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Pricing variance and volatility swaps: a Monte Carlo simulation technique benchmarked to two closed-form solutions

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Abstract

Our paper introduces an innovative variance reduction technique to improve Monte Carlo (MC) simulation when pricing variance and volatility swaps. This technique previously applied to the pricing of bond options, speeds up the convergence of estimators and improves MC results benchmarked to two closed-form solutions: Demeterfi *et al.* (1999) and Javaheri *et al.* (2004). The variance reduction technique constrains the Wiener process inside upper and lower limits to speed up the convergence towards the 'true' strike values of the swaps obtained with the closed-form solutions. Market participants in needs of accurate numerical methods for pricing variance and volatility swaps will find our methodology appealing and easy to implement.

Keywords:

Variance reduction technique, Importance sampling, Variance swap, Volatility swap, Monte Carlo simulation.

JEL classification:

C15; C63; G13

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Valoración de swaps de varianza y volatilidad: **Una técnica de simulación Monte Carlo referenciada a dos soluciones en forma cerrada**

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Resumen

Este artículo presenta una innovadora técnica de reducción de la varianza para mejorar la simulación Monte Carlo (MC) a la hora de valorar swaps de varianza y volatilidad. Esta técnica, previamente aplicada a la valoración de opciones sobre bonos, acelera la convergencia de los estimadores y mejora los resultados MC cuando se referencia a dos soluciones en forma cerrada: Demeterfi *et al.* (1999) y Javaheri *et al.* (2004). La técnica de reducción de la varianza propuesta limita el proceso de Wiener dentro de las bandas superior e inferior para acelerar la convergencia hacia los 'verdaderos' valores de ejercicio de los swaps obtenidos con las soluciones en forma cerrada. Los participantes en el mercado, necesitados de metodos numéricos exactos para valorar los swaps de varianza y volatilidad de los swaps, encontrarán esta metodología atractiva y fácil de implementar.

Palabras clave:

Técnica de reduccón de la varianza, muestreo por importancia, swap de varianza, swap de volatilidad, simulación Monte Carlo.

1. Introduction

Variance and volatility swaps are over-the-counter derivatives that allow market participants to speculate on or hedge risks associated with the variance (volatility) of an equity index, an interest rate, an exchange rate, or any financial security or market indicator. The swap specifications are: 1) the maturity; 2) the variance (volatility) strike price; 3) the realized variance (volatility) during the life of the swap and 4) the notional amount. The latter is not exchanged between the two parties. At maturity of the swap, one leg will pay the realized variance (volatility) whereas the other leg will pay a fixed amount, the strike price. The difference between the realized variance (volatility) and the strike price, adjusted for the notional amount, provides the net payoff that will be settled in cash. The two parties may pay margins during the life of the swap. At the stage of pricing variance and volatility swaps (i.e. pricing the strike price), our paper proposes a variance reduction technique coupled with Monte Carlo (MC) simulation introduced by Rostan and Rostan (2012) for the pricing of bond options. We use MC simulation to simulate the solution of the stochastic differential equation of the variance presented by Heston (1993). We test separately Euler and Milstein discretizations. We benchmark the MC results to two closed-form solutions:

1. Demeterfi *et al.* (1999b) solution, which replicates a portfolio of options.
2. Javaheri *et al.* (2004) solution, which is derived from the general partial differential equation based on the GARCH(1,1) stochastic volatility process.

Our study is organised as follows: the literature review highlights the evolution of variance and volatility swaps in the literature. The methodology section presents the different models used in this study, their implementation and their inputs and outputs. We wrap up our results and we make final comments in the two last sections.

2. Literature review

As OTC instruments, variance and volatility swaps have been promoted by leading investments banks. Demeterfi *et al.* (1999a) produced quantitative research notes on volatility swaps for Goldman Sachs. Understanding the potential of these instruments, JP Morgan published introductory notes on variance swaps (Bossu *et al.*, 2005).

Since their early phase, the competitors of variance and volatility swaps have been futures and options on volatility. The London based subsidiary of the Swedish exchange, OMLX, launched volatility futures in 1997 followed by the German Swiss Exchange Eurex in 2002. Volumes were disappointing since volatility traders were already using combined static positions in options with dynamic trading in the underlying asset instead of volatility futures (Carr and Madan, 2002). The Chicago-based CME launched futures on VIX[1] in 2004 and options on VIX in 2006. VIX options volumes have increased exponentially since then, whereas VIX futures volumes have been traded modestly. Other exchanges have launched options and futures on volatility but with limited success. Compared to exchange-traded options and futures on volatility index, volatility and variance swaps are traded on the OTC market, therefore they offer flexible notional value and maturity. In addition, the choice of the underlying asset is not limited to a volatility index. The drawbacks of volatility and variance swaps are their lack of liquidity and absence of secondary market, and the counterparty risk. There is no clear estimate of variance and volatility swaps volumes since they belong to the opaque OTC market but Biscamp and Weithers (2007) assessed that 'variance trading has roughly doubled every year for the past few years'.

Regarding the pricing of volatility and variance swaps, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004) models have been two academic references. We describe the models in the methodology section. Other authors proposed closed-form solutions. D'Ippoliti *et al.* (2010) presented 'a stochastic volatility jump-diffusion model for pricing derivatives with jumps in both spot return and volatility underlying dynamics'. This model explicitly obtains the fair delivery price for variance swaps. Goard (2011) proposed a new time-dependent stochastic model for the dynamics of variance in the analytic solutions of variance and volatility swaps. Finally, Song-Ping *et al.* (2011) presented 'a closed-form exact solution for the partial differential equation (PDE) system based on the Heston's two-factor stochastic volatility model' embedded in the framework proposed by Little and Pant (2001). Besides closed-form solutions, several authors proposed numerical solutions or improvements of models. D'Halluin *et al.* (2003) described a computational framework for pricing variance swaps using numerical PDE methods under jump diffusion processes. Elliot *et al.* (2007) emphasized that 'the parameters of Heston's stochastic volatility model depend on the states of a continuous-time observable Markov chain process, which can be interpreted as the states of an observable macroeconomic factor'. With incomplete markets, there is more than one equivalent martingale pricing measure. Elliott et al (2007) used 'a regime switching Esscher transform to determine a martingale pricing measure for the valuation of variance and volatility swaps in this incomplete market'. Finally, Jordan and Tier (2009) considered the problem of pricing the variance swap when the underlying asset follows the Constant Elasticity of Variance[2] process. 'A hedging argument is used to replicate the variance swap in part using the log contract'.

[1] *The VIX Index is an implied volatility index that measures the market's expectation of the 30-day S&P 500 index volatility implied in the prices of near-term S&P 500 index options; introduced in 1993 by CME.*

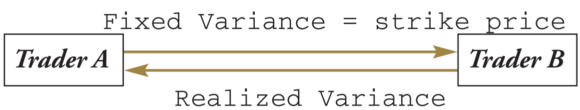
[2] *The CEV model assumes that the volatility of the stock price is no more constant but it is a function of the underlying asset price. Cox (1975) was the pioneer of the Constant Elasticity of Variance (CEV) diffusion model.*

3. Methodology

We price variance and volatility swaps with numerical methods. We apply the plain MC simulation to the Heston equation of variance and we test MC simulation coupled with an innovative variance reduction technique proposed by Rostan and Rostan (2012). We benchmark the results to the Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004) closed-form solutions. The methodology section is organized as follow: sections 3.1 and 3.2 discuss the exchanges of cash-flows between market participants for variance and volatility swaps. Section 3.3 discusses the special behavior of the variance. Section 3.4 presents the models under review; section 3.5 describes the database. In order to understand the intuition behind models, we first discuss the exchanges of cash-flows between market participants for variance and volatility swaps.

3.1 Variance swap

At maturity of the swap, the realized variance (volatility) determines the net amount that one party will pay to the other. In the following example, Trader A hedges its portfolio against an expected increase of the portfolio variance during the life of the variance swap.



Demeterfi *et al.* (1999a) define the variance swap as a forward contract on the annualized variance where Trader A pays a fixed amount K_{var} (strike) and receives the realized annualized variance σ_R^2 at maturity. The gain is $N\left(\sigma_R^2 - K_{var}\right)$ where N is the nominal value of the swap. The measure of σ_R^2 may be defined in discrete or continuous time:

$$\sigma_R^2 = \frac{1}{T-1}\sum_{i=1}^{T}\sigma_i^2 \qquad \text{(Discrete time)}$$

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma(t)dt \qquad \text{(Continuous time)}$$

Market participants quote the notional amount by volatility point, e.g. N = \$500,000 / (volatility point)². K_{var} may be quoted in different forms, for example the square of volatility (15%)².

3.2 Volatility swap

The easiest way to trade volatility is to enter in a volatility swap. The volatility swap (Demeterfi *et al.*, 1999a) is a forward contract on the annualized volatility. The "payoff" at maturity of the contract is $N\left(\sigma_R - K_{vol}\right)$, where σ_R is the annualized realized volatility (of the S&P 500 for example) and the strike price K_{vol} is the fixed annualized volatility. It is quoted such as volatility, for example 15%.

Market participants quote the notional amount N quoted in volatility point, e.g. N = \$500,000 / (volatility point).

3.3 Special behavior of the variance

The variance has two specific features: 1) Similar to interest rates, the variance is mean reverting around its long-term average. The variance at a high level is deemed to decrease, whereas the volatility at a low level is deemed to increase; 2) the variance is often negatively correlated with its underlying asset price. For example, the variance is generally high following a large drop in the asset price. Note that this is referred as the "leverage effect": a negative return increases variance by more than a positive one. A negative return on a stock implies a reduction in equity value (the underlying asset) this implies that the firm becomes more levered (i.e. implying an increase the debt to equity ratio assuming that the debt remain constant) and thus riskier.

With a mean-reverting behavior, our intuition has focused on a bounded distribution of the innovation term ε in the equation of the variance as Rostan and Rostan (2012) did for the interest rate. During MC simulation, ε is drawn from a Normal distribution $N(0,1)$. To speed up the convergence of the simulation, we simply reduce the interval of drawing, for instance $\varepsilon \in [-1,1]$.

3.4 Models under review

The closed-form solutions of Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004) are our two benchmarks. Our study focuses on short-term maturities (1-month, 3-months, 6-months and 1-year). For longer maturities, analytical solutions computed from different models usually converge, as illustrated in the results section with the convergence of the two analytical solutions for 6-month and 1-year maturities. The following sections describe the models under review.

3.4.1 Demeterfi, Derman, Kamal and Zhou (1999) model

The value of a variance swap is typically zero at inception of the swap (T= 0). This means that the expected payoff at maturity T discounted at inception is equal to zero:

$$E\left[e^{-n}\left(\sigma_T^2 - K_{var}\right)\right] = 0 \tag{1}$$

With a strike price K_{var}, a risk-free rate r, on the period $(0,T)$, with $n = rT$. Equation 1 means that pricing a variance swap implies finding the strike price K_{var}. Demeterfi *et al.* (1999b) proposed to price the variance swap by replicating a portfolio of options.

Pricing a variance swap

Demeterfi *et al.* (1999b) use the standard model of Black and Scholes (1973) in order to replicate an asset with a portfolio of options. We present the intuition behind the model by setting the risk free rate to zero. We assume that at time t, we hold a standard option with a strike price K and an expiry date T. The option price computed with the Black and Scholes model is equal to C_{BS}. Define ν as the exposure or sensitivity of the option price to the variance of the underlying asset, such

$$\nu = \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}\,exp\left(-d_1^2/2\right)}{2\sigma\sqrt{2\pi}} \tag{2}$$

where S is the price of the underlying asset, σ^2 is the variance of the underlying asset, $\tau = (T - t)$ is the time to maturity of the option, and d_1 is defined by:

$$d_1 = \frac{ln\left(S/K\right) + \left(\sigma^2 t\right)/2}{\sigma\sqrt{\tau}} \tag{3}$$

In equation 2, Demeterfi *et al.* (1999b) refer to ν as "variance Vega"[3] and proved that a portfolio built with options with weights inversely proportional to K^2, produces a variance Vega virtually independent of the underlying asset price S. This is true as long as S remains within the set of strike prices prevailing on the market and as long as strike prices are distributed evenly and are closed to each other. Let us consider $\prod$ a portfolio of options, including options with different strike prices K, a maturity T and weights inversely proportional to K^2.

The price of a variance swap can be written:

$$K_{var} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S^*}e^{n} - 1\right) - ln\frac{S^*}{S_0} + e^{n}\int_0^{s}\frac{P(K)dK}{K^2} + \int_s^{\infty}\frac{C(K)dK}{K^2}\right] \tag{4}$$

where $P(K)$ and $C(K)$ are the value of puts and calls with a strike price K at $T = 0$. Demeterfi *et al.* (1999b) proposed the following approximation to equation 4 when the strike price K is available for a discrete interval:

$$K_{var} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S^*}e^{n} - 1\right) - ln\frac{S^*}{S_0}\right] + e^{n}\prod\nolimits_{CP} \tag{5}$$

The term $\prod_{CP}$ is equal to the value of the portfolio of options at $T = 0$, with a payoff at time T given by:

$$f(S_T) = \frac{2}{T}\left[\frac{S_T - S^*}{S^*} - ln\frac{S_T}{S^*}\right] \tag{6}$$

[3] *This measure of sensitivity "variance Vega" is almost identical to the well-known measure Vega. The only difference between the definition of Vega and equation 2 is the presence of σ at the denominator of equation 2.*

In order to approximate the «payoff» $f(S_T)$, a series of calls with strike prices $K_0 < K_{1c} < K_{2c} < \ldots$ is needed, as well as a series of puts with strike prices $K_0 > K_{1p} > K_{2p} > \ldots$, where $K_0 = S^*$. The weight of each option contract is given by:

$$\omega_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n,p} - K_{n+1,p}} - \sum_{i=0}^{n-1} \omega_p(K_{i,p})$$

$$\omega_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} \omega_c(K_{i,c}) \qquad (7)$$

With these weights, we can compute the portfolio value $\prod_{CP}$:

$$\prod_{CP} = \sum_n \omega_p(K_{n,p}) P(S, K_{n,p}) + \sum_n \omega_c(K_{n,c}) C(S, K_{n,c}) \qquad (8)$$

By substituting $\prod_{CP}$, r, T, S_0 and S^* in equation 5, we obtain an estimator of the quote K_{var}. This is the method proposed by Demeterfi *et al.* (1999b) in order to compute a quote of variance swap.

Pricing a volatility swap

To price a volatility swap, Demeterfi *et al.* (1999b) propose the following approximation of the payoff of a volatility swap with a maturity T:

$$\sigma_{t,T} - K_{vol} \approx \frac{1}{2K_{vol}} \left(\sigma_{t,T}^2 - K_{vol}^2 \right) \qquad (9)$$

Equation 9 states that volatility swap with a notional value N=1 corresponds to a variance swap with a notional value of $N = \frac{1}{2K_{vol}}$ and a strike price K_{vol}^2. This approximation assumes that the strike price of a volatility swap is $K_{vol} = \sqrt{K_{var}}$. However, Demeteri *et al.* (1999b) assert that this approximation seems only to work when the future realized volatility is closed to K_{vol}. In order to tackle this problem, Javaheri *et al.* (2004) proposed an alternative way to estimate a volatility swap.

3.4.2 Javaheri, Wilmott and Haug (2004) model

Javaheri *et al.* propose a closed-form solution to price volatility and variance swaps strike prices, respectively K_{vol} and K_{var}. This solution is based on the GARCH(1,1) process. Javaheri *et al.* model assumes that the instantaneous variance υ of the underlying asset, in the framework of a continuous GARCH(1,1) process, follows the mean-reverting process:

$$d\upsilon = \kappa(\theta - \upsilon)dt + \gamma \upsilon dZ \qquad (10)$$

where κ: adjustment speed.

θ: The long-term mean average of the variance.

dZ: The Wiener process, defined by $dZ = \varepsilon\sqrt{dt}$, where $\varepsilon \sim N(0,1)$

γ : The volatility of the underlying asset volatility.

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The discrete version of equation 10 is described by Engle and Mezrich (1995) as:

$$\upsilon_{n+1}=(1-\alpha-\beta)V+\alpha u_n^2+\beta\upsilon_n \quad (11)$$

Where V is the long-term variance, u_n^2 the adjusted trend of the underlying asset return at time n,α the associated weight to u_n^2 and β the associated weight to υ_n.

There is an equivalence between the coefficients estimated by the GARCH(1,1) process of the discrete case (11) and those of the continuous case (10) with:

$$\theta=\frac{V}{dt} \quad (12)$$

$$\kappa=\frac{1-\alpha-\beta}{dt} \quad (13)$$

$\gamma=\alpha\sqrt{\frac{\xi-1}{dt}}$, where ξ the Pearson kurtosis of u_n (i.e. the fourth moment of u_n).

Pricing a variance swap

At its inception, the value of the variance swap should be equal to zero, i.e. K_{var} is equal to the risk-neutral expected value of the realized variance during the life of the contract. We define the risk-neutral expected value by the function $F(\upsilon, I, T)$ where I is the variance of the returns during the life of the contract and υ the instantaneous variance at each point of the time axis. We write I as:

$$I=\frac{1}{T}\int_0^T \upsilon(t)dt \quad (14)$$

The function $F(\upsilon, I, T)$ may be solved using the backward Feynman-Kac equation, with the following general form:

$$\frac{\partial}{\partial t}+\frac{1}{2}g(\upsilon)^2\frac{\partial^2}{\partial\upsilon^2}+f(\upsilon)\frac{\partial}{\partial t}+\upsilon\frac{\partial}{\partial I}=0 \quad (15)$$

Thus, for the function $F(\upsilon, I, T)$, the differential equation can be written:

$$\frac{\partial F}{\partial t}+\frac{1}{2}\gamma^2\upsilon^2\frac{\partial^2 F}{\partial\upsilon^2}+\kappa(\theta-\upsilon)\frac{\partial F}{\partial t}+\upsilon\frac{\partial F}{\partial I}=0 \quad (16)$$

with $F(\upsilon, I, T)=I$. Solving the differential equation, we find the expected value of I, equal to K_{var} using the definition of a variance swap:

$$F(\upsilon, I, t)=\theta\left(T-t\frac{e^{-\kappa(T-t)}-1}{\kappa}\right)+\frac{1}{2}\left(1-e^{-\kappa(T-t)}\right)\upsilon+I=K_{var} \quad (17)$$

Pricing a volatility swap

In the Demeterfi *et al.* (1999b) model, we have seen that the variance swap quote could be estimated by $K_{vol}=\sqrt{K_{var}}$. However, as acknowledged by their authors, the approximation seems only to work when the future realized volatility is close to K_{vol}. Javaheri *et al.* (2004) derive a closed-form solution for K_{vol}, using the Brockhaus and Long (2000) approximation. This approximation is a second-order Taylor expansion of the function of the square root of the variable υ around the point $\upsilon_0=E[\upsilon]$.

$E[\sqrt{\upsilon}]$ is approximated by:

$$E[\sqrt{\upsilon}]=\sqrt{E(\upsilon)}-\frac{\text{var}(\upsilon)}{8E[\upsilon]^{3/2}} \quad (18)$$

To estimate K_{vol}, we need to know its expected value $F(\upsilon, I, T)$ but also the variance of I. Using a new function $G(\upsilon, I, T)$, to represent $E[I^2]$, we can define $var(I)$ as:

$$var(I)=E[I^2]-E[I]^2=G-F^2 \quad (19)$$

We only need to determine the form of the function $G(\upsilon,I,T)$. By using the same method than the one for solving $F(\upsilon,I,T)$, Javaheri *et al.* (2004) find an analytical solution to this function and therefore obtains the adjusted quote of the volatility swap by using the Brockhaus and Long (2000) approximation:

$$K_{vol}=\sqrt{F}-\frac{G-F^2}{8(F)^{3/2}} \quad (20)$$

Where $\sqrt{F}$ is the non-adjusted quote of the volatility swap and $\frac{G-F^2}{8(F)^{3/2}}$ the adjustment for convexity.

3.4.3 Monte Carlo Simulation

Pricing a variance swap

The theoretical definition of the realized variance can be given by the following continuous integral:

$$V=\frac{1}{T}\int_0^T \sigma^2(t,\ldots,)dt \quad (21)$$

The pricing of a variance swap quote can be directly found by computing the risk-neutral expected value of equation 21:

$$K_{var}=\frac{1}{T}E\left[\int_0^T \sigma^2(t,\ldots,)dt\right] \quad (22)$$

In order to model the integral of equation 21, we propose to use the differential equation of the Heston (1993) model. Heston states that the processes generating price and variance are defined as:

$$\begin{aligned} dS&=\mu S dt+\sqrt{\upsilon} S dZ_1 \\ d\upsilon&=\kappa(\theta-\upsilon)dt+\sigma\sqrt{\upsilon}dZ_2 \end{aligned} \quad (23)$$

Where S is the price of the underlying asset, υ the instantaneous variance, κ the adjustment speed, θ the long-term variance and σ the volatility of the variance. Z_1 and Z_2 are two correlated Wiener processes. The correlation between dZ_1 and dZ_2 is defined as:

$$corr\ (dZ_1,\ dZ_2)=\rho dt \tag{24}$$

Since we just need to model the variance, only the second equation 23 will be used. Gatheral (2006) presents the Euler and Milstein discretization of the second equation 23. We test both discretizations in our paper.

The Euler discretization is:

$$\upsilon_{i+1}=\upsilon_i-\kappa\ (\upsilon_i-\theta)dt+\sigma\sqrt{\upsilon_i}\sqrt{\Delta t}Z \tag{25}$$

where $Z \sim N(0,1)$. In equation 25, the variance follows a Cox-Ingersoll-Ross (CIR, 1985) process, which is an arithmetic Brownian motion where negative variances can occur. As Gatheral (2006) explains, in a CIR process, depending upon the initial variance and the parameters, i.e. long-term mean and speed of mean reversion, the probability of realizing negative values is a lot lower than, for example, a Vasicek (1977) process. In practice, the reflective hypothesis is used to avoid the problem. Thus, the absolute value of υ_{i+1} is computed before going to the next iteration. However, this type of discretization requires very short intervals to insure the convergence of the simulation. The Milstein discretization can solve the problem of weak convergence and negative variance generated by the Euler discretization. Although more computationally-demanding, the Milstein discretization reduces considerably the negative variances generated by the algorithm. By using Milstein discretization, we can write the differential equation of the variance:

$$\upsilon_{i+1}=(\sqrt{\upsilon_i}+\frac{\sigma}{2}\sqrt{\Delta t}Z)-\kappa(\upsilon_i-\theta)\Delta t-\frac{\sigma}{4}\Delta t \tag{26}$$

Therefore, we obtain a quote of a variance swap by computing the integral of equation 22 after simulating the instantaneous variance (equations 25 or 26).

Equation 22 is the expected value of integrals of instantaneous variances on the time interval $[0,T]$. Therefore, to estimate the integral $\frac{1}{T}\int_0^T \sigma^2(t,\ldots,)dt$, we need to simulate the instantaneous variance over the time interval $[0,T]$ for each trajectory and compute the average of the simulated instantaneous variances of all trajectories. This computation is similar to the payoff of an Asian option.

Pricing a volatility swap

Javaheri *et al.* (2004) state that the quote of volatility swaps can be approximated by using the Brockhaus and Long (2000) approximation:

$$K_{vol} = \sqrt{F} - \frac{G - F^2}{8(F)^{3/2}} \tag{27}$$

where F represents the expected value of the realized variance, which is obtained with a Monte Carlo simulation. To estimate G, the expected value of the realized variance square, we can use the following equivalence:

$$G = var(I) + F^2 \tag{28}$$

where I is the realized variance. Here, $var(I)$ can be easily computed by taking the variance of the Monte Carlo estimate of the quote K_{var}. Thus, we are able to compute F and G; therefore, we can find the estimate for the quote K_{vol} using the Monte Carlo simulation.

Improving Monte Carlo simulation with an innovative variance reduction technique

The proposed variance reduction technique (Rostan and Rostan, 2012) is based on a bounded distribution of the innovation term ε. During MC simulation, ε is drawn from a Normal distribution $N(0,1)$. To speed up the convergence of the simulation, the interval of drawing is simply reduced, for instance with $\varepsilon \in [-1,1]$. This is "Importance Sampling" since the distribution is constrained inside upper and lower limits.

To implement the variance reduction, the following lines are added to the simulation algorithm in Matlab, for example with $\varepsilon \in [-0.5,+0.5]$:

```
epsilon=norminv((normcdf(0.5)-(1- normcdf(0.5)))*rand+(1- normcdf(0.5)));
```

The corresponding VBA algorithm with $\varepsilon \in [-0.5,+0.5]$ is:

```
epsilon = Application.NormSInv((Application.NormSDist(0.5) - _
(1 - Application.NormSDist(0.5))) * Rnd + (1 - Application.NormSDist(0.5)))
```

3.4.4 Building the database

The dataset is composed of market observations at their close during May 2011 of: 1) S&P 500 index values and 2) S&P 500 index Mid-price options traded on the CBOE. We derive the variance and volatility swaps from these values for four different maturities, 1-month, 3-months, 6-months and 1-year.

4. Results

Considering the four swap maturities under review, 1-month, 3-months, 6-months and 1-year (refer to Figures 1 to 4), the two analytical methods return values of variance swap strikes ranging from 0.43% to 4.7%. Demeterfi *et al.* (1999b) model, most of the time, returns higher values than Javaheri *et al.* (2004) model.

We observe that the values of the strikes of longer-term swaps (6-month and 1-year) priced with the two analytical solutions of Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004) converge. In the appendix, Tables 3 to 6 compile the values of the variance and volatility swaps and Figures 5 to 8 plot the values.

Figure 1. Pricing 1-month variance swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri et al. (2004), and MC simulation. Sample period: May 2011.

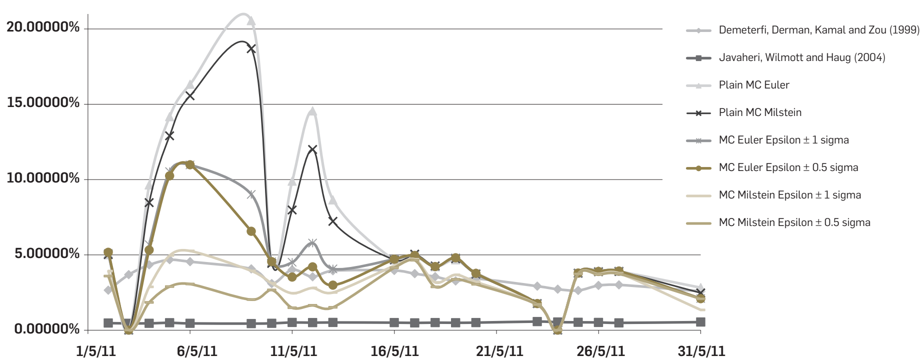
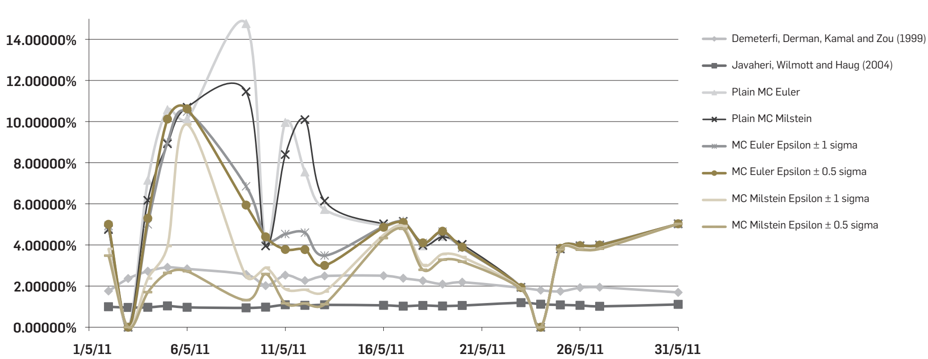


Figure 2. Pricing 3-month variance swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011



For each maturity, from 1 month to 1 year, we observe that closed-form solutions do not vary much over the one-month sample period. Strike values obtained with MC simulations are far more volatile than analytical values. As expected, MC simulations coupled with our new variance reduction technique display less volatile results than Plain MC over the period. In addition, the variance reduction technique, most of the time, returns values closer to analytical values (our benchmarks) than values obtained with Plain MC.

Figure 3. Pricing 6-month variance swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011

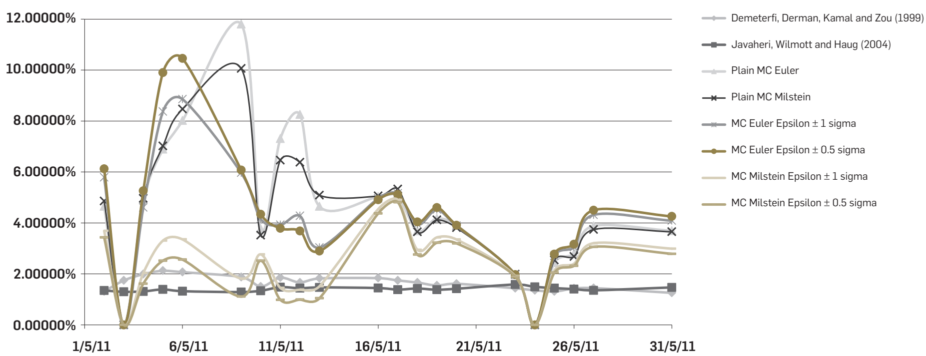
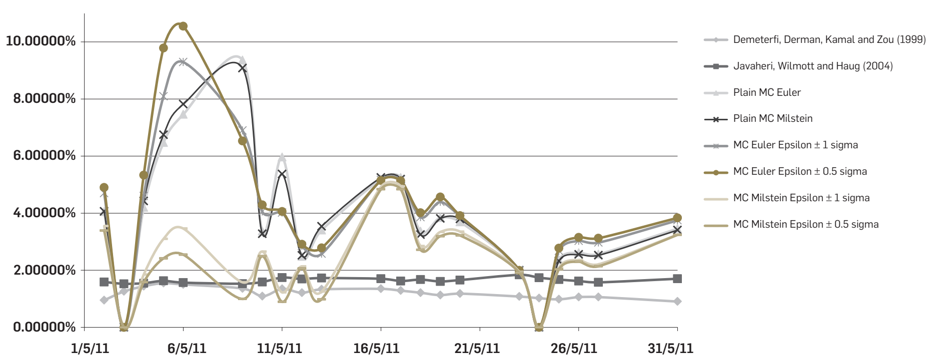


Figure 4. Pricing 1-year variance swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011



In order to assess the efficiency of the variance reduction technique, we apply the Root Mean Square Error (RMSE) criteria. We compute RMSE by differentiating the midpoint of the strikes obtained with the two analytical solutions and the swap strike obtained with MC simulation, using equation 29:

$$RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\text{Midpoint strike}_{analytical\ solutions} - \text{Swap strike}_{numerical\ solution})^2} \qquad (29)$$

Referring to Tables 3 to 6 in the appendix, RMSE values are lower when obtained with the variance reduction technique than with Plain MC, except with Euler discretization for 1-year variance and volatility swaps. Euler discretization is improved for all other maturities (1-, 3-, 6-month) especially when the variance reduction technique uses epsilon from the interval [-1,+1]. For epsilon between [-0.5,+0.5], the technique works less efficiently.

Milstein discretization coupled with the variance reduction technique provides the best results: RMSE values are the lowest regardless of the swap maturity, and we observe that the smaller the interval of epsilon (from ±1 to ±0.5), the more accurate the swap value. Based on the RMSE criteria, on average over the four maturities, the variance reduction technique coupled with Milstein discretization increases the convergence of a Plain MC (decreases RMSE) by 56% using epsilon between [-1,+1], by 63% using epsilon between [-0.5,+0.5] for variance swaps and increases, for volatility swaps, the convergence by 40% using epsilon between [-1,+1] and by 45% using epsilon between [-0.5,+0.5].

However, we would like to stress the challenging calibration process during the MC simulation: it takes between 4 to 10 minutes to calibrate the Heston equation for each maturity and each day of the sample period, using either the methods of $RMSE, %RMSE or IVRMSE presented by Rouah and Vainberg (2007) and described in the Appendix section below. The 3- and 6-month calibrations do not converge all the time or swap values resulting from calibration are way higher than expected (e.g. swap value of 9% instead of 1%). When the calibration method does not converge or when we obtain abnormal parameters of the equation, we use the equation of another maturity (e.g. 1-month equation for the pricing of a 3-month swap). In addition, for two days of our sample, it is impossible to calibrate the Heston (1993) equation with the three calibration methods for all swap maturities (1-month to 1-year). We conclude that the calibration process is a long and challenging process and that it requires judgment of the operator (e.g. using one maturity for another).

5. Conclusion

We test a numerical solution based on the MC simulation of the Heston (1993) model coupled with an innovative variance reduction technique (Rostan and Rostan, 2012) that we benchmark to two closed-form solutions: Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004). We show that, for the four maturities under review, i.e. 1-month, 3-month, 6-month and 1-year, the MC simulation using Milstein discretization coupled with the variance reduction technique always performs better than Plain MC simulation for both variance and volatility swaps. The innovative technique is based on a bounded distribution of the innovation term ε drawn from the $N(0,1)$. This technique speeds up the convergence of the simulation and improves swap values since these values are always closer to closed-form solutions than values obtained with Plain MC simulation. In addition, it works best when limits are placed at plus or minus 0.5 standard deviations with Milstein discretization. Euler discretization offers mixed results when coupled with the variance reduction technique. One drawback of the MC simulation method applied to Heston (1993) equation is the calibration process, which is long to run and it does not converge all the time.

Song-Ping *et al.* (2011) have recently presented a closed-form solution to price variance and volatility swaps. Further work may be to compare their solution to our numerical method. Other studies on the topic of pricing variance and volatility swaps could compare the innovative technique to other variance reduction techniques such as control variate, conditioning, stratified sampling, splitting, quasi-MC or could integrate our new technique to Least-Squares Monte Carlo Method developed by Longstaff and Schwartz (2001).

Finally, we could test other distributions than Gaussian in the pricing of variance and volatility swaps, such as Student's t distribution. In that respect, Hou and Suardi (2011) used the Student's t distribution for interest rate innovation and claim that 'it is consistent with the widely observed non-normal short-term interest rate distribution'. It would be interesting to test if short-term interest rate distribution shares some similarities with variance distribution.

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8. Appendices

8.1 Technical notes for implementing Demeterfi *et al.* (1999b) model

Among inputs of the model, the frontier price S^* corresponds to the price such as calls used in the model have strike prices lower than S^*: $K_0 < K_{1c} < K_{2c} < \ldots < S^*$ and puts have strike prices higher than S^*: $S^* > K_{1p} > K_{2p} > \ldots$. In our paper, we use $S = S^*$, to mimic the original authors. The cost of the portfolio is equal to the cost in USD to build the replication portfolio Π_{CP}.

8.2 Technical notes for implementing Javaheri *et al.* (2004) model

Similar to Demeterfi *et al.* (1999b) model, the Javaheri *et al.* (2004) model needs initial parameters. The parameters I, α, β, ω, *Kurtosis, dt* are inputs of the model. In order to compute inputs, we use a 1-year daily time series of the S&P 500 index prices, mid quotes at the day close. We maximize the log-likelihood function of the GARCH(1,1) using the MLE method (Maximum Likelihood Estimator). We use the Nelder-Mead optimization algorithm that several commercial softwares include in their package such as Matlab. The VBA code is provided by Rouah and Vainberg (2007).

8.3 Technical notes for implementing MC Simulation

The implementation of MC simulation is simple. We simulate the stochastic differential equation of Heston (1993) represented by the second equation 23. We either discretize equation 23 with Euler (equation 25) or Milstein (equation 26) methods.

In our paper, we have decided to simulate 1,000 times for each maturity and method. We need to estimate the parameters of equation 23: κ the adjustment speed, θ the long-term variance and σ the volatility of the variance. We use the RMSE (Root Mean Square Error) method of calibration. In fact, this method has three approaches that converge: $RMSE, %RMSE and IVRMSE.

The estimated parameters of the RMSE method minimize the following RMSE function:

$$RMSE(\theta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} e_i(\theta)^2} \tag{C.1}$$

with θ the parameters to estimate and e the error term, i.e. the difference between market data and estimated data provided by the model. When we use $RMSE, e_i is the difference between the market option prices and the prices computed by the analytical solution of Heston (1993). For further details on RMSE methods, refer to Rouah and Vainberg (2007). Concerning %RMSE, e_i is e_i of $RMSE divided by the market price. Finally, for IVRMSE, e_i is the difference between the implied volatility obtained from market data and the implied volatility obtained from Heston (1993) option pricing model by the bissection method. In order to minimize the RMSE functions, we use the Nelder-Mead algorithm.

The parameters estimation requires initial parameters. We use the ones proposed by Gatheral (2006) when he replicates the volatility surface of the S&P 500 index with the Heston model:

Table 1. Initial parameters proposed by Gatheral (2006)

Rho (ρ):	-0.7165
Kappa (κ):	1.3253
Theta (θ):	0.0354
Volatility of variance (σ):	0.3877
Instantaneous variance (VO):	0.0174
Lambda (λ):	

In order to calibrate the model using the RMSE method, we need to use Put or Call with a given strike price, market price and implied volatility and a maturity identical to the swap that we price. We use a similar interpolation method than in Demeterfi *et al.* (1999b) to approximate market data when the swap maturity does not match an option maturity.

After minimization of the loss function, Table 2 displays the parameter estimates obtained on May 30, 2011 during the 3-month swaps valuation process:

Table 2. Parameters estimates of Heston (1993) equation on May. 30, 2011 during the 3-month swaps valuation process

	%RMSE
Loss function value:	0.071505
Rho (ρ):	0.274658
Kappa (κ):	2.332364
Theta (θ):	0.057711
Volatility of variance (σ):	0.012205
Instantaneous variance (VO):	0.038065
Lambda (λ):	0.000000

It takes between 3 and 10 minutes to calibrate the model using either RMSE method. A thousand simulations with plain MC run in 3 seconds; a thousand simulations with the innovative variance reduction technique with $\varepsilon \in [-0.5,+0.5]$ run in 29 seconds with an Intel Core 2 Duo CPU, E8400@3.00GHZ, 1.94 GB of RAM.

Table 3. **Pricing 1- and 3-month variance swaps using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.**

Variance Swap maturity: 1-month	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	2.6%	0.5%	1.6%	2.8%	2.5%	2.1%	2.1%	2.1%	1.3%
27/05/2011	3.0%	0.5%	1.7%	3.9%	3.9%	3.9%	3.9%	3.8%	3.7%
26/05/2011	3.0%	0.5%	1.7%	4.0%	3.9%	3.9%	3.9%	3.7%	3.7%
25/05/2011	2.6%	0.5%	1.6%	3.8%	3.8%	3.8%	3.8%	3.7%	3.7%
24/05/2011	2.7%	0.5%	1.6%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	2.9%	0.6%	1.7%	1.8%	1.8%	1.8%	1.8%	1.7%	1.7%
20/05/2011	3.4%	0.5%	2.0%	3.8%	3.7%	3.8%	3.8%	3.2%	3.0%
19/05/2011	3.3%	0.5%	1.9%	4.7%	4.7%	4.8%	4.8%	3.7%	3.4%
18/05/2011	3.5%	0.5%	2.0%	4.2%	4.3%	4.2%	4.2%	3.2%	2.9%
17/05/2011	3.7%	0.5%	2.1%	4.9%	5.0%	5.0%	5.0%	4.7%	4.6%
16/05/2011	4.0%	0.5%	2.2%	4.6%	4.7%	4.7%	4.7%	4.3%	4.1%
13/05/2011	3.9%	0.5%	2.2%	8.6%	7.2%	4.1%	3.0%	2.5%	1.5%
12/05/2011	3.5%	0.5%	2.0%	14.6%	12.0%	5.8%	4.2%	2.8%	1.6%
11/05/2011	4.0%	0.5%	2.3%	9.9%	8.0%	4.5%	3.5%	2.4%	1.5%
10/05/2011	3.1%	0.5%	1.8%	4.5%	4.4%	4.5%	4.5%	3.1%	2.7%
9/05/2011	4.1%	0.4%	2.2%	20.5%	18.7%	9.0%	6.6%	3.9%	2.0%
6/05/2011	4.5%	0.4%	2.5%	16.3%	15.6%	11.0%	11.0%	5.3%	3.1%
5/05/2011	4.7%	0.5%	2.6%	14.2%	12.9%	10.5%	10.2%	5.0%	2.9%
4/05/2011	4.3%	0.5%	2.4%	9.6%	8.5%	5.7%	5.3%	2.9%	1.8%
3/05/2011	3.7%	0.4%	2.1%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	2.7%	0.5%	1.6%	5.0%	5.0%	5.1%	5.2%	3.9%	3.6%
			RMSE:	7.0%	6.2%	3.7%	3.4%	1.7%	1.4%
Variance Swap maturity: 3-month	**Demeterfi, Derman, Kamal and Zou (1999b)**	**Javaheri, Wilmott and Haug (2004)**	**Mid-Point**	**Plain MC Euler**	**Plain MC Milstein**	**MC Euler Epsilon ± 1 sigma**	**MC Euler Epsilon ± 0.5 sigma**	**MC Milstein Epsilon ± 1 sigma**	**MC Milstein 0.5 sigma**
31/05/2011	1.7%	1.1%	1.4%	5.0%	5.0%	5.0%	5.0%	5.0%	5.0%
27/05/2011	1.9%	1.0%	1.5%	4.0%	4.0%	4.0%	4.0%	3.8%	3.8%
26/05/2011	1.9%	1.1%	1.5%	4.0%	4.0%	4.0%	4.0%	3.8%	3.8%
25/05/2011	1.7%	1.1%	1.4%	3.8%	3.8%	3.8%	3.8%	3.8%	3.8%
24/05/2011	1.8%	1.1%	1.5%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	1.9%	1.2%	1.6%	2.0%	1.9%	1.9%	1.9%	1.9%	1.8%
20/05/2011	2.2%	1.1%	1.6%	3.9%	4.0%	3.9%	3.9%	3.4%	3.2%
19/05/2011	2.1%	1.0%	1.6%	4.4%	4.4%	4.6%	4.7%	3.6%	3.3%
18/05/2011	2.3%	1.1%	1.7%	4.0%	4.0%	4.1%	4.1%	3.0%	2.8%

17/05/2011	2.4%	1.0%	1.7%	5.2%	5.1%	5.1%	5.1%	4.9%	4.8%
16/05/2011	2.5%	1.1%	1.8%	5.0%	5.0%	4.9%	4.9%	4.5%	4.3%
13/05/2011	2.5%	1.1%	1.8%	5.7%	6.1%	3.5%	3.0%	1.7%	1.1%
12/05/2011	2.3%	1.1%	1.7%	7.5%	10.1%	4.6%	3.8%	1.8%	1.2%
11/05/2011	2.5%	1.1%	1.8%	10.0%	8.4%	4.5%	3.8%	1.9%	1.2%
10/05/2011	2.0%	1.0%	1.5%	4.1%	4.0%	4.3%	4.4%	2.9%	2.6%
9/05/2011	2.6%	0.9%	1.8%	14.8%	11.5%	6.9%	5.9%	2.5%	1.3%
6/05/2011	2.8%	1.0%	1.9%	10.1%	10.7%	10.5%	10.6%	9.9%	2.7%
5/05/2011	2.9%	1.0%	2.0%	10.6%	8.9%	9.0%	10.1%	3.9%	2.6%
4/05/2011	2.7%	1.0%	1.8%	7.1%	6.2%	5.0%	5.3%	2.4%	1.7%
3/05/2011	2.4%	1.0%	1.7%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	1.8%	1.0%	1.4%	4.8%	4.7%	4.9%	5.0%	3.8%	3.5%
			RMSE:	5.1%	4.7%	3.6%	3.7%	2.6%	1.8%

Table 4. Pricing 6-month and 1-year variance swaps using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.

Variance Swap maturity: 6-month	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	1.3%	1.5%	1.4%	3.7%	3.7%	4.1%	4.3%	3.0%	2.8%
27/05/2011	1.4%	1.4%	1.4%	3.9%	3.7%	4.3%	4.5%	3.2%	3.1%
26/05/2011	1.4%	1.4%	1.4%	2.8%	2.7%	3.0%	3.2%	2.4%	2.3%
25/05/2011	1.3%	1.4%	1.4%	2.4%	2.5%	2.7%	2.8%	2.1%	2.0%
24/05/2011	1.4%	1.5%	1.4%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	1.4%	1.6%	1.5%	2.0%	2.0%	2.0%	2.0%	1.9%	1.9%
20/05/2011	1.6%	1.4%	1.5%	3.9%	3.8%	3.9%	3.9%	3.3%	3.2%
19/05/2011	1.5%	1.4%	1.5%	4.2%	4.1%	4.5%	4.6%	3.4%	3.2%
18/05/2011	1.7%	1.4%	1.5%	3.7%	3.6%	3.9%	4.0%	2.9%	2.8%
17/05/2011	1.7%	1.4%	1.6%	5.1%	5.3%	5.2%	5.1%	4.9%	4.8%
16/05/2011	1.8%	1.4%	1.6%	5.0%	5.1%	5.0%	4.9%	4.5%	4.4%
13/05/2011	1.8%	1.5%	1.6%	4.6%	5.1%	3.0%	2.9%	1.5%	1.0%
12/05/2011	1.7%	1.4%	1.6%	8.3%	6.4%	4.3%	3.7%	1.4%	1.0%
11/05/2011	1.9%	1.5%	1.7%	7.3%	6.5%	3.9%	3.8%	1.5%	1.0%
10/05/2011	1.5%	1.3%	1.4%	3.8%	3.5%	4.2%	4.3%	2.8%	2.5%
9/05/2011	1.9%	1.3%	1.6%	11.8%	10.1%	6.0%	6.1%	1.8%	1.1%
6/05/2011	2.1%	1.3%	1.7%	8.0%	8.5%	8.9%	10.5%	3.4%	2.6%
5/05/2011	2.1%	1.4%	1.8%	6.9%	7.0%	8.4%	9.9%	3.3%	2.5%
4/05/2011	2.0%	1.3%	1.7%	4.9%	5.0%	4.6%	5.3%	2.0%	1.6%
3/05/2011	1.7%	1.3%	1.5%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	1.3%	1.3%	1.3%	4.6%	4.9%	5.8%	6.1%	3.7%	3.4%
			RMSE:	4.0%	3.7%	3.3%	3.7%	1.6%	1.5%

Variance Swap maturity: 1-year	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	0.9%	1.7%	1.3%	3.5%	3.4%	3.7%	3.8%	3.2%	3.2%
27/05/2011	1.1%	1.6%	1.3%	2.6%	2.5%	3.0%	3.1%	2.2%	2.1%
26/05/2011	1.1%	1.6%	1.3%	2.6%	2.6%	3.0%	3.2%	2.4%	2.3%
25/05/2011	1.0%	1.7%	1.3%	2.4%	2.3%	2.7%	2.8%	2.1%	2.0%
24/05/2011	1.0%	1.7%	1.4%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	1.1%	1.8%	1.5%	2.0%	2.0%	2.0%	2.0%	1.9%	1.9%
20/05/2011	1.2%	1.7%	1.4%	3.7%	3.8%	3.9%	3.9%	3.4%	3.2%
19/05/2011	1.1%	1.6%	1.4%	3.8%	3.8%	4.4%	4.6%	3.3%	3.2%
18/05/2011	1.2%	1.7%	1.4%	3.4%	3.2%	3.9%	4.0%	2.8%	2.7%
17/05/2011	1.3%	1.6%	1.5%	5.2%	5.2%	5.2%	5.1%	4.9%	4.8%
16/05/2011	1.3%	1.7%	1.5%	5.2%	5.2%	5.2%	5.1%	4.9%	4.8%
13/05/2011	1.3%	1.7%	1.5%	3.4%	3.5%	2.6%	2.8%	1.3%	1.0%
12/05/2011	1.2%	1.7%	1.5%	2.5%	2.5%	2.8%	2.9%	2.1%	2.1%
11/05/2011	1.3%	1.7%	1.5%	6.0%	5.4%	4.0%	4.1%	1.2%	0.9%
10/05/2011	1.1%	1.6%	1.3%	3.4%	3.3%	4.1%	4.3%	2.7%	2.5%
9/05/2011	1.4%	1.5%	1.4%	9.4%	9.1%	6.9%	6.5%	1.6%	1.0%
6/05/2011	1.5%	1.6%	1.5%	7.5%	7.8%	9.3%	10.5%	3.5%	2.5%
5/05/2011	1.5%	1.6%	1.6%	6.5%	6.7%	8.1%	9.8%	3.1%	2.4%
4/05/2011	1.4%	1.5%	1.5%	4.2%	4.4%	4.6%	5.3%	1.8%	1.6%
3/05/2011	1.3%	1.5%	1.4%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	1.0%	1.6%	1.3%	4.0%	4.1%	4.7%	4.9%	3.5%	3.4%
			RMSE:	3.2%	3.2%	3.3%	3.7%	1.7%	1.5%

● **Table 5. Pricing 1- and 3-month volatility swaps using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.**

Volatility Swap maturity: 1-month	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	16.1%	7.3%	11.7%	16.8%	15.7%	14.5%	14.4%	14.4%	11.6%
27/05/2011	17.3%	6.9%	12.1%	19.7%	19.7%	19.8%	19.8%	19.4%	19.3%
26/05/2011	17.2%	7.1%	12.2%	19.9%	19.7%	19.7%	19.7%	19.3%	19.2%
25/05/2011	16.2%	7.2%	11.7%	19.4%	19.4%	19.5%	19.5%	19.3%	19.3%
24/05/2011	16.5%	7.3%	11.9%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	17.1%	7.5%	12.3%	13.3%	13.3%	13.3%	13.3%	13.0%	13.0%
20/05/2011	18.5%	7.1%	12.8%	19.5%	19.3%	19.4%	19.4%	17.9%	17.4%
19/05/2011	18.1%	7.0%	12.5%	21.6%	21.8%	21.9%	21.9%	19.2%	18.4%
18/05/2011	18.8%	7.1%	12.9%	20.5%	20.6%	20.5%	20.5%	17.8%	16.9%
17/05/2011	19.4%	6.9%	13.1%	22.1%	22.4%	22.3%	22.2%	21.7%	21.5%
16/05/2011	19.9%	7.0%	13.5%	21.6%	21.6%	21.7%	21.7%	20.7%	20.4%
13/05/2011	19.9%	7.1%	13.5%	29.4%	26.9%	20.2%	17.3%	15.8%	12.3%

12/05/2011	18.8%	7.0%	12.9%	38.1%	34.6%	24.0%	20.5%	16.7%	12.8%
11/05/2011	20.0%	7.1%	13.6%	31.5%	28.2%	21.2%	18.8%	15.6%	12.2%
10/05/2011	17.6%	6.7%	12.1%	21.2%	21.0%	21.3%	21.3%	17.6%	16.3%
9/05/2011	20.2%	6.6%	13.4%	45.3%	43.2%	30.0%	25.6%	19.8%	14.2%
6/05/2011	21.3%	6.7%	14.0%	40.4%	39.4%	33.1%	33.1%	22.9%	17.5%
5/05/2011	21.6%	7.0%	14.3%	37.6%	35.9%	32.4%	32.0%	22.3%	17.0%
4/05/2011	20.8%	6.7%	13.8%	31.0%	29.1%	23.8%	23.1%	16.9%	13.6%
3/05/2011	19.2%	6.7%	12.9%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	16.3%	6.8%	11.5%	22.4%	22.4%	22.6%	22.7%	19.8%	18.9%
			RMSE:	15.1%	14.0%	10.6%	9.9%	6.9%	6.0%

Volatility Swap maturity: 3-month	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	13.0%	10.5%	11.8%	22.4%	22.4%	22.4%	22.4%	22.4%	22.4%
27/05/2011	13.9%	10.1%	12.0%	19.9%	20.0%	20.0%	20.0%	19.6%	19.5%
26/05/2011	13.9%	10.3%	12.1%	20.0%	19.9%	19.9%	19.9%	19.5%	19.4%
25/05/2011	13.2%	10.4%	11.8%	19.6%	19.5%	19.6%	19.6%	19.5%	19.5%
24/05/2011	13.4%	10.6%	12.0%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	13.9%	10.9%	12.4%	14.0%	13.9%	13.9%	13.9%	13.7%	13.6%
20/05/2011	14.8%	10.3%	12.5%	19.8%	20.1%	19.6%	19.7%	18.5%	17.9%
19/05/2011	14.5%	10.1%	12.3%	21.0%	21.0%	21.5%	21.6%	18.9%	18.1%
18/05/2011	15.0%	10.3%	12.7%	19.9%	20.0%	20.2%	20.2%	17.4%	16.7%
17/05/2011	15.4%	10.1%	12.8%	22.7%	22.7%	22.6%	22.6%	22.2%	21.9%
16/05/2011	15.8%	10.3%	13.1%	22.3%	22.4%	22.1%	22.0%	21.2%	20.8%
13/05/2011	15.8%	10.4%	13.1%	23.9%	24.8%	18.7%	17.3%	13.2%	10.6%
12/05/2011	15.0%	10.3%	12.7%	27.5%	31.8%	21.4%	19.4%	13.5%	10.8%
11/05/2011	15.9%	10.4%	13.2%	31.6%	29.0%	21.3%	19.4%	13.6%	10.8%
10/05/2011	14.2%	9.9%	12.0%	20.4%	19.9%	20.8%	21.0%	17.0%	16.1%
9/05/2011	16.0%	9.7%	12.8%	38.4%	33.8%	26.2%	24.4%	15.7%	11.5%
6/05/2011	16.8%	9.8%	13.3%	31.8%	32.7%	32.4%	32.6%	31.4%	16.5%
5/05/2011	17.1%	10.1%	13.6%	32.5%	29.9%	30.0%	31.8%	19.9%	16.2%
4/05/2011	16.5%	9.8%	13.1%	26.7%	24.8%	22.4%	23.0%	15.4%	13.0%
3/05/2011	15.4%	9.8%	12.6%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	13.3%	9.9%	11.6%	22.0%	21.8%	22.2%	22.4%	19.5%	18.6%
			RMSE:	12.6%	12.2%	10.3%	10.3%	7.9%	6.6%

Table 6. Pricing 6-month and 1-year volatility swaps using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.

Volatility Swap maturity: 6-month	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	11.2%	12.1%	11.7%	19.2%	19.1%	20.2%	20.6%	17.3%	16.7%
27/05/2011	12.0%	11.6%	11.8%	19.7%	19.3%	20.8%	21.2%	17.9%	17.5%
26/05/2011	12.0%	11.8%	11.9%	16.7%	16.4%	17.5%	17.8%	15.5%	15.2%

25/05/2011	11.4%	12.0%	11.7%	15.6%	15.9%	16.4%	16.7%	14.6%	14.3%
24/05/2011	11.6%	12.2%	11.9%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	12.0%	12.6%	12.3%	14.0%	14.1%	14.1%	14.0%	13.8%	13.8%
20/05/2011	12.7%	11.9%	12.3%	19.8%	19.5%	19.7%	19.8%	18.3%	17.9%
19/05/2011	12.4%	11.7%	12.1%	20.4%	20.3%	21.3%	21.5%	18.5%	18.0%
18/05/2011	12.9%	11.9%	12.4%	19.2%	19.1%	19.9%	20.1%	17.2%	16.6%
17/05/2011	13.2%	11.7%	12.5%	22.5%	23.1%	22.7%	22.6%	22.1%	21.9%
16/05/2011	13.5%	12.0%	12.8%	22.4%	22.5%	22.3%	22.2%	21.3%	20.9%
13/05/2011	13.5%	12.1%	12.8%	21.6%	22.6%	17.4%	17.0%	12.3%	10.2%
12/05/2011	12.9%	12.0%	12.4%	28.7%	25.2%	20.7%	19.2%	11.7%	10.0%
11/05/2011	13.6%	12.1%	12.9%	27.0%	25.4%	19.8%	19.5%	12.1%	9.9%
10/05/2011	12.2%	11.5%	11.9%	19.4%	18.8%	20.4%	20.8%	16.6%	15.9%
9/05/2011	13.7%	11.3%	12.5%	34.3%	31.7%	24.4%	24.7%	13.5%	10.5%
6/05/2011	14.4%	11.5%	12.9%	28.3%	29.1%	29.8%	32.3%	18.3%	16.0%
5/05/2011	14.6%	11.8%	13.2%	26.2%	26.5%	28.9%	31.5%	18.2%	15.9%
4/05/2011	14.1%	11.4%	12.8%	22.1%	22.3%	21.5%	22.9%	14.3%	12.7%
3/05/2011	13.2%	11.4%	12.3%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	11.5%	11.6%	11.5%	21.6%	22.0%	24.0%	24.8%	19.2%	18.5%
			RMSE:	10.9%	10.4%	9.8%	10.3%	6.1%	5.8%

Volatility Swap maturity: 1-year	Demeterfi, Derman, Kamal and Zou (1999b)	Javaheri, Wilmott and Haug (2004)	Mid-Point	Plain MC Euler	Plain MC Milstein	MC Euler Epsilon ± 1 sigma	MC Euler Epsilon ± 0.5 sigma	MC Milstein Epsilon ± 1 sigma	MC Milstein 0.5 sigma
31/05/2011	9.5%	13.0%	11.3%	18.6%	18.5%	19.4%	19.6%	18.0%	18.0%
27/05/2011	10.3%	12.5%	11.4%	16.0%	15.9%	17.2%	17.7%	14.9%	14.6%
26/05/2011	10.3%	12.7%	11.5%	16.2%	16.0%	17.4%	17.8%	15.4%	15.2%
25/05/2011	9.9%	12.9%	11.4%	15.4%	15.2%	16.4%	16.7%	14.5%	14.3%
24/05/2011	10.1%	13.2%	11.6%	N/A	N/A	N/A	N/A	N/A	N/A
23/05/2011	10.4%	13.5%	12.0%	14.1%	14.1%	14.1%	14.1%	13.9%	13.8%
20/05/2011	10.9%	12.9%	11.9%	19.2%	19.5%	19.7%	19.8%	18.3%	17.9%
19/05/2011	10.6%	12.7%	11.6%	19.6%	19.5%	21.0%	21.4%	18.3%	17.9%
18/05/2011	11.0%	12.9%	12.0%	18.3%	18.0%	19.6%	20.0%	16.9%	16.5%
17/05/2011	11.3%	12.7%	12.0%	22.9%	22.8%	22.7%	22.7%	22.2%	22.0%
16/05/2011	11.6%	13.0%	12.3%	22.8%	22.9%	22.7%	22.7%	22.2%	22.0%
13/05/2011	11.5%	13.1%	12.3%	18.5%	18.8%	16.0%	16.7%	11.2%	9.9%
12/05/2011	11.0%	13.0%	12.0%	15.7%	15.9%	16.9%	17.1%	14.6%	14.4%
11/05/2011	11.6%	13.2%	12.4%	24.4%	23.2%	20.0%	20.1%	11.1%	9.5%
10/05/2011	10.4%	12.6%	11.5%	18.4%	18.1%	20.1%	20.7%	16.3%	15.8%
9/05/2011	11.7%	12.3%	12.0%	30.6%	30.1%	26.3%	25.6%	12.6%	10.0%
6/05/2011	12.3%	12.5%	12.4%	27.3%	28.0%	30.5%	32.5%	18.6%	16.0%
5/05/2011	12.4%	12.7%	12.6%	25.4%	26.0%	28.4%	31.3%	17.6%	15.5%
4/05/2011	12.0%	12.4%	12.2%	20.5%	21.0%	21.4%	23.1%	13.6%	12.6%
3/05/2011	11.2%	12.3%	11.8%	N/A	N/A	N/A	N/A	N/A	N/A
2/05/2011	9.8%	12.6%	11.2%	20.1%	20.2%	21.7%	22.1%	18.8%	18.4%
			RMSE:	9.5%	9.5%	9.8%	10.4%	6.3%	6.0%

Figure 5. Pricing 1-month volatility swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.

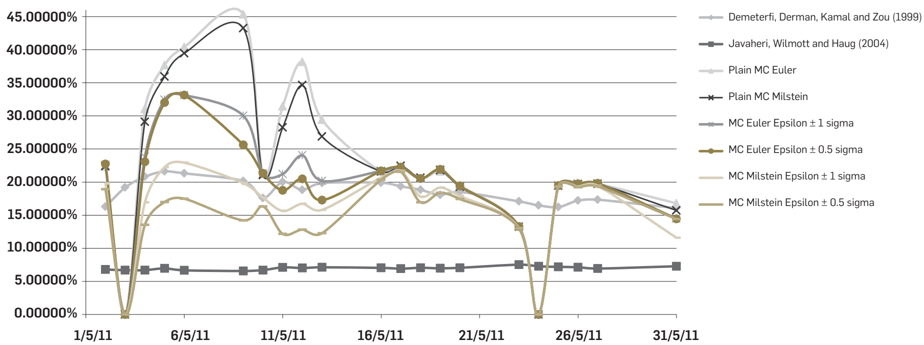

Figure 6. Pricing 3-month volatility swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.

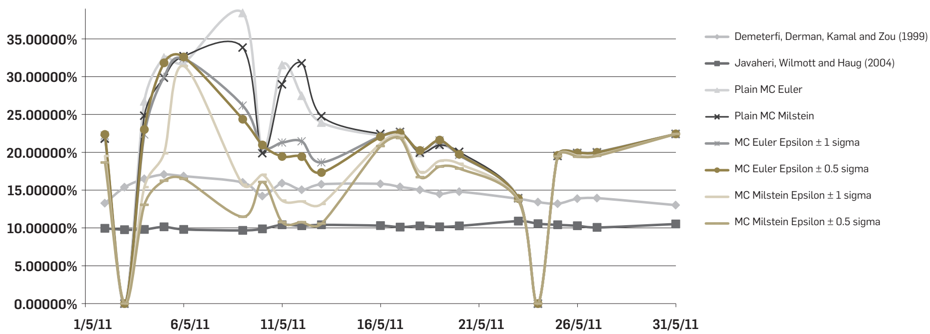

Figure 7. Pricing 6-month volatility swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.

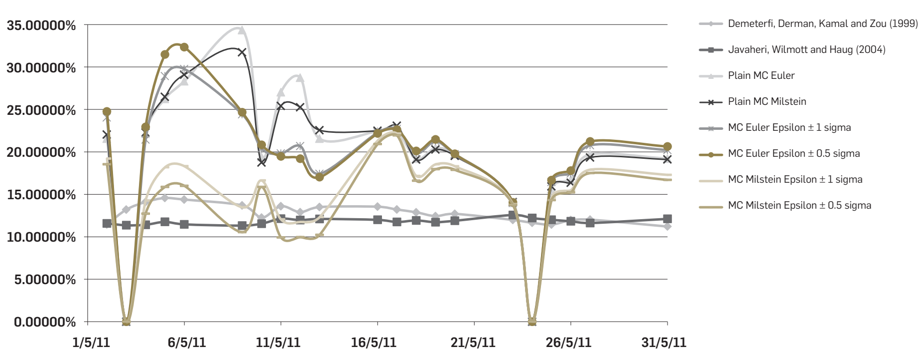

Figure 8. **Pricing 1-year volatility swap using two analytical solutions, Demeterfi *et al.* (1999b) and Javaheri *et al.* (2004), and MC simulation. Sample period: May 2011.**