process: (1) the first step is replacing conditionals $(\alpha\rightarrow\beta)$ with $(\neg\alpha\vee\beta)$, and biconditionals $(\alpha\leftrightarrow\beta)$ with $((\neg\alpha\vee\beta)\wedge(\neg\beta\vee\alpha))$, so we get a well known logically equivalent sentence that includes only the connectives ¬, ∨, ∧; followed by (2) introducing negations by means of the well known logical laws of negation; followed by (3) renaming all bounded variables, to avoid repeated variables in different quantifications. (4) Transforming the formula to get the Prenex Normal Form (*PNF*), and finally (5) eliminating, from left to right, universal quantifiers by introducing "witnesses" that are new Skolem constants or functions.[7]

We can summarize and "name" the steps of this procedure (which shows how to get $SFV(\varphi)$ from φ) in the following way: replacing conditionals (*RC*), replacing biconditionals (*RB*), introduction of negations (*IN*), renaming quantified variables (*RV*), prenexing all the quantifiers (*PNF*),[8] dual Skolemizing ($D(x,t)$) where "x" is the eliminated universal variable and "t" is the new Skolem constant or function that substitutes it. We finally get $SFV(\varphi)$.[9]

Logical equivalence is denoted by "≡". It is important to keep in mind that the steps of dual Skolemization, consisting of universal quantifier's elimination, differ from the other steps in that they do not preserve logical equivalence. To illustrate this, we give an example of the application of this algorithm using the abbreviations introduced before:

$$
\begin{array}{ll}
\varphi=[\exists y\forall xP(x,y)\wedge\neg\exists z\forall x\forall yQ(z,x,f(y))] & \\
\equiv[\exists y\forall xP(x,y)\wedge\forall z\exists x\exists y\neg Q(z,x,f(y))] & IN \\
\equiv[\exists y\forall xP(x,y)\wedge\forall z\exists v\exists w\neg Q(z,v,f(w))] & RV \\
\equiv\exists y\forall x\forall z\exists v\exists w[P(x,y)\wedge\neg Q(z,v,f(w))] & PNF \\
Sk_1=\exists y\forall z\exists v\exists w[P(g(y),y)\wedge\neg Q(z,v,f(w))] & D(x,g(y)) \\
Sk_2=\exists y\exists v\exists w[P(g(y),y)\wedge\neg Q(h(y),v,f(w))] & D(z,h(y))
\end{array}
$$

Thus

$$
\begin{array}{l}
PNF(\varphi)=\exists y\forall x\forall z\exists v\exists w[P(x,y)\wedge\neg Q(z,v,f(w))] \\
SFV(\varphi)=\exists y\exists v\exists w[P(g(y),y)\wedge\neg Q(h(y),v,f(w))]
\end{array}
$$

Skolem theorem establishes that a formula φ is logically valid if and only if the Skolem Form for Validity of φ ($SFV(\varphi)$) is logically valid. We say that φ and $SFV(\varphi)$ are *equivalid formulas*. Formally:

THEOREM 3.3 (*Skolem Theorem*). For any formula φ:

a) $\vDash\varphi$ if and only if $\vDash SFV(\varphi)$.

b) $\vDash[\varphi\rightarrow SFV(\varphi)]$.

c) There are formulas ψ such that $\nvDash[SFV(\psi)\rightarrow\psi]$.

For a proof of Skolem theorem cf. [Malitz (1979), p.157] or [Amor (2006), p.92].

As a second example, we will apply the algorithm to a formula we will use in Section IV: Let $\varphi=[\exists x\forall yP(x,y)\rightarrow\forall y\exists xP(x,y)]$. Then

$$
\begin{array}{ll}
\varphi\equiv[\neg\exists x\forall yP(x,y)\vee\forall y\exists xP(x,y)] & RC \\
\equiv[\forall x\exists y\neg P(x,y)\vee\forall y\exists xP(x,y)] & IN \\
\equiv[\forall x\exists y\neg P(x,y)\vee\forall z\exists wP(w,z)] & RV \\
\equiv\forall x\exists y\forall z\exists w[\neg P(x,y)\vee P(w,z)] & PNF \\
Sk_1=\exists y\forall z\exists w[\neg P(c,y)\vee P(w,z)] & D(x,c) \\
Sk_2=\exists y\exists w[\neg P(c,y)\vee P(w,f(y)] & D(z,f(y))
\end{array}
$$

Thus:

$$
\begin{array}{l}
PNF(\varphi)=\forall x\exists y\forall z\exists w[\neg P(x,y)\vee P(w,z)] \\
SFV(\varphi)=\exists y\exists w[\neg P(c,y)\vee P(w,f(y)]
\end{array}
$$

DEFINITION 3.2 A *grounded term* is a term without variables. Let ψ be a formula without quantifiers, then a *grounded instance* of ψ is a formula obtained by substituting all free variables of ψ by grounded terms.

THEOREM 3.4 (*Herbrand Theorem*) If $\psi(x_1,\ldots,x_n)$ is a formula without quantifiers then, $\vDash(\exists x_1\ldots\exists x_n\psi)$ if and only if there exists a finite series $\psi_1,\ldots,\psi_m$, of grounded instances of ψ, such that $\vDash\psi_1\vee\ldots\vee\psi_m$.

Note that the "if" part is actually a direct consequence of the following elementary result: if $\psi(x_1,\ldots,x_n)$ is a formula without quantifiers and $\psi_1,\ldots,\psi_m$ is any finite series of grounded instances of ψ, then $(\psi_1\vee\ldots\vee\psi_m)\vDash(\exists x_1\ldots\exists x_n\psi)$. This result will be used latter. The "only if" part of Herbrand theorem is the strong one and it is proved using the compactness theorem. For a proof, see [Malitz (1979), p.185] or [Amor (2006), p.95].

Now let us go through the semantic approach proof of the soundness–completeness theorem. Let $\Sigma\vDash\varphi$ be a logical consequence of a formula φ from a set of formulas Σ. Then by the compactness theorem there is some $\Gamma\subseteq\Sigma$, Γ finite, such that $\Gamma\vDash\varphi$. Let $\Gamma=\{\alpha_1,\ldots,\alpha_m\}$. As we know, this is equivalent to $\alpha_1,\ldots,\alpha_m\vDash\varphi$. So, by applying *m* times the elementary semantic property:

$$\Sigma,\alpha\vDash\varphi \text{ if and only if } \Sigma\vDash(\alpha\rightarrow\varphi),$$

we see that $\alpha_1,\ldots,\alpha_m\vDash\varphi$ is equivalent to:

$$\vDash(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$$

Now let $\gamma=(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$ be the above logically valid formula. The sentence $\forall\gamma$ is the universal closure of the formula γ, that is, if $\gamma=\gamma(x_1,\ldots,x_n)$ where $x_1,\ldots,x_n$ are all the free variables in γ, then $\forall\gamma=\forall(\gamma(x_1,\ldots,x_n))=\forall x_1\ldots\forall x_n\gamma(x_1,\ldots,x_n)$. Since γ is equivalent in terms of logical validity (or equivalid) to its *universal closure* $\forall\gamma$, the sentence $\forall\gamma$ is also logically valid.

On the other hand, by Skolem theorem, the logical validity of the sentence $\forall\gamma$ is equivalent to the logical validity of $SFV(\forall\gamma)$,[10] and this is in turn equivalent, by Herbrand theorem, to the existence of a series of grounded instances of the matrix ψ of $SFV(\forall\gamma)$, whose disjunction is logically valid. In other words, we have the following semantic argument:

$\Sigma\vDash\varphi$ iff $\{\alpha_1,\ldots,\alpha_m\}\vDash\varphi$, for some $\{\alpha_1,\ldots,\alpha_m\}\subseteq\Sigma$ and $m\in\mathbb{N}$
iff $\vDash(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$
iff $\vDash\gamma$ since $\gamma=(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$
iff $\vDash\forall\gamma$
iff $\vDash SFV(\forall\gamma)$
iff $\vDash(\exists x_1\ldots\exists x_k\psi)$ with $SFV(\forall\gamma)=(\exists x_1\ldots\exists x_k\psi)$ and $\psi=\psi(x_1,\ldots,x_k)$ without quantifiers
iff there are grounded instances $\psi_1,\ldots,\psi_n$ of ψ, such that $\vDash(\psi_1\vee\ldots\vee\psi_n)$

We have to expand the original language with an infinite denumerable number of new constants and function symbols for each arity. This is necessary in order to have enough grounded instances of formulas without quantifiers.

Let us start by defining our system in such a way that its axioms are precisely all the formulas that share the properties of the formula $(\psi_1\vee\ldots\vee\psi_n)$ obtained with the semantic argument displayed above. In other words, the axioms will be all the logically valid disjunctions of grounded instances of formulas without quantifiers.

We must say that even though axioms defined in this way are not instances of axiom schemas, there are an infinite number of them. On the other hand, since an axiom is by definition logically valid, then whether a formula of the proposed form is or is not an axiom must be effectively decidable. But this is the case, because disjunctions of grounded instances of formulas without quantifiers are propositional formulas, and so their logical validity is effectively decidable in propositional logic.

Now, we go reverse in the previous semantic argument. We define as the first inference rule the reversed step in Herbrand theorem. However we do so in the general form $(\psi_1\vee\ldots\vee\psi_n)\vDash(\exists x_1\ldots\exists x_k\psi)$ introduced by the observation that followed Herbrand theorem. Then from any disjunction of grounded instances of substitution of a formula $\psi(x_1,\ldots,x_k)$ without quantifiers, we obtain the existential quantification of ψ. This existential quantification of ψ is precisely $SFV(\forall\gamma)$. This rule, whose premise could be logically valid (axiom) or not, will be called *Herbrand rule* or *HB*:

$$\frac{\psi_1\vee\ldots\vee\psi_n}{\exists x_1\ldots\exists x_k\psi}$$

where $\psi=\psi(x_1,\ldots,x_k)$ and every ψ_i $(1\leq i\leq n)$ is a grounded instance of ψ.

Let us define now, as the second inference rule, the reversed step in Skolem theorem: from SFV($\forall\gamma$), we obtain $\forall\gamma$. This rule will be called *Skolem rule* or *SK*:

$$\frac{SFV(\forall\gamma)}{\forall\gamma}$$

Let us define now, as the third inference rule, the reversed step in the universal closure equivalidity property "$\vDash\gamma$ if and only if $\vDash\forall\gamma$". But let us do so in the more general form $\forall\gamma\vDash\gamma$, so that from $\forall\gamma$ we obtain the variable instantiation γ. This rule applied to any formula of the form $\forall\gamma$ (that is, a sentence, logically valid or not), will be called *Variable Instantiation rule* or *VI*:

$$\frac{\forall\gamma}{\gamma}$$

Observe that $\vDash(\forall\gamma\rightarrow\gamma)$ and so $\forall\gamma\vDash\gamma$, but that generally $\nvDash(\gamma\rightarrow\forall\gamma)$. However, once again, we are speaking about two equivalid formulas: $\vDash\forall\gamma$ iff $\vDash\gamma$.

Remember that $\gamma=(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$. Now let us define, as the fourth inference rule, the reversed step in the elementary semantic property "$\Gamma,\alpha_i\vDash\varphi$ iff $\Gamma\vDash(\alpha_i\rightarrow\varphi)$" that was applied *m* times in our semantic argument. Each time in reverse will be an application of the following rule: from α and (α→φ) we get φ, with α and φ any formulas. This rule is called *Modus Ponens* or *MP*:

$$\frac{\alpha,(\alpha\to\varphi)}{\varphi}$$

Notice that up to here we have, by definition, a schema of formal derivation of the formula φ from the set of formulas Σ (given in four plus $2m$ steps, with $m\geq 0$). All this within the axiomatic system we have just defined, motivated by our previous semantic argument. The following list of $4+2m$ formulas shows the formal derivation of the formula φ from the set of formulas Σ:

1. $(\psi_1\vee\ldots\vee\psi_n)$	Axiom
2. $(\exists x_1\ldots\exists x_k\,\psi)$	Rule *HB* to 1
3. $\forall\gamma$	Rule *SK* to 2 $(\exists x_1\ldots\exists x_k\psi)=SFV(\forall\gamma)$
4. γ	Rule *VI* to 3
4. $(\alpha_1\to(\alpha_2\to\ldots\to(\alpha_m\to\varphi)\ldots))$	This is the same formula γ
4+1. α_1	Hypothesis
⋮	
$4+m$. α_m	Hypothesis
$4+m+1$. $(\alpha_2\to\ldots\to(\alpha_m\to\varphi)\ldots))$	Rule *MP* to 4+1,4
⋮	
$4+m+(m-1)$. $(\alpha_m\to\varphi)$	Rule *MP* to $4+(m-1)$, $4+m+(m-2)$
$4+2m$. φ	Rule *MP* to 4+m, 4+m+(m–1)

Therefore $\alpha_1,\ldots,\alpha_m\vdash\varphi$, by definition, from 1 to 4+2m and hence $\Sigma\vdash\varphi$ by elementary property of monotony.

It should be clear by now that the compactness theorem, the elementary semantic property "$\Gamma,\alpha_i\models\varphi$ iff $\Gamma\models(\alpha_i\to\varphi)$", the universal closure semantic property, Skolem and Herbrand semantic theorems, justify completeness and *weak soundness* of the given system in the case $\Sigma=\varnothing$, $m=0$, $\gamma=\varphi$ and only four steps. Since decidability is a main feature of axiomatic systems, it is very important to point out that the axioms and inference rules we defined are indeed decidable.

Let us call *S* the axiomatic system of Malitz where the axioms are the same as the ones defined by us, but the inference rules are only *HB* and *SK* and the definition of derivation is the traditional one without restriction in the application of rules. This system satisfies weak soundness–completeness only for sentences. However it does not satisfy the strong version. To show this we have the following examples:

1) Consider the formula $\forall xP(x)$. Since $SFV(\forall xP(x))=P(c)$, using the rule *SK* we get $P(c)\vdash_S\forall xP(x)$. However $P(c)\not\models\forall xP(x)$. This shows that *S* does not satisfy soundness.

2) On the other hand we have that $\forall xP(x)\models P(c)$. However an analysis of all possible formal proofs in S shows that $\forall xP(x)\nvdash_S P(c)$, because if $\forall xP(x)\vdash_S P(c)$, then there is a derivation of $P(c)$ from $\forall xP(x)$ in S where $\forall xP(x)$ is an hypothesis that cannot be a formal theorem (because $\nvDash\forall xP(x)$) and *HB* or *SK* cannot be applied to it because of its form and with *VI* we get only $P(x)$. So, we can eliminate $\forall xP(x)$ from the derivation and we get $\vdash_S P(c)$. But $\nvDash P(c)$, contradicting weak soundness of S. Cf. [Amor (2001), pp.64–66]. This means that S does not satisfy completeness.

It seems natural to wonder how we can modify the system S so that it satisfies the strong theorem. The answer to this question goes through two general metalogical results about axiomatic systems that satisfy the *weak* soundness–completeness property. It should be mentioned here that these results are proved using the compactness theorem cf. [Amor (2003)]. The first one refers to the metalogical relation between completeness and Modus Ponens (*MP*):

THEOREM 3.5 If S is an axiomatic system that satisfies *weak* completeness, then S satisfies completeness if and only if S satisfies *MP*.

The second refers to the metalogical relationship between soundness and the Deduction Metatheorem (*DMT*):

THEOREM 3.6 If S is an axiomatic system that satisfies *weak* soundness and completeness, then S satisfies soundness if and only if S satisfies the *DMT*.

COROLLARY 3.7 If S is an axiomatic system that satisfies weak soundness–completeness, then S satisfies soundness–completeness if and only if S satisfies *MP* and *DMT*.

For a proof of these results, cf. [Amor (2003)].

These general metalogical relations clearly show that given a system that satisfies *weak* soundness–completeness, in order to satisfy the soundness–completeness theorem, the system must also satisfy both, Modus Ponens and the Deduction Metatheorem. Notice then that the system S of Malitz cannot satisfy *MP* and *DMT*. So these two properties are not a matter of personal preference or practical convenience. In fact, they can be considered intrinsic structural properties of first order logic that play a fundamental role in soundness and completeness.

Observe that in general $\nvDash(SFV(\forall\gamma)\rightarrow\forall\gamma)$ but here we have $\models(\forall\gamma\rightarrow SFV(\forall\gamma))$. However this two formulas are equivalid: $\models SFV(\forall\gamma)$ iff $\models\forall\gamma$. Now we impose *a restriction to the application* of the second inference rule called Skolem rule (or rule *SK*). In this case, the restriction imposed is that the inference rule *SK* cannot be applied to hypotheses or to formulas that depend on hypotheses. This restriction is motivated by the semantic restriction pointed out just after Skolem theorem and it is precisely the restriction to the Skolem rule what guarantees that the logical consequences are preserved. With this restriction as part of our definition of derivation, our axiomatic system will satisfy soundness and equivalently the Deduction Metatheorem.

IV. THE AXIOMATIC SYSTEM *MA*

Following what has been said, we can give our system *MA* (for Malitz–Amor) that fulfills soundness–completeness. Language has been expanded with infinite denumerable–many new constants and function symbols of each arity.

DEFINITION 4.1. The *axiomatic system MA* is given as follows:

a) *Axioms of MA*: all logically valid disjunctions $(\psi_1 \vee \ldots \vee \psi_n)$ of grounded instances of formulas ψ without quantifiers are axioms of *MA*. All ground terms of the expanded language are used here.

b) *Inference Rules of MA*:

1) *Herbrand Rule*, *HB*:

$$\frac{\psi_1 \vee \ldots \vee \psi_n}{\exists x_1 \ldots \exists x_k \psi}$$

with $\psi(x_1,\ldots,x_k)$ a formula without quantifiers, and each ψ_i is a ground instance of ψ.

2) *Skolem Rule*, *SK*:

$$\frac{SFV(\forall\varphi)}{\forall\varphi}, \text{ if } \vdash_{MA} SFV(\forall\varphi)$$

with $SFV(\forall\varphi)=(\exists x_1 \ldots \exists x_k \psi)$ the Skolem Form of Validity for $\forall\varphi$, where $\psi=\psi(x_1,\ldots,x_k)$ is a formula without quantifiers.

3) *Variable Instantiation*, *VI*:

$$\frac{\forall\varphi}{\varphi}$$

for φ any formula.

4) *Modus Ponens*, *MP*:

$$\frac{\alpha, (\alpha \rightarrow \varphi)}{\varphi}$$

for α and φ formulas.

A *derivation of a formula* α *from a set of formulas* Σ *in* MA,[11] is a finite list of n formulas $\alpha_1,\ldots,\alpha_n$, with $n\geq 1$; such that $\alpha_n=\alpha$ and for all i $(1\leq i\leq n)$, either α_i is an axiom of *MA* or α_i is a hypothesis (formula of Σ), or α_i is obtained from earlier formulas in the list by means of the inference rules *HB*, *VI*, *MP*, of *MA*, or there exists a list of formulas $\delta_1,\ldots,\delta_m$ such that $\delta_m=\alpha_i$ and such that:

1) For all i $(1\leq i\leq m)$ $\delta_i=\alpha_k$ for some k $(1\leq k\leq n)$.

2) If $1 \leq k \leq j \leq m$ and $\delta_k = \alpha_s$ and $\delta_j = \alpha_p$ then $1 \leq s \leq p \leq$ n.
3) $\delta_1, \ldots, \delta_m$ is a formal proof of *MA*, using only the axioms and the four rules *HB, SK, VI, MP*, but *without* hypotheses.

If such a derivation exists, this is denoted by $\Sigma \vdash_{MA} \alpha$ and it is read "α is derivable from Σ in the system *MA*".

Observe that the sequence $\delta_1, \ldots, \delta_m$ is a sequence that is a part of the original one keeping its order and that there are neither formulas of Σ that are not axioms nor formulas obtained from formulas of Σ that are not axioms, because it is a formal proof.

This definition of derivation corresponds with the intuitive idea that a formula is derived from a set of formulas in the system determined by the formal theorems of our system (which of course include axioms) and with inference rules that are applicable to all formulas that are *HB*, *VI* and *MP*. Notice as well that the above definition retrieves rigorously our intuitive original idea of the restriction: *do not apply rule SK to hypotheses and to formulas which depend on hypotheses*.

In order to understand better the definition of the system *MA*, we give an example of a derivation in *MA*:

$$\exists x \forall y\, (Px,y) \vdash_{MA} \forall y \exists x\, (Px,y)$$

First, observe that $\psi_1 = [\neg P(c,f(c)) \vee P(c,f(f(c)))]$ is the grounded instance $\{y/f(c), w/\text{c}\}$ of the formula $\psi = [\neg P(c,y) \vee P(w,f(y))]$. Second, that $\psi_2 = [\neg P(c,c) \vee P(c,f(c))]$ is the grounded instance $\{y/c, w/\text{c}\}$ of the same formula ψ. Finally, the disjunction of these two grounded instances $[\neg P(c,f(c)) \vee P(c,f(f(c)))] \vee [\neg P(c,c) \vee P(c,f(c))]$ is logically valid since it is a tautology of the propositional form $(\neg A \vee B \vee \neg C \vee A)$. Hence $\psi_1 \vee \psi_2$ is an axiom of *MA*.

Let $\varphi = (\exists x \forall y P(x,y) \rightarrow \forall y \exists x P(x,y))$ then $SFV(\varphi) = \exists y \exists w (\neg P(c,y) \vee P(w,f(\text{y})))$.[12] Then the derivation is the following:

1. $[\neg P(c,f(c)) \vee P(c,f(f(c)))] \vee [\neg P(c,c) \vee P(c,f(c))]$	Axiom of *MA* ($\psi_1 \vee \psi_2$)
2. $\exists y \exists w (\neg P(c,y) \vee P(w,f(\text{y})))$	Rule *HB* to 1 ($SFV(\varphi)$)
3. $[\exists x \forall y P(x,y) \rightarrow \forall y \exists x P(x,y)]$	Rule *SK* to 2 (φ)
4. $\exists x \forall y P(x,y)$	Hypothesis
5. $\forall y \exists x P(x,y)$	MP to 4, 3

LEMMA 4.1 (*Weak soundness–completeness of the system MA*). For every formula φ:

$$\vDash \varphi \text{ if and only if } \vdash_{MA} \varphi.$$

Proof: We will use compactness and Skolem theorems. The proof comes directly from the definition of the system *MA*, based on the semantic argument of Section III. Let φ a formula and suppose $\vDash \varphi$. Then $\vDash \forall \varphi$ because the universal closure semantic property, then $\vDash SFV(\forall \varphi)$ because of Skolem theorem, then $\vDash (\exists x_1 \ldots \exists x_k \psi)$ where $SFV(\forall \varphi) = (\exists x_1 \ldots \exists x_k \psi)$ for some $\psi = \psi(x_1, \ldots, x_k)$ without quantifiers. Then there are grounded instances $\psi_1, \ldots, \psi_n$ of ψ, such that $\vDash (\psi_1 \vee \ldots \vee \psi_n)$ because of Herbrand theorem, so by definition $(\psi_1\vee\ldots\vee\psi_n)$ is an axiom of *MA* and the succession of four formulas $(\psi_1\vee\ldots\vee\psi_n, \exists x_1\ldots\exists x_k\psi, \forall\varphi, \varphi)$ is by definition a formal proof of φ in *MA*, justified respectively per "axiom", "rule *HB*", "rule *SK*" and "rule *VI*", therefore $\vdash_{MA}\varphi$.

On the other hand, suppose $\vdash_{MA}\varphi$ and let $(\alpha_1,\alpha_2,\ldots,\alpha_m=\varphi)$ be a formal proof of φ in *MA*. We show by mathematical induction on i that for all $i\leq m$, $\vDash\alpha_i$. In the case $i=m$ we are done. Suppose $\vDash\alpha_j$ for all $j<i\leq m$. There are five cases for α_i: α_i is an axiom, α_i is obtained by *HB*, α_i is obtained by *SK*, α_i is obtained by *VI* and α_i is obtained by *MP*. Since there are no hypotheses, the restriction to application of rule *SK* is not used at all.

i) If α_i is an axiom then $\vDash\alpha_i$.

ii) If α_i is obtained by *HB* from $\alpha_j=(\psi_1\vee\ldots\vee\psi_n)$ with $j<i$ and $\alpha_i=(\exists x_1\ldots\exists x_k\psi)$ then by inductive hypothesis $\vDash(\psi_1\vee\ldots\vee\psi_n)$. Then because of Herbrand theorem $\vDash(\exists x_1\ldots\exists x_k\psi)$, that is $\vDash\alpha_i$.

iii) If α_i is obtained by *SK* applied to $\alpha_j=SFV(\alpha_i)$ with $j<i$, then by inductive hypothesis $\vDash SFV(\alpha_i)$. Then because of Skolem theorem, $\vDash\alpha_i$.

iv) If α_i is obtained by *VI* applied to $\alpha_j=\forall\alpha_i$ with $j<i$ then by inductive hypothesis $\vDash\forall\alpha_i$. Then because of universal closure semantic property, $\vDash\alpha_i$.

v) If α_i is obtained by *MP* applied to α_j and $\alpha_k=(\alpha_j\rightarrow\alpha_i)$ with $j,k<i$ then by inductive hypothesis $\vDash\alpha_j$ and $\vDash(\alpha_j\rightarrow\alpha_i)$. Then we conlude, $\vDash\alpha_i$.

We will now show that *MA* satisfies soundness–completeness. Notice that all this has been proven in a semantic way, using only the necessary syntactic definitions and elementary syntactic properties that all axiomatic systems satisfy purely by definition.

THEOREM 4.2 There is an axiomatic system (*MA*), such that for any set of formulas $\Sigma\cup\{\varphi\}$, satisfies:

$$\Sigma\vDash\varphi \text{ if and only if } \Sigma\vdash_{MA}\varphi$$

Proof: We will use compactness and Skolem theorems. Let *MA* be the system we have just defined and let $\Sigma\cup\{\varphi\}$ be any set of formulas.

($\Rightarrow$) We suppose $\Sigma\vDash\varphi$. Then, by the Compactness Theorem, there is $\Gamma\subseteq\Sigma$, Γ finite, such that $\Gamma\vDash\varphi$. Let $\Gamma=\{\alpha_1,\ldots,\alpha_m\}$. Thus, by the Semantic property applied m times $\vDash(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$. Therefore, by the Weak Completeness of *MA* (Lemma 4.1), $\vdash_{MA}(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$. But then, by *MP* applied m times, $\alpha_1,\ldots,\alpha_m\vdash_{MA}\varphi$. Hence, there is $\Gamma\subseteq\Sigma$, Γ finite, such that $\Gamma\vdash_{MA}\varphi$ and, therefore by the Monotony general property $\Sigma\vdash_{MA}\varphi$.

($\Leftarrow$) We suppose $\Sigma\vdash_{MA}\varphi$. Then by definition there is a finite list $\alpha_1,\alpha_2,\ldots,\alpha_n=\varphi$, which is a formal derivation of φ from Σ in *MA*. We will show by mathematical induction on i that for all $i\leq n$, $\Sigma\vDash\alpha_i$. In the case $i=n$ we are done.

We suppose $\Sigma\vDash\alpha_j$ for all $j<i\leq n$. We show it for α_i. There are six cases: α_i is an axiom, α_i is a hypothesis of Σ, α_i is obtained by *HB*, α_i is obtained by *SK*, α_i is obtained by *VI* and α_i is obtained by *MP*.

i) If α_i is an axiom of *MA* then $\vDash\alpha_i$ and so $\Sigma\vDash\alpha_i$.

ii) If α_i is a hypothesis of Σ then obviously $\Sigma\vDash\alpha_i$.

iii) If α_i is obtained by *HB* from $\alpha_j=(\psi_1\vee\ldots\vee\psi_m)$ with $j<i$ and $\alpha_i=(\exists x_1\ldots\exists x_k\psi)$ then by inductive hypothesis $\Sigma\vDash(\psi_1\vee\ldots\vee\psi_m)$. But we have $(\psi_1\vee\ldots\vee\psi_m)\vDash(\exists x_1\ldots\exists x_k\psi)$, so $\Sigma\vDash(\exists x_1\ldots\exists x_k\psi)$, that is $\Sigma\vDash\alpha_i$.

iv) If α_i is obtained by *SK* applied to $\alpha_j=SFV(\alpha_i)$ with $j<i$, then because of the definition of derivation (restriction to the application of the rule *SK* to $SFV(\alpha_i)$ in the given deduction), we know that $SFV(\alpha_i)$ is not an hypothesis and was obtained without use of hypotheses. So by definition we know that $\vdash_{MA}SFV(\alpha_i)$ and then by weak soundness of the Lemma 4.1 we have $\vDash SFV(\alpha_i)$. Using now Skolem theorem we have that $\vDash\alpha_i$ and finally we can conclude $\Sigma\vDash\alpha_i$.

v) If α_i is obtained by *VI* applied to $\alpha_j=\forall\alpha_i$ with $j<i$ then by inductive hypothesis $\Sigma\vDash\forall\alpha_i$. But we know that $\forall\alpha_i\vDash\alpha_i$, and we conclude $\Sigma\vDash\alpha_i$.

vi) If α_i is obtained by *MP* applied to α_j and $\alpha_k=(\alpha_j\rightarrow\alpha_i)$ with $j,k<i$ then by inductive hypothesis $\Sigma\vDash\alpha_j$ and $\Sigma\vDash(\alpha_j\rightarrow\alpha_i)$. But $\alpha_j,(\alpha_j\rightarrow\alpha_i)\vDash\alpha_i$ and we conclude $\Sigma\vDash\alpha_i$.

Notice that the restriction in the definition of derivation, introduced for the application of rule *SK*, was fundamental in case (iv).

COROLLARY 4.3 (*Deduction Metatheorem for the system MA*) For all $\Sigma\cup\{\alpha,\beta\}$ set of formulas:

$$\text{if } \Sigma,\alpha\vdash_{MA}\beta, \text{ then } \Sigma\vdash_{MA}(\alpha\rightarrow\beta)$$

It is an immediate consequence of the main theorem and the metalogical result presented in Section III.

V. HEURISTICS AND CONCLUSIONS

In this section we give a description of the intuitive ideas and the heuristics that helped us to obtain this axiomatic system. The definition of our axiomatic system *MA* came out as an *ad hoc* system, from a semantic point of view, because its axioms and inference rules do not have syntactic grounds. They rather respond to a reversed semantic process that we outline next. For any formula φ and any set of formulas Σ, we start with a logical consequence φ from Σ. Because of the compactness theorem, this can be thought as an implicative form formula, say A, where $A=(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$ is logically valid. So the sentence $\forall A=\forall(\alpha_1\rightarrow(\alpha_2\rightarrow\ldots\rightarrow(\alpha_m\rightarrow\varphi)\ldots))$ is also logically valid. On the other hand, a special Skolem form $B=SFV(\forall A)$ is algorithmically obtained. B is a logically valid sentence because of Skolem theorem. Then we get a logically valid grounded propositional sentence C, without free variables and without quantifiers, whose logical validity is decidable because it is a propositional sentence.

Once we had this *ad hoc* system we proved directly soundness by mathematical induction and got the Deduction Metatheorem as consequence, even though the construction of the system was highly inspired by it. In fact it must be noticed that it is precisely the restriction to the application to the Skolem rule what guarantees that the logical consequences are preserved.

It seems then we have successfully given an affirmative answer to the problem of establishing a semantic approach proof for the Soundness–Completeness theorem that may be thought of as a corollary of the compactness and Skolem theorems. Finally it can be said that we have shown how powerful semantic ideas are to prove syntactic results.

Departamento de Matemáticas
Facultad de Ciencias, UNAM
Ciudad Universitaria, México D. F. 04510, México
E-mail: jaam@fciencias.unam.mx

NOTES

[*] I am quite grateful to two anonymous referees for their critics and to a referee of **teorema** for his (her) comments on a previous version of this paper, which highly improved it.

[1] For each formula there must be an effective procedure for deciding if it is a member of Δ or it is not.

[2] For each inference rule there must be an effective procedure for deciding if a formula follows or not from other formulas using that rule.

[3] In this case we say that α_i is a hypothesis of Σ.

[4] In fact if such a restriction exists it must be part of the very definition of formal derivation. Cf. [Amor (2004)].

[5] This work is part of that presented in [Amor (2001)].

[6] It is one way of the double implication: If Σ is *S*–consistent then Σ is satisfiable. Cf. [Henkin (1949)].

[7] These functions (called "*Skolem*") depend on the variables of the existential quantifiers previous to the universal quantifier that is being eliminated. In case that there is no existential quantifier previous to the universal quantifier that is being eliminated, the substitution is made by a constant (called "*Skolem*"). Cf. [Amor (2004)].

[8] Up to here, all the generated sentences are logically equivalent between each other.

[9] The Skolem Form of Validity *SFV*, could be given in terms of the so–called Skolem Form of *Satisfaction*, written *SFS*. The following establishes, for the interested reader, the dual relation between the two mentioned Skolem forms: if φ is a formula, then $SFV(\varphi)=PNF[\neg(SFS(\neg\varphi))]$ as well as $SFS(\varphi)=PNF[\neg(SFV(\neg\varphi))]$.

[10] Observe that $\forall\gamma$ is a sentence, and then $SFV(\forall\gamma)$ is also a sentence. This is important for the use of Herbrand theorem if it is applied to the sentence $SFV(\forall\gamma)$ that is of the form $(\exists x_1 \ldots \exists x_k \psi)$.

[11] I am grateful to the anonymous referee who suggested me this definition that corresponds with the original idea, but it is more rigorous and more intuitive. I am indebted to her or him because this definition improved the proof.

[12] Cf. the second example of Section III.

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