AESTIMATIO, THE IEB INTERNATIONAL JOURNAL OF FINANCE, 2012. **5**: 58-87

Discrete Affine Term Structure Models Applied to German and Greek Government Bonds

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RECEIVED: 4 JANUARY 2012
ACCEPTED: 31 JULY 2012

Abstract

This paper has two parts. The first part will explore and document discrete time affine term structure models in a similar setup as seen in the celebrated papers from Backus, Foresi, Telmer (1998 and 1996) and Backus, Telmer and Wu (1999). However, the paper will concentrate on the multifactor case under Vasicek (1977) and Cox-Ingersoll-Ross (1985) and unify some of the notation taking into account some of the developments seen on Duffie and Kan (1996), Piazzesi (2010) and Cochrane (2005) as well as Singleton (2006). The second half concentrates in calibrating the models and presents discussion of results, which are encouraging. When the economy is booming risk free assets' yields are expected to flatten and when the economy is under recession risk free assets' yields such as German sovereign bonds are expected to steepen. A different picture is observed for Greek Government bonds, which we show are governed mainly by deficit-to-GDP ratio, unemployment rate and debt-to-GDP ratio. Greece, in times of financial distress exhibits a downward sloping yield curve and yields are highly correlated to increases in unemployment and increases to its sovereign debt-to-GDP ratio. For the case of Greece it is also observed that a deterioration of the *budget deficit*-to-GDP ratio results in a fall in Greek government yields, however, a deterioration of the *debt*-to-GDP ratio together with an increase in unemployment more than offset this effect, resulting in an overall rise in the yields and hence, in a further deterioration of the financial position.

Keywords:

Macroeconomic releases, Term structure of interest rates; Dynamic factors; Affine term structure models.

JEL classification:

E12; E43; E44; E52; G12.

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AESTIMATIO, THE IEB INTERNATIONAL JOURNAL OF FINANCE, 2012. **5**: 58-87

Modelo afín discreto de estructura de tipos de interés **aplicado a bonos alemanes y griegos**

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Resumen

Este artículo consta de dos partes. La primera explora y documenta los modelos afín de estructura de tipos de interés en tiempo discreto siguiendo una metodología similar a la publicada en Backus, Foresi y Telmer (1998 y 1996) y Backus, Telmer y Wu (1999). Sin embargo, este trabajo tiene un enfoque multivariante mediante procesos de Vasicek (1977) y de Cox-Ingersoll-Ross (1985), y unifica la notación teniendo en cuenta los logros publicados en Duffie y Kan (1996), Piazzesi (2010) y Cochrane (2005), así como Singleton (2006). La segunda parte de este artículo se concentra en calibrar estos modelos y presentar una discusión sobre sus resultados, que son alentadores. En este trabajo demostramos que cuando la economía se halla en un momento de expansión la curva de los tipos de interés sin riesgo, como los observados en bonos del gobierno alemán, se aplana; y cuando la economía se encuentra en un estado de recesión la curva de los tipos sin riesgo se vuelve más pronunciada. Por el contrario, el caso de los bonos griegos es diferente, demostrándose que están más bien gobernados por la ratio déficit presupuestario/PIB, la tasa de desempleo y la ratio deuda pública/ PIB. En el caso de Grecia, en tiempos de dificultades financieras, la curva de tipos de interés tiene pendiente negativa y está altamente correlacionada con la tasa de desempleo y la ratio de deuda soberana/PIB. También se observa que un deterioro en la ratio déficit presupuestario/ PIB tiene como resultado una caída en los rendimientos de los bonos. Sin embargo, el deterioro del ratio deuda pública/PIB junto con el aumento de la tasa de desempleo provoca un aumento en el rendimiento de los bonos que sobrepasa holgadamente la caída de los tipos como consecuencia del deterioro de la ratio déficit presupuestario/PIB, por lo que al final el resultado es un deterioro general en la posición financiera del estado griego.

Palabras clave:

Datos macroeconómicos, estructura temporal de tipos de interés; factores dinámicos; modelos afín de estructura temporal.

1. Introduction

This essay will document some algebra and concepts seen in the continuous time affine term structure literature and plug them into the discrete time approach. The paper's starting point is the celebrated papers from Backus, Foresi, Telmer (1996-98) and incorporates the developments seen on the continuous time approach as documented in Piazzesi (2010), Cochrane (2005), Singleton (2006). We will then go and calibrate the models using Interbank as well as German and Greek govies.

This research concentrates on the multifactor cases of affine term structure models, as the weaknesses seen on the one factor models under Vasicek (1977) and CIR (1985) are already very well documented in Backus, Foresi and Telmer (1998).

Most of the empirical evidence on affine term structure literature has been mainly confined to US data. This research fits a discrete time affine term structure model using European macroeconomic data for the German govies and uses Greek unemployment as well as Greek debt and deficit to GDP ratios for the Greek yield curve. We also focus discussion of results with special attention to the economic policy, as well as portfolio management implications.

This paper is organised as follows, section 2 introduces some of the notation with reference to latest developments seen in Piazzesi (2010) and Ang and Piazzesi (2003), Cochrane (2001), Singleton (2006) and Duffie and Kan (1996). In section 3, the Vasicek (1977) model is discussed under the multifactor setup. Section 4 presents the CIR (1985) which is adapted to fit the affine model. Section 5 presents a generalised version of affine term structure models a la Duffie and Kan (1996) but on a discrete version. Section 6 calibrates the models and main results are discussed and presented using interbank and German government yields. In section 7 we calibrate the Greek bonds and discuss some of the results. Finally, section 8 conclusion and final remarks are summarised.

2. Recalling Some Basic Concepts and Introducing New Ones

It is denoted $y_t^{(N)}$ for the yield of a zero coupon bond with maturity N in time t. For the time being and without loss of generality it will be assumed that $N=1$ and hence, for convenience, the 1 period yields can be specified as a function of the stochastic discount factor as follows:

$$y_t^{(N)} = -ln\ E\left[m_{t+1}\right]. \tag{1}$$

The right hand side of (1) refers to the stochastic discount factor. Specifications on equation (1) are referred to as *pricing kernel* by the dynamic asset pricing literature.

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The problem is that this is not observed and it can only be inferred via observable yields. Assumptions made on how (1) looks like are crucial and, depending on the author, it could lead to results which exhibit somehow different setups.

The present value of a bond is specified as follows:

$$E\left[P_t^{(N+1)}\right] = E\left[m_{t+1}P_{t+1}^{(N)}\right]$$

For which the natural log notation will be used, thus implying

$$ln\left[P_t^{(N+1)}\right] = ln\left[m_{t+1}\right] + ln\left[P_{t+1}^{(N)}\right]. \quad (2)$$

For $ln\left[P_t^{(N+1)}\right]$ being the natural log present value of a bond in time t with maturity N+1, which will equate the addition between the log stochastic discount factor and the redemption value of the bond in t+1.

Seminal research such as Piazzesi (2010), Ang and Piazzesi (2003) and Ang, Dong and Piazzesi (2004) show there is a link between the discount factor and macroeconomic variables. In the process of specifying this link, they not only intend to link the short rate to macroeconomic variables but also establish assumptions about how macroeconomic variables — or so called state variables — are interlinked to the stochastic discount factor.

A notation common seen in Piazzesi (2010) as well as in Singleton (2006) and Cochrane (2005) is that the short rate is a linear function of state variables, thus,

$$r_t^f = \gamma_0 + \gamma_1' x_t. \quad (3)$$

Equation (3) is not accounted for in any of the Backus, Foresi and Telmer (1998) and (1996) and Backus, Telmer and Wu (1999). For those not familiar with the notation r_t^f denotes the short rate, γ_0 is a scalar constant term, γ_1' is a $1\times k$ vector of coefficients describing how the short rate responds to shocks on independent state variables x_t. Finally x_t is a $k \times 1$ vector. Notice that " ' " is used to denote for the transpose of a vector or a matrix.

The results obtained from the multifactor version documented in Backus, Foresi and Telmer (1998) work very well for an average yield curve but require some changes, should the researcher wish to understand movements in the yield curve, i.e. steepening, flattening and/or twists as a result of changes in the state variables, simply because under their settings the state variables are on average zero so that at the end the yield curve depends on parameter $A(N)$ only.

Another aspect which is accounted for in the literature is the behaviour of the state variables. This has two components: 1) the specification of the mean reversing process and 2) the specification of the random error term. The novelty of this work also lies in plugging Piazzesi (2010) and Cochrane (2005) into the Backus-Foresi-Telmer (1996) and (1998). Results differ mainly because authors have different specifications and different assumptions about the mean reversing process as well as the specification of the random error term and the stochastic discount factor.

Another common aspect seen in the affine term structure literature is that log prices are linear functions of state variables. A possible specification could be:

$$-ln\lfloor P_{t+1}^{(N)}\rfloor = A(N)+B(N)'x_{t+1}. \tag{4}$$

This is only a *guess*, as the functional form of (4) is not known. However, the literature appears to agree on this, as seen on Piazzesi (2010), Singleton (2006), Cochrane (2005), as well as in Backus-Foresi-Telmer (1996) and (1998) and seminal papers of Duffie and Kan (1996).

From our guess in (4) we wish to find a closed solution and estimate the parameters $A(N)$ and $B(N)'$. Once we have these parameters all we need to do is to plug them into the following yield curve and taking into account for several maturities (4) would now boil down to

$$y_{t+1}^{(N)} = \frac{A(N)}{N} + \frac{B(N)'}{N} x_{t+1}. \tag{5}$$

In the next sections the multi-factor models are dicussed, for different stochastic processes governing the behaviour of the state variables and the discount factor as accounted under the Vasicek (1977), Cox-Ingersoll-Ross (1985) and generalized affine term structure models a la Duffie-Kan (1996) asset classes.

3. Multifactor Affine Term Structure under Vasicek

A good starting point is to use the pricing kernel a la Backus-Foresi-Telmer (1998) which here is combined with the Vasicek (1977) process a la Piazzesi (2010). A possible specification would be like:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1} \tag{6}$$

$$-ln\left[m_{t+1}\right] = \delta + r_t^f + \lambda' \varepsilon_{t+1} \tag{7}$$

Equation (6) is the classical mean reversing process whereby x_{t+1} and its mean being both a $k\times 1$ vector of independent state variables and x_t being its 1 period lag. σ_x is a diagonal $k\times k$ matrix of standard deviations of the state variables. ε_{t+1} is a $k\times 1$ vector of random error terms with classical normal assumptions of mean zero and variance 1.

Equation (7) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (7) was originally the univariate Vasicek (1977) case. In this essay we transform this specification and adapt it for the multifactor case of a k-dimensional vector of state variables. In addition the short rate r_t^f is replaced by (3) which will bring this closer to the Cochrane (2005) and Piazzesi (2010) results. Same as in Backus-Foresi-Telmer (1998) δ is specified as follows:

$$\delta = \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 \tag{8}$$

Specification for (8) is fortuitous, the only aim is to normalise the stochastic discount factor so that it becomes the inverse of the short rate. Notice that with (8), now (7) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - r_t^f - \lambda'\varepsilon_{t+1}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - r_t^f$$

$$Var\left[-\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - r_t^f - \lambda'\varepsilon_{t+1}\right] = \sum_{i=1}^{k}\lambda_i^2$$

And assuming $E[ln\, x] = \mu(x) + ½\sigma^2(x)$, which yields

$$E[ln\, m_{t+1}] = -r_t^f$$

Backus-Foresi-Telmer (1998) multifactor under the Vasicek (1977) case set the x_t to zero and δ is replaced by the mean of the short rate. Here it will not be required to do this because the short rate follows as described in (3). This will make possible to generate any yield curve at any point in time, whereas Backus-Foresi-Telmer (1998) could only produce an average yield curve, therefore it will be possible to generate any yield curve and study the risk premium λ_i in time series fashion, should we wish to do so.

Here it is shown how to get there. Starting first with equation (2) and substituting the right hand term for (7) and (4) which boils down to:

$$-ln\left\lfloor P_t^{(N+1)}\right\rfloor = -\delta - r_t^f - \lambda'\varepsilon_{t+1} - A(N) - B(N)'x_{t+1} \tag{9}$$

The intention is to compute the present value recursively using what it is known from (2) for some guess of coefficients from (4). Since $P_{t+1}^{(N)} = 1$ and $A(N{=}0) = B(N{=}0)' = 0$, which means this can be solve recursively, as for 1 period would imply $A(N{=}1) = \gamma_0$ and $B(N{=}1)' = \gamma_1'$ which means that equals the short rate as described in (3). Now for any set of state variables the resulting yield curve can be computed. As this author is trying to compute the coefficients for maturity N, all is needed is to use (2) to compute the present value of an N+1 maturity bond.

As discussed earlier, modifications to the Backus-Foresi-Telmer (1998) version are added by replacing δ for (8), r_t^f for (3) and replacing x_{t+1} for the Vasicek (1977) process described in (6).

$$ln\left[P_t^{(N+1)}\right] = -\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 - \gamma_0 - \gamma_1' x_t - \lambda'\varepsilon_{t+1} - A(N) - B(N)'\left[x_t + \Phi(\bar{x} - x_t) + \sigma_x \varepsilon_{t+1}\right] \quad (10)$$

The constant terms and the terms multiplying x_t and ε_{t+1} are grouped, so that at the end it would look something like this

$$ln\left[P_t^{(N+1)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x}\right) - (\gamma_1' + B(N)'(I-\Phi))x_t - (\lambda' + B(N)'\sigma_x)\varepsilon_{t+1}' \quad (11)$$

The reader should remember what is known from (2), so that the conditional moments on (11) can satisfy,

$$E\left[ln m_{t+1} + ln P_{t+1}^{(N)}\right] = -\left(\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x}\right) - (\gamma_1' + B(N)'(I-\Phi))x_t \quad (12)$$

and

$$Var\left[ln\, m_{t+1} + ln P_{t+1}^{(N)}\right] = (\lambda' + B(N)'\sigma_x)^2 \quad (13)$$

Recalling that the implied present value of a fixed income security yields

$$-E\left[ln P_t^{(N+1)}\right] = -E\left[ln\, m_{t+1} + ln P_{t+1}^{(N)}\right] - \frac{1}{2}Var\left[ln\, m_{t+1} + ln P_{t+1}^{(N)}\right] \quad (14)$$

Substituting (12) and (13) into (14) yields

$$-E\left[ln P_t^{(N+1)}\right] = \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2 + \gamma_0 + A(N) + B(N)'\Phi\bar{x} + (\gamma_1' + B(N)'(I-\Phi))x_t - \frac{1}{2}(\lambda' + B(N)'\sigma_x)^2 \quad (15)$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (4) yields,

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - (\lambda' + B(N)'\sigma_x)^2\right) \quad (16)$$

$$B(N+1)' = (\gamma_1' + B(N)'(I-\Phi)) \quad (17)$$

All is needed is to replace (16) and (17) into (5) and solve numerically by fitting the curve to the observed yields by adjusting λ's for a given choice of maturities. All other parameters are obtained from observations. Backus-Foresi-Telmer (1998) estimated the λ's for two factor model and fit the mean yields for maturities 60 and 120 months. For each of the λ's it is possible to adjust the parameters to a desired maturity, the greater the number of λ's, the better the fit will be, as it would be possible to fit for more maturities resulting in a better fit of the parameters $A(N)$ and $B(N)$ to the observed curvature. This choice is rather arbitrary, as there is no more rule than the size of the autocorrelation coefficients. Hence, λ's from state variables which show greater persistence — thus with a greater degree of autocorrelation — are used to fit longer maturities and λ's linked to variables with a low autocorrelation coefficient are used for fitting shorter maturities, as they exhibit less persistence.

An important difference is that under the Backus-Foresi-Telmer (1998) setup (17) was equated to zero, as the means of x_t were equal to zero. Intuitively, parameters γ_0 and γ_{1i} under Backus-Foresi-Telmer (1998) are 0 and 1 respectively. Here these parameters are free and obtained empirically for which it will be shown that parameters $\gamma_0 \neq 0$ and $\gamma_{1i} \neq 1$ and the signs for parameters $B(N)_i$ from (5) depend on γ_{1i}.

4. Multifactor Affine Term Structure under Cox-Ingersoll-Ross

As in previous chapter the starting point is the pricing kernel a la Backus-Foresi-Telmer (1998) which combined with the CIR (1985) process a la Piazzesi (2010) yields:

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x \sqrt{x_t}\,\varepsilon_{t+1} \tag{18}$$

$$-ln\left[m_{t+1}\right] = \left(1 + \frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right) r_t^f + \lambda'\sqrt{x_t}\,\varepsilon_{t+1} \tag{19}$$

Equation (18) is the CIR mean reversing process whereby x_{t+1} and its mean being both a $k \times 1$ vector of independent state variables and x_t being its 1 period lag. σ_x is a diagonal $k \times k$ matrix of standard deviations of the state variables. ε_{t+1} is a $k \times 1$ vector of random error terms with classical normal assumptions of mean zero and variance 1.

Equation (19) is the stochastic discount factor as seen in Backus-Foresi-Telmer (1998), however here with somehow a different setting, as (19) was originally a univariate CIR (1985) case and here this specification is adapted for the multifactor case, as similar to the Vasicek (1977) discussed in previous section. Again, the short rate r_t^f is replaced by (3) which will bring this closer to the Duffie and Kan (1996), Piazzesi (2010), Cochrane (2005) and Singleton (2006) results.

The selection of the coefficient in (19) obeys the only purpose of normalising the stochastic discount factor so that it equates the inverse of the short rate, so that equation (19) results with the following conditional means and variance:

$$E\left[-\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)r_t^f-\lambda'\sqrt{x_t}\varepsilon_{t+1}\right]=-\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)r_t^f$$

$$Var\left[-\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)r_t^f-\lambda'\sqrt{x_t}\varepsilon_{t+1}\right]=\left(\sum_{i=1}^{k}\lambda_i^2\right)r_t^f$$

And assuming $E[ln\ x]=\mu(x)+\frac{1}{2}\sigma^2(x)$, which yields

$$E[ln\ m_{t+1}]=-r_t^f$$

Backus-Foresi-Telmer (1998) documented the discrete multifactor case for the CIR under a different setup. Their example is mainly limited to a two factor under Longstaff and Schwartz (1992) setup. Here, the aim is confined to a generalised version of a multifactor model under the CIR with a k-dimensional vector of state variables. As in previous Vasicek example, the generalised CIR multifactor case is specified as follows.

Starting with equation (2) and substituting the right hand term for (19) as well as (4) which boils down to

$$ln\left[P_t^{(N+1)}\right]=-\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)r_t^f-\lambda'\sqrt{x_t}\varepsilon_{t+1}-A(N)-B(N)'x_{t+1} \quad (20)$$

Same as for the Vasicek model the intention is to compute the present value recursively using what is known from (2) for some guess of coefficients from (4) . Since $P_{t+1}^{(N)}=1$ and $A(N=0)=B(N=0)'=0$, which means it can be solve recursively, as for 1 period it will imply $A(N=1)=\gamma_0$ and $B(N=1)'=\gamma_1'$ and thus equating the short rate as described in (3). Unfortunately this does not work because for it to work it would require $\gamma_0=0$ and γ_1' to be a $k\times 1$ elements equal to 1, which is not true empirically. So it necessary to modify the CIR case. To be more precise it will be necessary to sacrifice normality in order to be able to let a parameter λ's account for the discrepancies and thus enable the CIR model fit the observed values.

As discussed earlier modifications to the Backus-Foresi-Telmer (1998) version are added by replacing, r_t^f for (3) and replacing x_{t+1} for the CIR (1985) process already discussed in (18).

$$ln\left[P_t^{(N+1)}\right]=-\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\left(\gamma_0+\gamma_1'x_t\right)-\lambda'\sqrt{x_t}\varepsilon_{t+1}-A(N)-B(N)'\left[x_t+\Phi(\bar{x}-x_t)+\sigma_x\sqrt{x_t}\varepsilon_{t+1}\right] \quad (21)$$

Rearranging and collecting terms so that the constant terms and the terms multiplying x_t and ε_{t+1} are grouped, which would look something like this

$$\begin{aligned} ln\left[P_t^{(N+1)}\right] &= -\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)-A(N)-B(N)'\Phi\bar{x} \\ &-\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)x_t-(\lambda'+B(N)'\sigma_x)\sqrt{x_t}\varepsilon_{t+1} \end{aligned} \quad (22)$$

Recalling (2), equation (22) has the following conditional moments,

$$E\left[ln\,m_{t+1}+ln\,P_{t+1}^{(N)}\right]=-\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)-A(N)-B(N)'\Phi\bar{x}-\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)x_t \quad (23)$$

and

$$Var\left[ln\,m_{t+1}+ln\,P_{t+1}^{(N)}\right]=(\lambda'+B(N)'\sigma_x)^2x_t \quad (24)$$

Recalling (14), which is reproduced in (25) below,

$$-E\left[lnP_t^{(N+1)}\right]=-E\left[ln\,m_{t+1}+lnP_{t+1}^{(N)}\right]-\frac{1}{2}Var\left[ln\,m_{t+1}+lnP_{t+1}^{(N)}\right] \quad (25)$$

And now substituting (23) and (24) into (25) yields,

$$\begin{aligned} E\left[ln\,P_t^{(N+1)}\right] &= \gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)+A(N)+B(N)'\Phi\bar{x}+\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)x_t \\ &-\frac{1}{2}(\lambda'+B(N)'\sigma_x)^2x_t \end{aligned} \quad (26)$$

Rearranging the constant terms and the terms multiplying x_t and lining up with (4) yields,

$$A(N+1)=\gamma_0\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)+A(N)+B(N)'\Phi\bar{x} \quad (27.a)$$

$$B(N+1)'=\left(\left(1+\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2\right)\gamma_1'+B(N)'(I-\Phi)\right)-\frac{1}{2}(\lambda'+B(N)'\sigma_x)^2 \quad (28.a)$$

Same as in previous Vasicek model, all is needed to do now is to replace (27) and (28) into (5) and solve numerically by fitting the curve to the observed values by adjusting λ's for a given choice of maturities.

Backus-Foresi-Telmer (1998) do not account for (27.a) and (28.a). However, from their univariate case it is possible to intuit that $\gamma_0=0$ and γ_1' is a $1\times k$ elements equal to 1 which is not realistic. The CIR process described in equation (18) is slightly different whereby in their version $(I-\Phi)$ would be first order auto-regression coefficient ϕ. Under this paper's settings the use of the $(I-\Phi)$ brings it closer to the continuous time version described under the Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010) class of affine models.

However, notice that (27.a) and (28.a) only work if, and only if, $\gamma_0 = 0$ and γ_1' equals a $1\times k$ elements equal to 1. The problem stems from how (19) has been specified because γ_0 is not zero and elements in γ_1' are not one. So it will be necessary to change (19) and sacrifice the possibility of making (19) equal the short rate under normality. Thus, (19) will now look more like:

$$-ln\left[m_{t+1}\right] = r_t^f + \left(\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2\right)x_t + \lambda'\sqrt{x_t}\varepsilon_{t+1}$$

Now if what was shown in (20) to (28) is re-performed under the above setup, (27.a) and (28.a) would look more like

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} \qquad (27.b)$$

$$B(N+1)' = \gamma_1' + B(N)'(I-\Phi) + \frac{1}{2}\left[\sum_{i=1}^{k}\lambda_i^2 - (\lambda' + B(N)'\sigma_x)^2\right] \qquad (28.b)$$

Notice that now (27.b) and (28.b) does give the opportunity to solve by applying the recursion as in (2). Since $P_{t+1}^{(N)} = 1$ and $A(N{=}0) = B(N{=}0)' = 0$, and now $A(N{=}1) = \gamma_0$ and $B(N{=}1)' = \gamma_1'$ which means that under (27.b) and (28.b) the model equals the short rate as described in (3) for $N{=}1$. This paper will use (27.b) and (28.b) when calibrating the CIR model because the original (27.a) and (28.a) do not work for the reasons explained above.

5. Generalised Multifactor Affine Term Structure Duffie and Kan

The Vasicek (1977) and the Cox-Ingersoll-Ross (1985) are special cases of the generalised multifactor affine term structure models which were first developed by Duffie and Kan (1996) and translated into discrete time by Backus, Foresi and Telmer (1996). The intention will, as for the previous models, include some of the developments documented by Piazzesi (2010), Cochrane (2005) and Singleton (2006).

Under the generalised affine term structure state variables and the stochastic discount factor are specified as follows.

$$x_{t+1} = x_t + \Phi(\bar{x} - x_t) + \sigma_x\varepsilon_{t+1} \qquad (29)$$

$$-ln\left[m_{t+1}\right] = \delta + r_t^f + \lambda'\sigma_x\varepsilon_{t+1} \qquad (30)$$

$$\delta = \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 \qquad (31)$$

$$r_t^f = \gamma_0 + \gamma_1' x_t \cdot \tag{32}$$

$$\sigma_x = \Sigma s(x_t) \tag{33}$$

$$s_i(x_t) = \sqrt{s_{0i} + s_{1i}' x_t} \tag{34}$$

Equation (29) describes the stochastic process of the independent state variables. This is the usual mean reversing process whereby Δx_{t+1} is likely to be negative if x_t is above its mean and, is likely to be positive if x_t is below its mean. x_t and its mean are both k-dimensional vectors. Φ is a $k \times k$ matrix of diagonal elements Φ_i which represent the speed of adjustment at which each of x_{it} elements reverse to their means. σ_x is a diagonal $k \times k$ matrix comprising the volatility of the state variables. ε_{t+1} is a k-vector of shocks moving x_t away from its mean and with $\varepsilon_{i,t+1}$ elements being normally distributed with mean zero and variance 1.

Equations (30) and (31) describe the stochastic discount factor as seen in Backus-Foresi-Telmer (1998) which introduces some changes to the already discussed version shown in the Vasicek case equations (7) and (8), thus here with somehow a different setting, as (30) now includes a σ_x term.

Equation (32) which was already discussed in the introduction in (3) will be replaced by the short rate r_t^f so that it would get closer to the Piazzesi (2010) results.

The selection of (31) obeys the only purpose of normalising the stochastic discount factor so that it equates the inverse of the short rate.

Equation (33) and (34) describe the volatilities of the state variables. $s(x)$ is a diagonal $k \times k$ matrix with elements $s_i(x)$. Notice that by doing so it is possible to generalise for both Vasicek and the CIR cases. Because the Vasicek is a *Gaussian* process and CIR is a *square root* process. With (34) enabling for both cases, thus $s_{1i} = 0$ and $s_{0i} = 1$ for the Vasicek case, whereby the variance parameters in Σ are free. Alternatively, if it is wished to account for the CIR case, then set $s_{1i} = 1$ and $s_{0i} = 0$. Piazzesi (2010) and the celebrated paper from Duffie and Kan (1996) as well as Dai and Singleton (2000) remember us of the conditions required to obtain a unique solution to the stochastic differential equations, and these comprise the *Feller* and the *Lipschitz* conditions, for which the reader is encouraged to refer to Piazzesi (2010) page 706 for some examples on how this works. As for our discussion this is not of crucial relevance to this research.

Replacing (31) into (30) and adjusting the signs accordingly yields

$$ln\left[m_{t+1}\right] = -\frac{1}{2}\left(\sum_{i=1}^{k} \lambda_i^2\right)\sigma_x^2 - r_t^f - \lambda' \sigma_x \varepsilon_{t+1} \tag{35}$$

Equation (35) has the following conditional means and variance:

$$E\left[-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2-r_t^f-\lambda'\sigma_x\,\varepsilon_{t+1}\right]=-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2-r_t^f$$
$$Var\left[-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2-r_t^f-\lambda'\sigma_x\,\varepsilon_{t+1}\right]=\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2 \tag{35}$$

And assuming $E[ln\,x]=\mu(x)+½\,\sigma^2(x)$, which yields

$$E\left[ln\,m_{t+1}\right]=-r_t^f \tag{36}$$

Backus-Telmer-Wu (1999) documented an affine case for two variables under a slightly different setup. Now, similar to the previous Vasicek and CIR examples, the generalised version follows.

This starts again with equation (2) and substitute the right hand term for (35) and (4) which boils down to

$$ln\left[P_t^{(N+1)}\right]=-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2-r_t^f-\lambda'\sigma_x\,\varepsilon_{t+1}-A(N)-B(N)'x_{t+1} \tag{37}$$

Same as for the Vasicek and CIR models the intention is to compute the present value recursively using what is known from (2) for some guess of coefficients from (4). Since $P_{t+1}^{(N)}=1$ and $A(N=0)=B(N=0)'=0$, which means this can be solved recursively, as for 1 period would imply $A(N=1)=\gamma_0$ and $B(N=1)'=\gamma_1'$ and by doing so it the short rate is obtained as described in (3).

As discussed earlier, modifications to the Backus-Foresi-Telmer (1998) version will be added by replacing, r_t^f for (3) and replacing x_{t+1} for the general affine version process described in (29).

$$ln\left[P_t^{(N+1)}\right]=-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\sigma_x^2-(\gamma_0+\gamma_1'x_t)-\lambda'\sigma_x\,\varepsilon_{t+1}-A(N)-B(N)'\left[x_t+\Phi(\bar{x}-x_t)+\sigma_x\,\varepsilon_{t+1}\right] \tag{38}$$

Accounting now for (33) and (34), (38) would now look more like

$$\begin{aligned}ln\left[P_t^{(N+1)}\right]=&-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i}-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i}\,x_t-\gamma_0-\gamma_1'x_t-\lambda'\Sigma s(x)\varepsilon_{t+1}\\&-A(N)-B(N)'x_t-B(N)'\Phi\bar{x}+B(N)'\Phi x_t-B(N)'\Sigma s(x)\varepsilon_{t+1}\end{aligned} \tag{39}$$

Rearranging and collecting terms so that the constant terms and the terms multiplying x_t and ε_{t+1} are grouped, resulting in (39) being

$$ln\left[P_t^{(N+1)}\right]=-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i}-\gamma_0-A(N)-B(N)'\Phi\bar{x}$$
$$-\left(\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i}+\gamma_1'+B(N)'(I-\Phi)\right)x_t-\left[\lambda'+B(N)'\right]\Sigma s(x)\varepsilon_{t+1} \tag{40}$$

And has conditional moments,

$$E\left[ln\ m_{t+1}+P_{t+1}^{(N)}\right]=-\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i}-\gamma_0-A(N)-B(N)'\Phi\bar{x}$$
$$-\left(\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i}+\gamma_1'+B(N)'(I-\Phi)\right)x_t \tag{41}$$

and

$$Var\left[ln\,m_{t+1}+ln\,P_{t+1}^{(N)}\right]=\left[\lambda'+B(N)'\right]^2\Sigma'\Sigma s_{0i}+\left[\lambda'+B(N)'\right]^2\Sigma'\Sigma s_{1i}x_t \tag{42}$$

The implied present value of a fixed income security being

$$-E\left[ln\,P_t^{(N+1)}\right]=-E\left[ln\,m_{t+1}+ln\,P_{t+1}^{(N)}\right]-\frac{1}{2}Var\left[ln\,m_{t+1}+ln\,P_{t+1}^{(N)}\right] \tag{43}$$

So that substituting (41) and (42) into (43) yields

$$-E\left[ln\,P_t^{(N+1)}\right]=\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{0i}+\gamma_0+A(N)+B(N)'\Phi\bar{x}$$
$$+\left(\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2\right)\Sigma'\Sigma s_{1i}+\gamma_1'+B(N)'(I-\Phi)\right)x_t-\frac{1}{2}\left[\lambda'+B(N)'\right]^2\Sigma'\Sigma s_{0i}-\frac{1}{2}\left[\lambda'+B(N)'\right]^2\Sigma'\Sigma s_{1i}x \tag{44}$$

Rearranging and collecting terms, with the constant terms and the terms multiplying x_t being grouped. Finally, lining up with (4) results in

$$A(N+1)=\gamma_0+A(N)+B(N)'\Phi\bar{x}+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2-\left[\lambda'+B(N)'\right]^2\right)\Sigma'\Sigma s_{0i} \tag{45}$$

$$B(N+1)'=\gamma_1'+B(N)'(I-\Phi)+\frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2-\left[\lambda'+B(N)'\right]^2\right)\Sigma'\Sigma s_{1i} \tag{46}$$

Same as in previous Vasicek and CIR model, all is needed now is to replace (45) and (46) into (5) and solve numerically by fitting the curve to the observed values by adjusting λ's for a given choice of maturities.

Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) and (1996) document (45) and (46) in a somehow different setup. The pricing kernel described in (30) is normalised, so that under log normal conditions the stochastic discount factor equates the short rate, this is not so obvious in their case. A matrix Σ of free parameters is also included in the model similar to Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010), which is not included in Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) and (1996).

As in the previous cases, Backus-Telmer-Wu (1999) and Backus-Foresi-Telmer (1998) their versions show that $\gamma_0 = 0$ and γ_1' is vector of $1 \times k$ elements equal to 1. Finally, also the process described in equation (29) is slightly different whereby in their version $(I-\Phi)$ would be first order auto-regression coefficient ϕ. Under our setting the use of the $(I-\Phi)$ brings us closer to the continuous time version described under the Duffie and Kan (1996), Cochrane (2005) and Piazzesi (2010) class of affine models. However, Piazzesi (2010) does not account neither for λ's nor the volatility of the stochastic discount factor in a way that allows fitting the curve to observed yields.

Ideally, (45) and (46) would allow to identify back (16) and (17) as well as (27) and (28), but this does not quite match because of the $\Sigma'\Sigma$ multiplying both λ' and $B(N)'$ terms in (45) and (46). This is because of how the stochastic discount factor has been specified in (30) and (31). Under this setup we differ to the stochastic discount factor under Vasicek (1977) and CIR (1985) documented earlier in (7) and (19) mainly because in these specifications it did not account for the volatility σ_x as part of the stochastic discount factor, as depicted in (30) and (31). This has been mainly for convenience only. Thus, without loss of generality (45) and (46) are adapted slightly, thus yielding

$$A(N+1) = \gamma_0 + A(N) + B(N)'\Phi\bar{x} + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left[\lambda' + B(N)'\sigma_x\right]^2\right)s_{0i} \tag{47}$$

$$B(N+1)' = \gamma_1' + B(N)'(I-\Phi) + \frac{1}{2}\left(\sum_{i=1}^{k}\lambda_i^2 - \left[\lambda' + B(N)'\sigma_x\right]^2\right)s_{1i} \tag{48}$$

Notice that now when s_{0i} is 1 and s_{1i} is zero the model accounts for the Vasicek (1977) case discussed in Section 3 and when s_{0i} is zero and s_{1i} is 1 the model accounts for the CIR (1985) as discussed in section 4.

6. Calibrating Under the Discrete Approach

This paper calibrates the Vasicek (1977) and the CIR (1985) using macroeconomic data. The models discussed in sections 3 and 4, shown in (16) and (17) for the Vasicek process, and (27.b) and (28.b) under the CIR approach are fitted using monthly Euro-Zone Unemployment Rate, Euro-Zone M3, Euro-Zone Production Price Index and European Commission Consumer Confidence Index. Results are compared for estimated values of $a(N) = A(N)/N$ and $b(N) = B(N)/N$ with those observed via OLS published in Jakas (2011). This empirical work is based on monthly observations. EONIA, Euribor and German government yields have been obtained from Bloomberg. Most of the data series is only available since 1999. The period considered is from December 1999 until January 2010. This results in 122 observations. The yields are estimated under the restrictions (16) and (17) as well as (27.b) and (28.b) mentioned above and compared with the observed data.

Figure 1 below shows the coefficients $b(N)_i$ for OLS, Vasicek and CIR for different maturities. Vasicek and CIR show more persistence than OLS. Under OLS, coefficients fall faster and die away as maturity increases. Notice under OLS only the unemployment and producer prices show some persistence. In addition, they also exhibit a "humped" shape which this paper is not capable to reproduce under Vasicek or CIR and confirm the results seen on Backus, Telmer and Wu (1999). Not surprisingly, under the Vasicek as well as CIR approach, the coefficients are almost identical both models estimate virtually the same values. This is possible thanks to the use of more than one variable and the use of λ's to fit for the same maturities. The sign of the $b(N)_i$ coefficients are primarily governed by the estimated parameters γ_{1i}, which are — in turn — estimated via OLS by regressing the EONIA with the four factors (Unemployment, PPI, M3 and Consumer Confidence Index). Discussions on these results are shown in Jakas (2011).

Figure 2 shows that this behaviour is also observed for the coefficients $a(N)$. In general, it could be said that affine models exhibit coefficients which have a smoother behaviour across maturities than those seen under the OLS approach. $a(N)$ increases as maturities become longer. Under the OLS approach $a(N)$ is much steeper than those estimated under the Vasicek and CIR, and becomes negative for the Euribor 3 months, 6 months and the 2 year German government.

Economically, results are interpreted as follows: An increase in unemployment results in an increase in expected aggregate marginal utility with a the subsequent decrease in risk free assets' yields, as these are dear most in times of low consumption growth. An increase in consumer confidence results in a decrease in expected aggregate marginal utility growth and therefore, risk free yields are expected to increase as these assets act as a hedge in times when consumer confidence is low. An increase production prices has two implications, 1) an increase in production prices means that aggregate consumption growth is high, and in times when aggregate consumption growth is high risk free assets are dear less as these are negatively correlated with consumption growth, and investors are less risk averse and prefer riskier assets. 2) Taylor rules show that central bank policy will be to increase interest rates if production prices are expected to drift expected inflation away from target levels. Subsequently, an increase in interest rates results in an increase in yields, as financing becomes more expensive and margins between interest income and funding expense are tighter. An increase in monetary aggregates is expected to results in a fall in interest rates, as central banks conduct quantitative easing, yields fall mainly because the cost of financing generate higher margins and competition in the capital markets push bond prices up with the subsequent fall in yields. This behaviour is explained mainly by an increase in margins which in turn increases demand in the bond markets pushing prices up.

Figure 1. **Analysis of the $b(N)_i$ coefficients under OLS, Vasicek and CIR models**

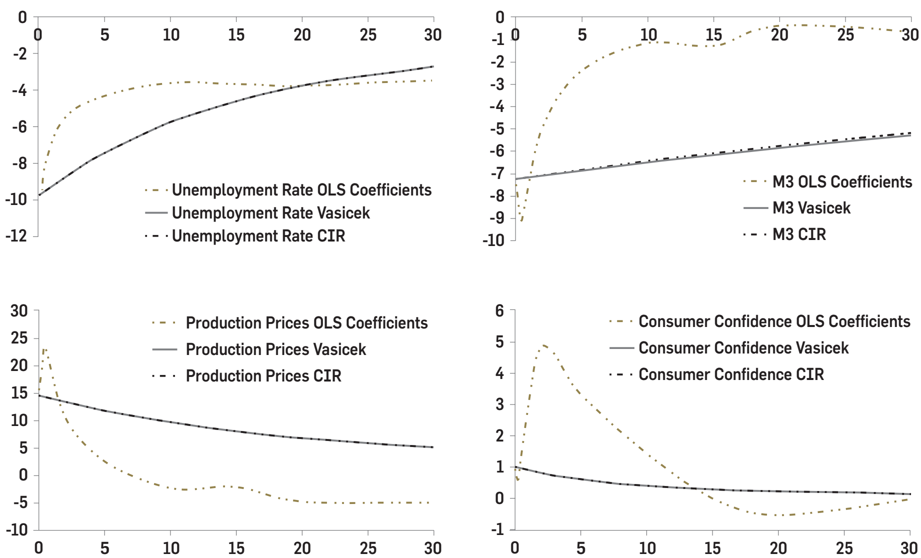
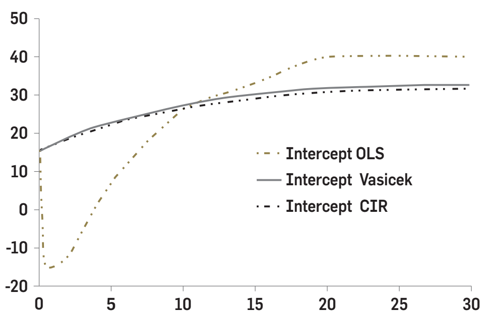


Figure 2. **Analysis of the $b(N)$ coefficient under OLS, Vasicek and CIR models**



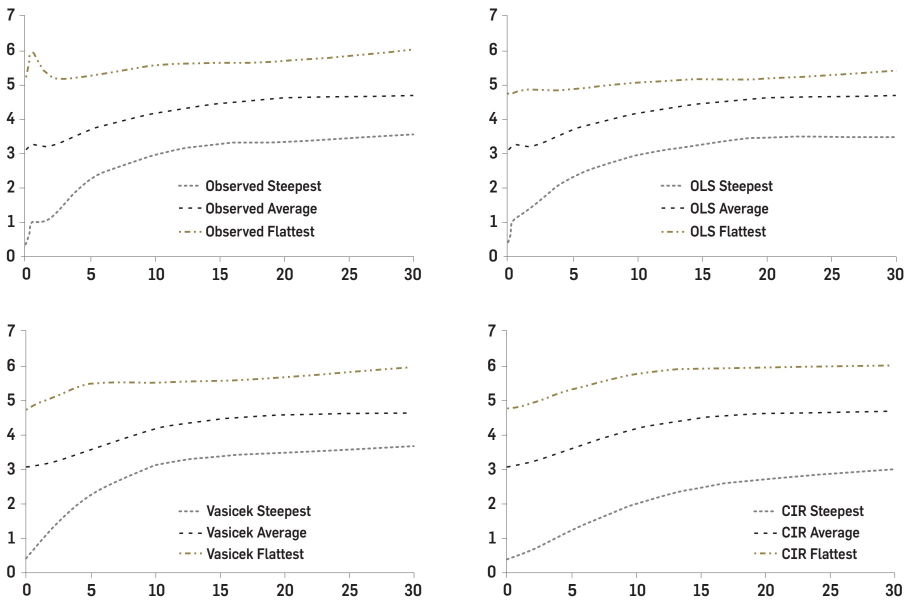
In order to analyse the models' ability to fit the curves with various shapes this research shows in figure 3 how the fitted curves look like at their flattest, mean or average, and steepest levels. In order to do this first the EONIA's historical highs, average and lows are identified and, subsequently the corresponding values observed for the four factors (unemployment, PPI, M3 and Consumer Confidence index). The model is capable of reproducing the observed sample data. Results have interesting policy implications which are outlined as follows: 1) the yield curve is at its flattest level when the overnight rate is at its highest value, unemployment is at its lowest level, production prices are at their highest levels, money supply (M3) is tight and consumer confidence levels is at its highs. And 2), when the yield curve is at its steepest the overnight rate is at its lowest, unemployment rate is at its highest levels, production prices are low, money supply is lax and consumer confidence level is the lowest.

Notice that an economy exhibiting the above mention behaviour of the yield curve would suggest that the government could take advantage by issuing new debt at low financing costs in times of low consumption growth in order to undertake countercyclical fiscal policies without increasing distortionary taxation. In times of low consumption growth risk free assets exhibit low yields and hence low financing costs. Alternatively, in times of high consumption growth, when risk free assets exhibit high yields and thus prices are low, governments should reduce debt growth outstanding via buy-back programmes and thus reduce current refinancing costs. In fact, governments should under such scenario take advantage of buying back at a low redemption price.

From a portfolio management perspective, in times when the curve is at its steepest, a representative investor will have the incentive to undertake a flattening strategy such as shorting the 2 year maturity bonds and long the 10 to 30 years, as the losses generated in the 10 to 30 years will be more than offset by the profits on the front end. In times when the curve is at its flattest, a representative investor will have the incentive to long the 2 year maturities and short the long end (10 to 30 years). The front end of the curve is relatively similar across models. The different models show discrepancies mainly on the long end. Hence, from the 10 year maturities onwards, OLS fails to describe the movement of the curve when yields are above their average. The observed, as well as the estimated via Vasicek and CIR describe a rather parallel shift whereas OLS shows more a flattening movement once yields have reached their means and are moving towards their flattest level.

In all cases it is shown that when the curve is at its flattest, yields are higher across the yield curve and when the yield curve is at its steepest the yields are at their lowest levels. All models agree with the observed yield curves that the front end is clearly more volatile and has a greater range of values than the long end. When yields are increasing, the yield curve is expected to become flatter if yields are currently below their means and, alternatively, when yields are falling, the yield curve is expected to become steeper if yields are above their mean values. This is mainly because the volatility in the yield curve is expected to fall as maturity increases. A possible way of describing this would be if it is assumed that the volatility and any point of the yield curve somehow obeys that $\sigma^{(N)}=\sigma^{(1)}/N$, thus the volatility and macroeconomic shock effects decrease as maturity increases.

Figure 3. **Fitting the yield curve**



The predictive-ability of the models is of special interest to this research. The intention is not only to explain economically the behaviour of the yield curve from a macroeconomic perspective but also understand how macroeconomic variables contribute to predict changes in the yield curve. Figures 4 and 5 show how the four factor Vasicek as well as the CIR models are capable of anticipating yield curve behaviour as a consequence of innovations in macroeconomic variables. The models are calibrated for the period starting from December 1999 to January 2010 and compared the estimated yields to those observed in the sample data. Results are encouraging, as fitted values are in line with the observable trends. Yields in the longer end are less predictable, compared to those seen in the front end however they still exhibit a high explanatory power and forecast quite well the underlying trends.

As maturities become longer, forecasts appear to be less convincing towards end of 2009 and beginning of 2010. However, even though not presented in figures 4-5, current data shows that yields have indeed fallen during 2011 to historic lows, Schatz (2 years German tresuries), Bobls (5 years German government) and Bunds (10 years German government bonds) reached historic lows below 2% for Bunds and even negative yields on less than 1 year German treasuries, thus showing that these models predicted in advance lower yield levels.

Figure 4. **Vasicek fitted versus observed yields**

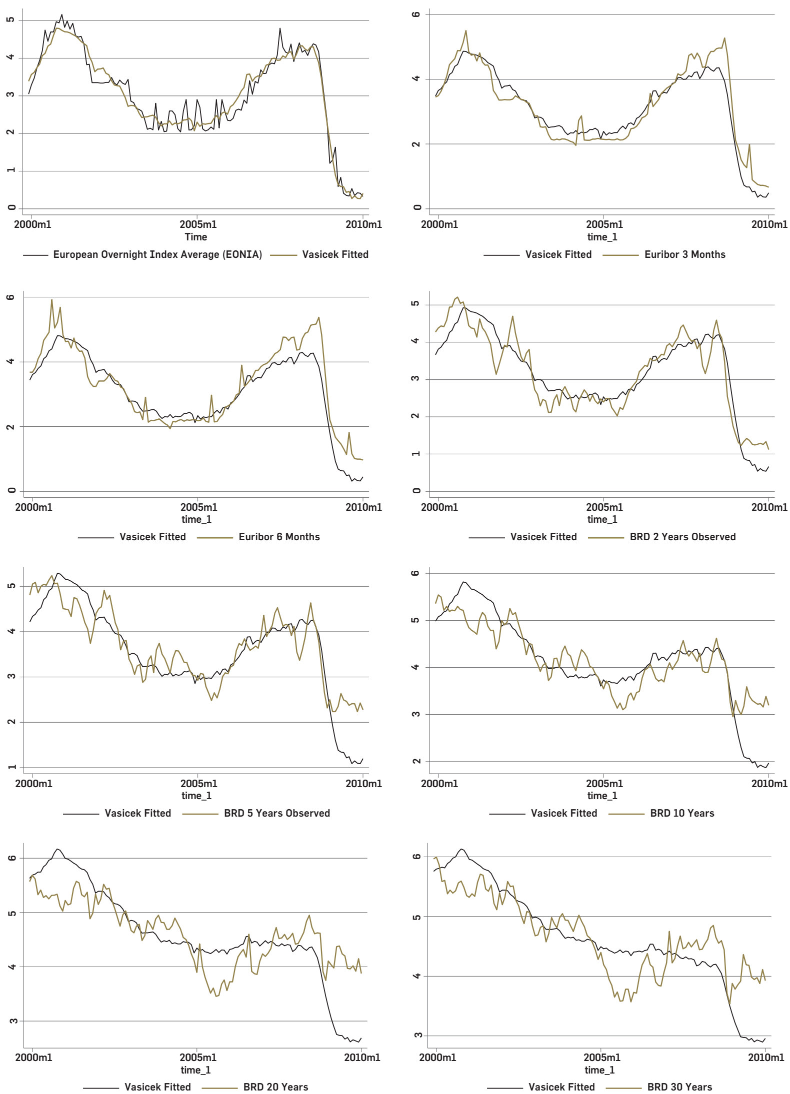
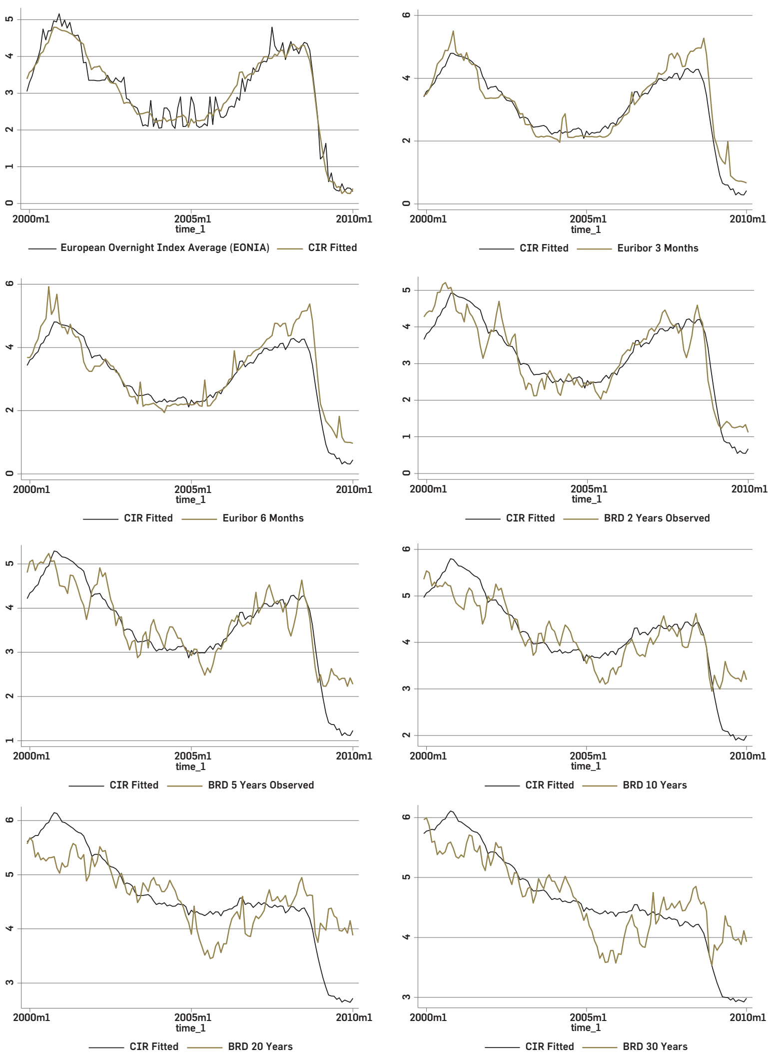


Figure 5. CIR fitted versus observed yields



7. Calibrating with Greek Government Bonds

In this section, we calibrate an affine term structure model under CIR with Greek government bonds. The study was mainly dependent on the availability of yield data for the period during Greek's financial collapse as well as previous periods where Greece enjoyed some stability. The model should be capable of accounting for all states of the economy. Full statistics were only available for 2, 5, 10, 15 year bonds available in Bloomberg historical data. The period used varies depending on the maturity, but overall we considered the period June 2001 to April 2012. The macroeconomic variables have been picked after surveying 50 market participants from various market leading financial institutions. We picked the top three: Greek government *budget deficit* to GDP ratio, Greek unemployment rate and Greek government *debt* to GDP ratio. Regression results show that all these factors were highly significant and that OLS as well as affine models perform very well, despite the volatility seen during the last two years.

Figures 6.1 and 6.2 below show the affine-CIR fitted versus the observed Greek Government yields, which are quite encouraging despite market conditions. Figure 6.2 shows that observed values exhibit more volatility than the fitted ones the closer we get to Greece's default. Figure 7 shows the OLS-fitted versus observed. The reason for doing this comparison is mainly to show the results stemming from a purely statistical perspective and see how much differ compared to an approach with deeper theoretical underpinnings, as seen in figure 6.1 under an affine-CIR approach.

Figures 6.2 and 7 show that the affine-CIR as well as the OLS models can reproduce the dramatic rise in yields shortly before Greece's collapse. Not so lucky appear to be the results seen for the 2 year yields which exhibits a slower growth rate compared to the observed data. However, this is not so disappointing, as the model has been able to account for a yield movement from 4% up to almost 50% levels. Remarkably, the 5, 10 and 15 years exhibit surprisingly good results, mainly because these have had a far lower impact compared to the 2 year yields. This research show that the macroeconomic variables used for calibrating the model explain very well yield dynamics.

In times of financial distress and when government bonds become risky assets, markets focus their attention more to the ability of governments to repay in the future and ratios such as debt-to-GDP as well as government deficit-to-GDP exhibit high explanatory power, in contrast to the German bonds, where markets look here more into unemployment as well as expected future consumption growth and less to debt-to-GDP or deficit-to-GDP ratios, because here German bonds act as a hedge for times when aggregate marginal utility growth is high and expected future consumption growth is low.

Figure 6.1. Greek government bond yields CIR-fitted versus observed yields

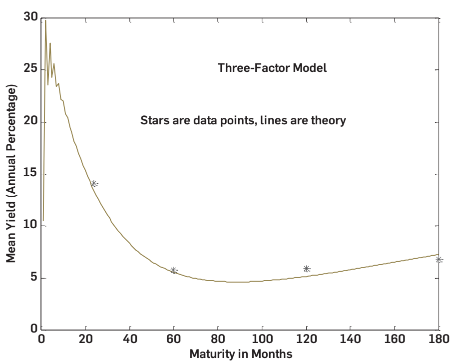


Figure 6.2. **Greek government bond yields CIR-fitted versus observed yields**

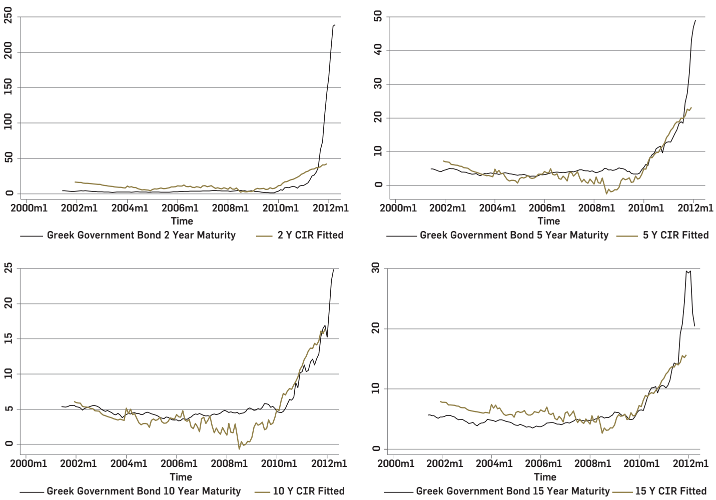
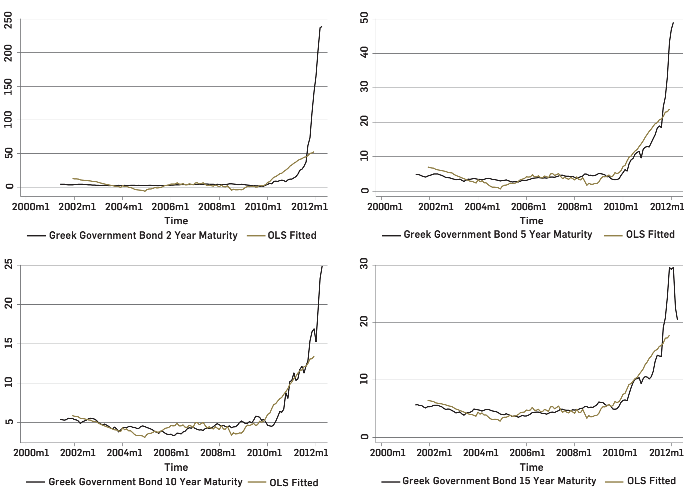


Figure 7. **Greek Government Bonds OLS fitted versus observed**



This paper also focus also some attention to the ability of the model to generate an average yield curve in times when the short yields are at their *lowest* and at their *highest* levels. Reason for doing this is simply to show the limitations of the model. Figure 8 below shows different coefficients for the parameters $A(N)/N$ and $B(N)_i/N$ which have been estimated calibrating a space state vector observed when the 2 year Greek Government yield was at its lowest level, at its mean and at its highest level. We see that the model struggles a bit when yields are at their highest levels. This should not surprise the reader, as these levels of yields are observed shortly before the Greek sovereign collapse and its subsequent default. Moreover, actually what is really struggling here is the Broydn function in Matlab rather than the model. In this paper we use this Matlab function to solve numerically equation (5) to adjust to observed yields (and to restrictions documented in (27.b) and (28.b)) by adjusting the vector λ' explained in previous sections. Parameters $A(N)/N$ and $B(N)_i/N$ exhibit virtually identical patterns and differ significantly only when yields are at their highest levels. Coefficients for Greek government budget deficit to GDP are negative which means that an increase in this ratio, hence a deterioration of its finances with respect to GDP, results in a fall in yields. A priori this might be seen as odd, but it makes sense if this is analysed together with the other two coefficients, thus looking into the size and the sign of the coefficients for unemployment and debt-to-GDP ratio. Unemployment and debt-to-GDP ratio show that these coefficients would more than offset any positive effect from the fall in yields as a result of an increase in the government deficit-to-GDP ratio.

Figure 8. Greek Government Bonds Affine-CIR estimated coefficients $A(N)/N$ and $B(N)_i/N$ fitting equation (5) to observed yields, restricted to (27.b) and (28.b) and when state vectors are $x_{i;min}$; $x_{i;mean}$; and $x_{i;max}$.

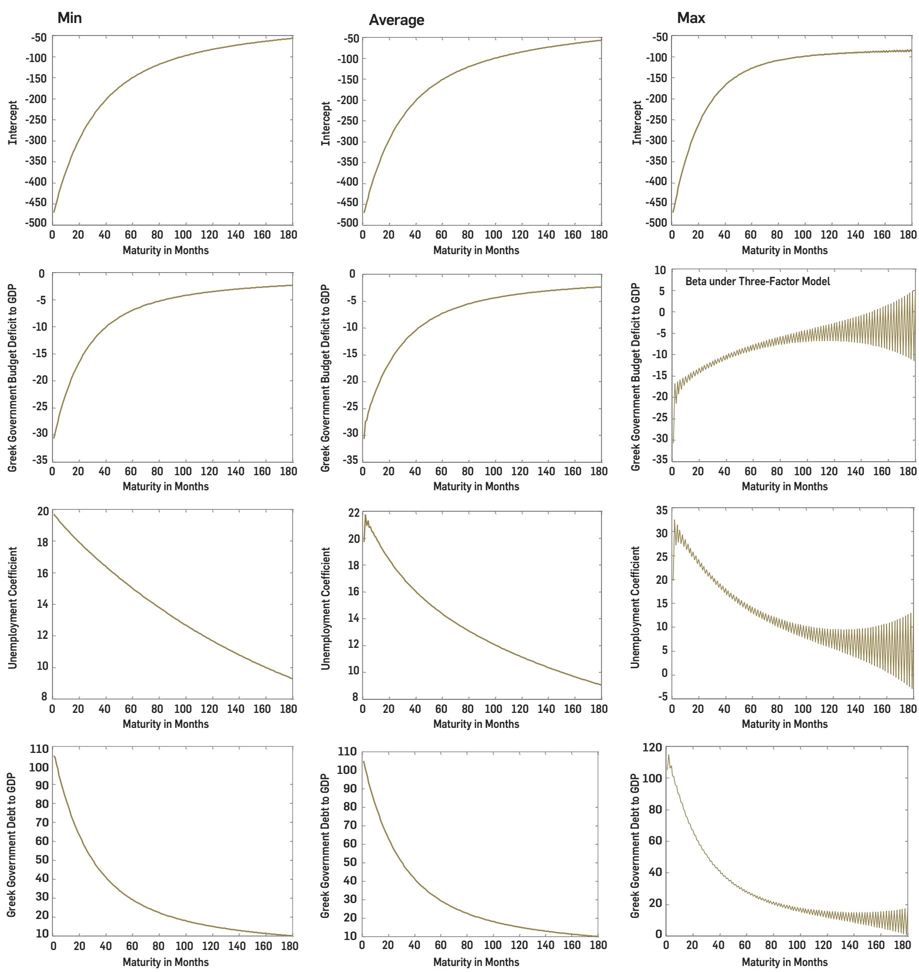


Table I shows some regression results. All variables used are very significant. OLS confirms our observation under the affine-CIR, thus if governments similar to the Greek case, engage in counter-cyclical fiscal policies that result in a deterioration of their government deficit with respect to GDP might still observe a fall in yields however, this is expected to be more than offset by an increase in yields as a consequence from a deterioration in their debt-to-GDP ratio. This implies that governments can run deficits to reduce unemployment only if the deterioration of its deficit does not result in a significant deterioration of their Debt to GDP ratio, as the deterioration of this ratio will more than offset any positive effect stemming from their countercyclical fiscal policies.

Table 1. **OLS Regression Results and Selected Diagnostics**

State Variables	Greek Government Sovereign Yields			
State Variables	**2 Years**	**5 Years**	**10 Years**	**15 Years**
Log Greek Gov. Deficit to GDP Ratio	-23.53547 (*t-stat:* -3.68) (*P*: 0.000)	-7.338331 (*t-stat:* -5.30) (*P*: 0.000)	-3.093097 (*t-stat:* -6.25) (*P*: 0.000)	-3.776979 (*t-stat:* -4.30) (*P*: 0.000)
Log Greek Unemployment Rate	18.56715 (*t-stat:* 3.04) (*P*: 0.003)	6.672775 (*t-stat:* 4.50) (*P*: 0.000)	3.40174 (*t-stat:* 5.99) (*P*: 0.000)	4.858906 (*t-stat:* 4.98) (*P*: 0.000)
Log Greek Government Debt to GDP Ratio	100.2863 (*t-stat:* 4.15) (*P*: 0.000)	39.68417 (*t-stat:* 7.90) (*P*: 0.000)	17.13523 (*t-stat:* 12.02) (*P*: 0.000)	24.58802 (*t-stat:* 7.59) (*P*: 0.000)
Intercept	-461.3932 (*t-stat:* -4.18) (*P*: 0.000)	-182.0015 (*t-stat:* -8.11) (*P*: 0.000)	-77.10492 (*t-stat:* -13.32) (*P*: 0.000)	-113.5292 (*t-stat:* -7.81) (*P*: 0.000)
Number of observations	121	121	121	121
***R*-squared**	0.5460	0.8158	0.8712	0.8183

8. Conclusions and Final Remarks

This paper documented some of the algebra and concepts seen in the continuous time affine term structure literature and plugged them into the discrete time approach. Starting point for this paper has been the celebrated papers from Backus, Foresi, Telmer (1996-98). In addition, some of the developments seen on the continuous approach as documented in Piazzesi (2010), Singleton (2006) and Duffie and Kan (1996) have been explored and adapted to the discrete time approach.

This research focused mainly on the multifactor cases of affine term structure models, as the weaknesses seen on the one factor models under Vasicek (1977) and CIR (1985) have been very well documented already in Backus, Foresi and Telmer

(1998). Novelty of this research is that the multifactor affine term structure models under the Vasicek (1977) and the CIR (1985) process were calibrated using observed Interbank and German sovereign yields and European macroeconomic data as well as Greek sovereign yields. For the European and German yield curve, we calibrate macroeconomic data such as Euro-Zone Unemployment rate, Euro-Zone Production Price Index, Euro-Zone monetary aggregates M3 and Euro-Zone Consumer Confidence Index. For the Greek yields curve we use Greece's sovereign budget deficit-to-GDP ratio, Greek unemployment rate and Greek sovereign debt-to-GDP ratio. The results are encouraging and the models fit the observed yields as well as give evidence of a reasonable predictive-ability.

Main findings can be summarised as follows: In the case of the interbank rates and German sovereigns, an increase in unemployment results in a fall in yields on risk free assets and the curve is expected to steepen with front end yields falling faster than the long end. An increase in production prices are expected to result in yield curve flattening, with yields in the front end increasing at a faster pace than the long end. An increase in monetary aggregate M3 is expected to result in yield curve steepening, with yields in the front end falling faster than the long end. Finally, an increase in the consumer confidence index is expected to result in yields flattening, with front end yields increasing faster than the long end. This means that when the economy is booming risk free assets' yields are expected to flatten and when the economy is under recession risk free assets' yields are expected to steepen. From a portfolio management perspective, a representative investor would have incentives to short risk free assets in times when yields are at their steepest levels and set a curve flattening strategy shorting the front end allocating greater weight than to the long end. A more conservative strategy would be to short 2 years versus long the 10 and 20 years onwards, as the profits from the front end are expected to outweigh the losses on the long end.

For the case of Greek government bonds the model shows that if governments engage in counter cyclical fiscal policies when unemployment is high, this will only be possible if these policies do not result in a significant deterioration of the debt-to-GDP ratio. Governments that exhibit a positive correlation of their yields to aggregate consumption growth need to ensure low deficits and debt burdens during booming periods so that they can still have capacity to issue new debt for the rainy days.

From the Greek case this paper shows that a deterioration of government's deficit-to-GDP ratio results in a fall in yields. This is mainly because the increase in spending helps to boom the economy. However, this is more than offset by the deterioration of its debt-to-GDP ratio, thus a deterioration of the latter ratio will more than offset any positive effects stemming from any budget-deficit-induced counter-cyclical policies.

We have learned that debt and deficit ratios can play a role in times of financial distress. However, more research is needed in order to understand what can be done once it's already too late, thus once governments have not done their homework and run unprecedented and unsustainable deficits, thus the question we should try to answer is: what can be done in order to avoid distortionary taxation and aggressive fiscal discipline that lead to more social unrest and further financial markets nervousness?. If economists do no find a solution to this, then we can just rather hand over the job to an accountant.

Acknowledgement

Very special thanks to Prof. Dr. Kaul from Saarland University for his guidance and suggestions.

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