Hedge ratios for short and leveraged ETFs

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Abstract

Exchange-traded funds (ETFs) exist for stock, bond and commodity markets. In most cases the underlying feature of an ETF is an index. Fund management today uses the active and the passive way to construct a portfolio. ETFs can be used for passive portfolio management, for which ETFs with positive leverage factors are preferred. In the frame of an active portfolio management the ETFs with negative leverage factors can also be applied for the hedge or cross hedge of a portfolio. These hedging possibilities will be analysed in this paper. Short ETFs exist with different leverage factors. In Europe, the leverage factors 1 (e.g. ShortDAX ETF) and 2 (e.g. DJ STOXX 600 Double Short) are offered while in the financial markets of the United States factors from 1 to 4 can be found. To investigate the effect of the different leverage factors and other parameters Monte Carlo simulation was used. The results show for example that higher leverage factors achieve higher profits as well as losses. In the case that a bearish market is supposed, minimizing the variance of the hedge seems not to obtain better hedging results, due to a very skewed return distribution of the hedge. The risk measure target-shortfall probability confirms the use of the standard hedge weightings, which depend only on the leverage factor. This characteristic remains when a portfolio has to be hedged instead of the underlying index of the short ETF. For portfolios that have a low correlation with the index return high leverage factors should not be used for hedging, due to the higher volatility and target-shortfall probability.

Resumen

Los fondos negociables en el mercado o EFTs (*Exchange Traded Funds*) existen en los mercados de valores – bonos – y de mercancías. En la mayoría de los casos tras un EFT hay un índice. La gestión actual de fondos utiliza formas activas y pasivas para construir una cartera de valores o *portfolio*. De esa forma, se opta por EFTs con factores de influencia positiva. En el marco de un portfolio activo también se pueden aplicar EFTs con factores de influencia negativa para la cobertura de una cartera de acciones. En este documento se analizarán estas posibilidades de cobertura. Existen EFTs con diferentes factores de cobertura. En Europa se ofrecen los factores de cobertura 1 (por ejemplo, ShortDAX ETF) y 2 (por ejemplo, DJ STOXX 600 Double Short), mientras que en los mercados financieros de EEUU encontramos factores del 1 al 4. Para analizar el efecto de los diferenteS factores de cobertura, así como de otros parámetros, utilizamos la Simulación Monte Carlo. Los resultados muestran, por ejemplo, que con factores con mayor cobertura

se obtiene tanto mayores beneficios como pérdidas. En el supuesto de un mercado con tendencia a la baja no parece que se minimice la variedad de la cobertura hasta que se consiguen mejores resultados, debido a una distribución muy sesgada de los rendimientos. Esa característica se mantiene también en el caso de que un portfolio tenga que ser cubierto o protegido, en lugar de utilizar el índice de un ETF *short* o reducido. Para aquellos portfolios con una baja correlación con los beneficios del índice, no deberían utilizarse factores de elevada cobertura, debido a la mayor volatilidad y probabilidad de déficit en los objetivos.

JEL-Classification-System: G11, G24, G32, C15

Key-Words: Portfolio Optimization, Hedging, Cross Hedge, Insurance and Immunization of Portfolios, Short Leveraged Exchange-Traded Funds (ETFs), Mean–Variance, Target-Shortfall Probability, Monte Carlo Simulation

1. Introduction

An exchange-traded fund (ETF) is a reconstructed index. If no index exists in some regions or sectors a portfolio whose structure is publicized can be the underlying feature of the ETF, too. These funds are traded every day at the exchange boards. Concerning volatility, an ETF can be denoted as being safer than a single stock of these markets. While a (long) ETF produces returns like the index or the underlying portfolio, short ETFs offer inverse returns. If an index loses 5% within one day, the short ETF with this underlying index would rise by 5%. A double short ETF would rise by 10% and a triple short ETF by 15% and so on. As described below, the rising of the short ETFs can be a little higher due to the payment of interest. Furthermore, a tracking error and a small management fee must be taken into consideration.

The reverse character of the short ETFs offers the possibility to apply short ETFs to hedging. The classic hedging instruments like short future and options produce specific results. A perfect hedge with a short future has a fixed return, which covers the cost of carry. There remains no chance to participate in profits when the prices of the stocks of the portfolio are rising. However, the value of the hedge would not be reduced if the prices are falling. In the case that the underlying index of the short future and the portfolio are not identical, an optimal hedge ratio must be computed (cross hedge). The perfect hedge can be denoted as immunization as the value of the hedge will not change when the prices in the market change. Options offer the possibility to construct portfolio insurance. In the case of the "protected put buying", losses do exist – but they are limited. The portfolio insurance does not exclude earning profits when stock prices are rising.

Using a short ETF for hedging is different in some ways from hedging with the derivative instruments futures or puts:

- The short ETF is not only a right that can be bought – it is an investment in the sense that more capital is needed. To reduce the capital, higher leverage factors can be used.
- Short ETFs do not have a duration like futures or puts, which need new contracts to roll on the hedging after some time.
- Dependent on the hedging period, short ETFs can offer a kind of immunization (for short time intervals T) and will change to a kind of insurance (for long periods T).[2] The strategy is neither bullish nor bearish; it is more a volatility-oriented strategy.[3] This changing character is illustrated in Figure 1, in which an index is hedged by a short ETF. Depending on the time interval T of the hedge, the function of the short ETF and thus the hedge result changes. As the short ETF pays additional interest i, the value of the short ETF in Figure 1 is placed a little higher than the value of the index when its profit is zero. The loss in the

[2] In general hedging inverse ETFs reduce the volatility, which was demonstrated by Hill, J. and Teller, S. (2010). They rebalanced the hedge of the S&P 500 by short and leveraged ETFs in a 7-month time interval.
[3] See Michalik Th., Schubert L. (2009).

case of insurance depends mainly on T, the leverage factor λ and the volatility σ of the index return. The function of a short ETF will be presented with the equation (2-3) in the following chapter.

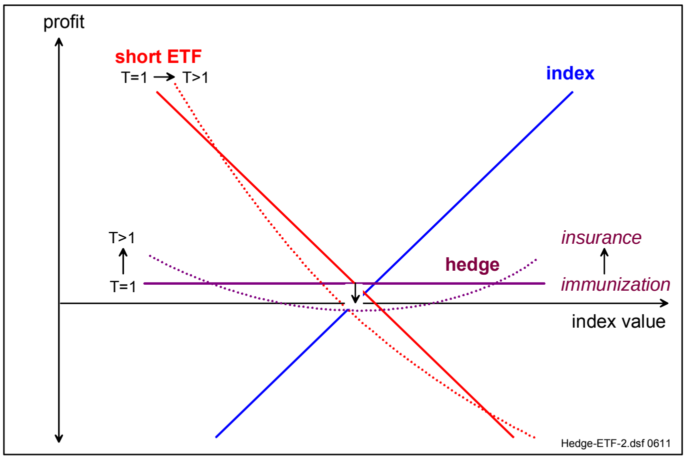

Figure 1: Hedging with short and leveraged ETFs: immunization or insurance

To investigate the effect of the different ETFs while hedging or cross hedging a portfolio of stocks (or an index), Monte Carlo simulation is used. Another question of hedging was considered by Alexander C. and Barbosa A., who looked for possibilities to hedge a portfolio of ETFs using future contracts.[4]

In the following chapter, the Monte Carlo simulation of asset values and their short and leveraged ETFs will be described. This approach was preferred as the use of empirical data often has the disadvantage that the data are restricted to particular types of ETFs,[5] e.g. in Germany, where until 2009 only short ETFs (leverage factor 1) were offered. The third chapter will depict the quality of the determined values. The effects of hedging with short and leveraged ETFs will be shown in chapter 4.

2. Monte Carlo simulation of short and leveraged ETFs

To gain some insights into the value of a short ETF, the prices of the underlying index were generated by **Monte Carlo simulation**. By this simulation a discrete "**random walk**"[6] was created for the index. The price development of the index depends on the annual expected return value μ and the (continuous compounded) volatility σ. For one day, the expected return is $\mu/360$ and respectively for the volatility $\sigma/\sqrt{360}$. For the simulation it is supposed that every day the stock prices are fixed at the exchange boards. Therefore, every day the price of the short ETF can change and has to be computed, too. The return r_t for day t = 1, …, T is simulated by y_t, a realization of the stochastic

[4] Alexander C., Barbosa A. (2007).
[5] Michalik Th., Schubert L. (2009).
[6] Deutsch H. P. (2004), pp. 26–34.

variable Y, which is standard normal distributed. The constructed value r_t is a realization of a stochastic variable $R \sim N(\mu/360, \sigma/360^{0.5})$:

$$r_t = \mu / 360 + \sigma / \sqrt{360} \cdot y_t \text{ with } Y \sim N(0,1). \tag{2-1}$$

The value I_0 of the index moves within one day (t=1) to

$$I_t = I_{t-1} e^{r_t} \text{ and respectively } I_1 = I_0 e^{r_t}. \tag{2-2}$$

The stochastic values of r_t determine the path of the price of the index within T days: $I_0, I_1, ..., I_T$. To compute the price of the short ETF, the ratios $I_1/I_0 \dots I_T/I_{T-1}$ were used (see (2-3)). The simulation uses a constant interest rate i=2% p.a. and respectively 2%/360 per day. To estimate the value S_t of the short ETF, the value of the last day S_{t-1} and the leverage factor λ must be taken into consideration. As the formula for the computation of the value of short ETFs only negative leverage factors are used; the sign of the factor is ignored in this paper (short ETF: λ=1, double short ETF: λ=2 etc.):

$$\boxed{S_t = \underbrace{S_{t-1} \cdot \left((\lambda+1) - \lambda \cdot \frac{I_t}{I_{t-1}} \right)}_{\text{Leverage Term}} + \underbrace{(\lambda+1) \cdot S_{t-1} \cdot \left(\frac{i}{360} \right)}_{\text{Interest Term}}}. \tag{2-3}$$

After T days, the prices I_T and S_T are determined by equations (2-1) to (2-3). Different time jumps for fixing the prices over the weekend etc. were ignored by the supposition that the prices of the index and the short ETF are fixed every day.

The computation for T days was repeated 10 million times to obtain the examples depicted below in chapters 3 and 4. For the simulation the programming language Delphi 4.0 was used. The creation of the random numbers was performed by the RandG(0,1) function of the unit “Math”.

The results of the simulation of I_T and the simultaneously determined S_T by equation (2-3) were used to compute the <u>hedge of the index by short ETF</u> $H_T=I_T+S_T$. If this hedge should have minimal variance, the weightings x_I for the part of the budget invested in t=0 in the index and x_S in the short ETF were determined by the cross-hedge equation:[7]

$$\boxed{x_I = \frac{s_S^2 - \text{cov}(r_I, r_S)}{s_I^2 + s_S^2 - 2 \cdot \text{cov}(r_I, r_S)}} \text{ and } x_S = 1 - x_I. \tag{2-4}$$

In the case of a hedge <u>for one day (T=1), a perfect hedge is possible</u> if the weightings

$$\boxed{x_I = \frac{\lambda}{\lambda+1}} \text{ and } \boxed{x_S = \frac{1}{\lambda+1}} \tag{2-5}$$

are used.[8] This standard hedge applied for one day offers an immunization of the index value. For short ETFs (λ=1) the weightings are x_I=1/2 and x_S=1/2 and for double short ETFs (λ=2) x_I=2/3 and

[7] See e.g. Grundmann W., Luderer B. (2003), p. 146.
[8] Proof: see appendix A1.

x_S=1/3. Higher leverage factors reduce the amount for the hedging instrument. For λ=4 the weighting is x_S=1/5, which means that only 20% of the budget has to be invested in the leveraged ETF.

The cross-hedge function (2-4) can be used when T>1, if doubts exist that the standard hedge weightings do not minimize the risk. Then the weightings can deviate from the standard hedge weightings (2-5) dependent on the supposed development of the index prices and respectively the annual continuous compounded mean return μ and the volatility σ. Furthermore, the time interval T will be considered and analysed in a chapter below. Table A3 in appendix A3 shows the results of the variation of the parameters μ and λ. These results contain information about the weighting of x_I and x_S for the "minimal variance hedge" (MVH), the expected return $r_I=(I_T/I_0-1)$ of the index and r_S of the short ETF after T days, the correlation corr of r_I and r_S and the variances of these values s_I^2 and s_S^2. The covariance of the cross-hedge equation (2-4) can be determined by the function $cov(r_I,r_S)=corr\cdot s_I\cdot s_S$. With this information the complete efficient frontier of the mix of an index with a short ETF can be computed.

Instead of an index, usually an asset or a portfolio has to be hedged. Therefore, the simulation of the path of the index prices and the determination of the value of the short ETF must be expanded to the simulation of the path of a portfolio. The returns of this portfolio and the index normally have a correlation ρ<1. For the simulation of the index and the ρ-correlated portfolio, the formula

$$y_{tP} = \rho \cdot y_t + \sqrt{1-\rho^2} \cdot y'_t \tag{2-6}$$

was applied to obtain random numbers that generate ρ-correlated returns.[9] The variable y_t is a realization of a stochastic variable Y, as described in (2-1). The value y'_t signifies a realization of an analogously defined stochastic variable Y'. The result of equation (2-6) is the realization y_{tP} of a stochastic variable Y_P, which has a correlation ρ with variable Y. The values y_{tP} (t=1, …, T) are used to construct the return of the portfolio and y_t for the return of the index. As the variables Y and Y' have an expected value of 0 and variance of 1, the variable Y_P will have these parameters, too (see appendix (A2-2)). With equation (2-1) and the stochastic variables Y and Y_P the returns of the index and the portfolio can be computed. The path of the value of the portfolio P_0, …, P_T has to be computed analogously to equation (2-2).

According to the well-known relationship[10]

$$\beta_P = \rho \cdot \frac{\sigma}{\sigma_P} \tag{2-7}$$

the β_P value of a portfolio can be designed by the selection of the standard deviation σ_P of the portfolio. For equal standard deviations $\sigma=\sigma_P$ the simulation will generate a portfolio with low

9 Proof: see appendix A2.
10 See e.g. Bamberg, G., Baur, F. (1996), p. 44.

systematic risk, as $\beta_P=\rho$ with ($\rho\leq1$). Furthermore, the α_P value of the portfolio using equation (2-7) would be:

$$\boxed{\alpha_P = \mu_P \cdot \rho \cdot \frac{\sigma}{\sigma_P} \cdot \mu} \,. \tag{2-8}$$

Tables A4-1 and A4-2 in appendix A4 show a different MVH from the parameters ρ=0.95, 0.90, 0.85, 0.80, 0.75. Now, instead of r_I and s_I^2, the tables contain the expected return of the portfolio $r_P=(P_T/P_0-1)$ and the variance s_P^2 of these portfolio returns for a given time interval T and leverage factor λ. As above, the completely efficient frontier of the mix of a portfolio and a short ETF can be computed by the information in these tables.

3. Quality of the Monte Carlo simulation

As the random walk of a path t = 0, …, T produces index prices I_T that are lognormal distributed, the expected mean[11] $E(I_T)$ must be

$$E(I_T) = I_0 \cdot e^{\left(\mu+\frac{\sigma^2}{2}\right)\cdot\frac{T}{360}} \text{ and the variance} \tag{3-1}$$

$$Var(I_T) = I_0 \cdot e^{\left(2\cdot\mu+\sigma^2\right)\frac{T}{360}} \cdot (e^{\sigma^2\cdot\frac{T}{360}} - 1)\,. \tag{3-2}$$

A good simulation should generate good estimations of the expected parameters: $E(I_T) \approx I_0 \cdot (1+ r_I)$ and $Var(I_T) \approx I_0 \cdot s_I^2$. To test these two equations, a simulated example with the following parameters is used: μ=5%, σ=50%, T=300 and the initial value I_0=100. The simulated values are r_I=0.157022 and s_I^2=0.310205. Applying the parameters to the equations (3-1) and (3-2) gives very similar values to the results of the simulation:

$$E(I_{300}) = 100 \cdot e^{\left(0.05+\frac{0.5^2}{2}\right)\cdot\frac{300}{360}} = 115.70033 \approx 100 \cdot (1+0.157022) = 115.7022 \text{ and} \tag{3-3}$$

$$Var(I_{300}) = 100 \cdot e^{\left(2\cdot0.05+0.5^2\right)\frac{300}{360}} \cdot (e^{0.5^2\cdot\frac{300}{360}} - 1) = 31.006454 \approx 100 \cdot 0.310205 = 31.0205\,. \tag{3-4}$$

The used expected return μ=5% and variance σ^2=25% are continuously compounded. These parameters can also be estimated by the values $\ln(I_T/I_0)=\ln(I_T/100)$. The mean and variance of these values achieved by simulation are 0.041613 and 0.209600, respectively. To obtain estimations for the annual mean and respectively variance these values must be multiplied by the factor 1.2=360/300. This product for the mean is $0.041613\cdot1.2=0.0499356\approx0.05$ and respectively for the continuously compounded variance $0.209600\cdot1.2=0.25152\approx0.25$.

[11] Luenberger, D. G. (1998), p. 309.

The simulation seems to offer very good estimations regarding the expected mean and acceptable estimations for the variance.

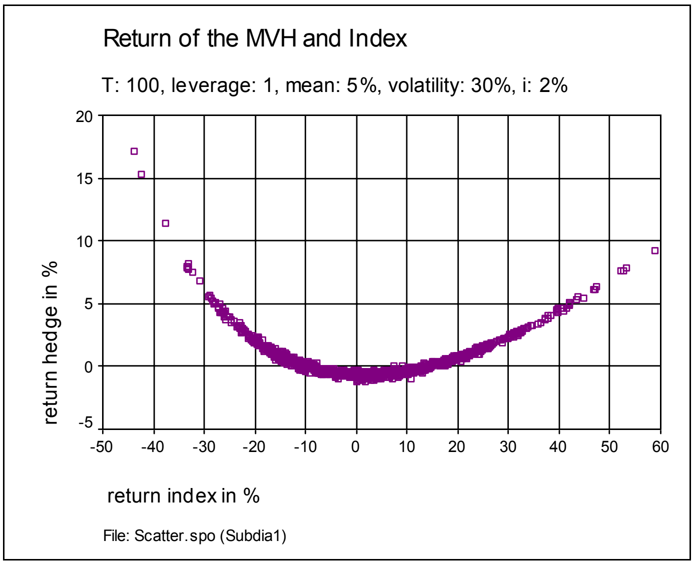

Figure 3-1: Return of an index (μ=5%, σ=30%) and the hedge with T=100

Additionally to the numerical quality of the simulation, a visual comparison of simulated data with empirical data shows a good fit of the simulated data to the returns in the financial market. In the scatter plot of Figure 3-1 a set of 1000 simulated returns of an index is plotted with the return of a standard hedge. The index returns achieved within T=100 days have a continuous compounded mean of μ=5% and a volatility of σ=30%. For the hedge $H_0=I_0+S_0$ a short ETF with a leverage factor λ=1 is used with the budget weightings x_I=0.5 and x_S=0.5 in t=0. The interest rate is fixed as i=2%.

While Figure 3-1 contains simulated values, the scatter diagram of Figure 3-2 was created by empirical data of the German stock index DAX with data of the decade 2000 to 2009.[12] The scatter plot of the simulated and the empirical data depicts a common relationship between the return of the index and that of the hedge: the plot has the shape of a sickle. Strong increasing and decreasing index prices effect a positive hedge return. Due to times of higher and times of lower mean and volatility of the DAX return, the empirical diagram does not have the same shape as the simulated one. In this time interval the “dot-com” and the “real estate” crises caused strong decreasing prices and mean returns, respectively, in the German stock market.

[12] See Michalik T., Schubert L. (2009), p. 9. The empirically generated scatter plot uses T=100 calendar days and the EONIA as interest rate i_t. As the short DAX ETF was generated synthetically, the depicted returns are not reduced by transaction costs and tracking errors. This scatter plot contains 2241 points.

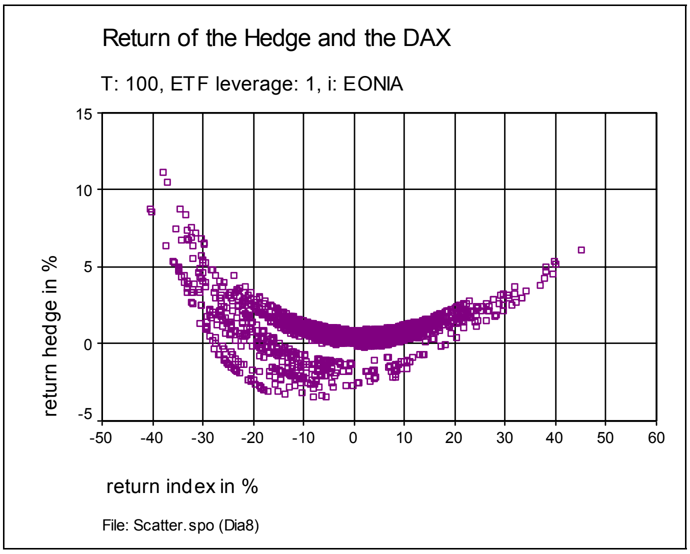

Figure 3-2: Return of the DAX and the hedge with T=100

4. Standard and optimized hedge

The use of weightings x that depend only on the leverage factor λ (see equation (2-5)) will be denoted as a standard hedging solution. For the optimized hedge, a supposition about the development of the market index is taken, e.g. the market will be bearish measured by mean μ of the index. Under this assumption the mix of an index or portfolio with a short or leveraged ETF will be selected, which minimizes the risk of the hedge. As risk measures, the variance and the target shortfall probability (TSP) will be applied.

The first chapter investigates the standard hedge approach. It shows the effect when the leverage factor λ, the time horizon T, the volatility σ or the mean μ changes. In the second part the standard hedge will be applied to hedge a portfolio. In this case, the effect of the correlations ρ between the return of the index and the portfolio will be in focus. The following chapters compare the results of the standard hedge with those of the optimized hedge. All the numerical results were generated by 10 million iterations, while the depicted scatter plots were simulated by at least 3000 iterations. Tables 4.1-1 to 4.3-2b contain for a specific configuration of the parameters λ, T, μ and σ the following information. After the head line, the weighting (x_I, x_S) is depicted. When a portfolio has to be hedged the weighting will be ($x_{Portfolio}$, x_S). The next line, denoted by "correlation", refers to the return of the index and its short ETF (in contrast, the correlation of the return of an index and a portfolio will always be denoted by the letter ρ). In the last two lines the mean return r_{Hedge} and the volatility s_{Hedge} of the standard hedge are shown. For the optimized hedge there will be analogous r_{MVH} and s_{MVH} (and respectively r_{MPH} and TSP) when the variance (and respectively the TSP) is minimized.

4.1 Standard hedge of an index for different parameters λ, σ, T and μ

The decision to use lower or higher leverage factors for hedging the value of an index has an effect on the liquidity. While the leverage factor of λ=1 needs x_S=0.5 and respectively 50% of the budget to hedge the value of an index, in the case of λ=4 only x_S=0.2 and respectively 20% of the budget are necessary to achieve a hedge (see equations (2-5)). As Table 4.1-1 shows, the correlation between the index return and the ETF return is negative, as expected, due to the inverse returns of the applied ETFs. For higher leverage factors, the absolute value of the correlation is shrinking. In the example of Table 4.1-1 the time interval T=100. Therefore, the correlation is not -1 as it would be for T=1. While the mean return of the hedge is weakly increasing when higher leverage factors are used, the standard deviation is growing stronger. Figure 4.1-1 depicts scatter plots of the return of the index and the hedge for different leverage factors. High positive and negative index returns always offer positive hedge returns, which will be higher for higher leverages. The price of these higher returns is losses, which are deeper for high leverage factors. For λ=1, the minimal return of the hedge in the scatter plot with 1000 points is -1.14%, for λ=2 the minimal return is -2.94% and for λ=3 and respectively λ=4 the returns are -4.57% and respectively -6.35%. This return also depends on other parameters, as the following figures will illustrate.

Leverage λ	**1**	**2**	**3**	**4**
x_I	0.5000	0.6667	0.7500	0.8000
x_S	0.5000	0.3333	0.2500	0.2000
Correlation	-.9755	-.9456	-.9048	-.8538
r_{Hedge}	0.5784%	0.6014%	0.6220%	0.6463%
s_{Hedge}	1.80%	3.55%	5.28%	7.05%

Table 4.1-1: Hedge for different leverage factors (T=100, μ=5%, σ=30%) with correlations

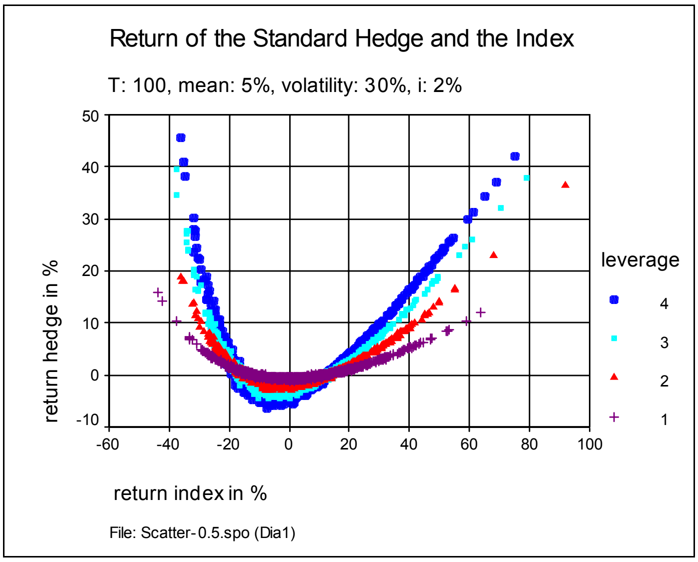

Figure 4.1-1: Return of an index (μ=5%, σ=30%) and the hedge with different leverage factors

Like the leverage factor, the volatility is responsible for high or low returns of the hedge. An obvious difference in the scatter plot of Figure 4.1-2 is the return of the index. A small volatility of for example σ=10% causes a small range of returns. In this figure for T=100 only the leverage factor of λ=1 is used. Table 4.1-2 shows for this leverage the development of the correlation and of the standard deviation s_{Hedge} of the hedge. For higher volatility of the return of the index, the correlation decreases and the standard deviation s_{Hedge} increases. Of the volatility σ=50% of the index return only s_{Hedge}=5.06% remains in the hedge.

Volatility σ	**10%**	**30%**	**50%**
x_I	0.5000	0.5000	0.5000
x_S	0.5000	0.5000	0.5000
Correlation	-.9973	-.9755	-.9048
r_{Hedge}	0.5618%	0.5784%	0.6513%
s_{Hedge}	0.20%	1.80%	5.06%

Table 4.1-2: Hedge for different volatilities (T=100, μ=5%, λ=1) with correlations

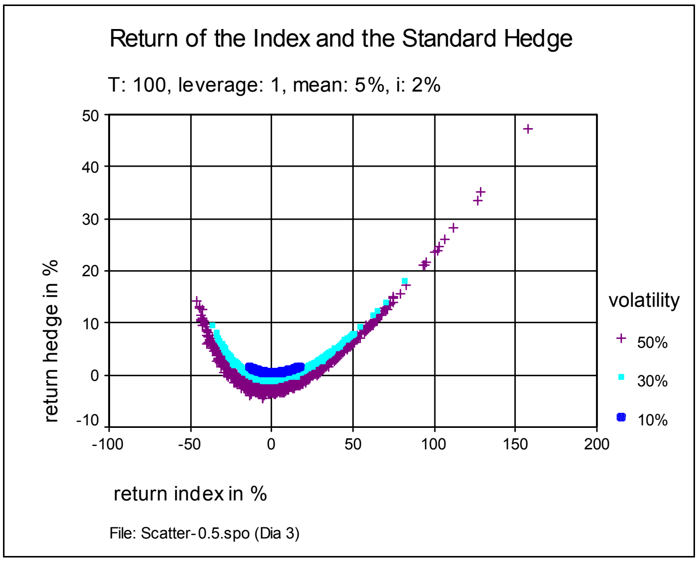

Figure 4.1-2: Return of an index (μ=5%, λ=1) and the hedge with different volatilities

An additional determinant of the development of the hedge return is the time T. In the figures above T=100 was used.[13] In Table 4.1-3 different time intervals T are considered for the leverage factors λ=1, μ=5% and σ=30%. As expected, for a higher T the negative correlation becomes smaller and the standard deviation s_{Hedge} increases.

In Figure 4.1-3 the scatter plots for the different Ts show that in time the minimal return of the hedge becomes smaller, as in the case of high volatility or higher leverage factors. For T=50, the minimal return of the hedge in the scatter plot with 1000 points is -0.65%; for T=100 and respectively T=300 this return is -1.12% and respectively -2.83%.

[13] As we investigated the investor's behaviour buying short and leveraged ETFs during the real estate crisis 2008/2009, the majority (85%) of the investors held the ETF for less than 100 days. For higher leverage factors a significantly lower holding time T was observed (see Flood, Ch. (2010)).

Time **T**	**50**	**100**	**300**
x_I	0.5000	0.5000	0.5000
x_S	0.5000	0.5000	0.5000
Correlation	-0.9879	-0.9755	-0.9281
r_{Hedge}	0.2830%	0.5784%	1.8766%
s_{Hedge}	0.88%	1.80%	5.66%

Table 4.1-3: Hedge for different time intervals T (λ=1, μ=5%, σ=30%) with correlations

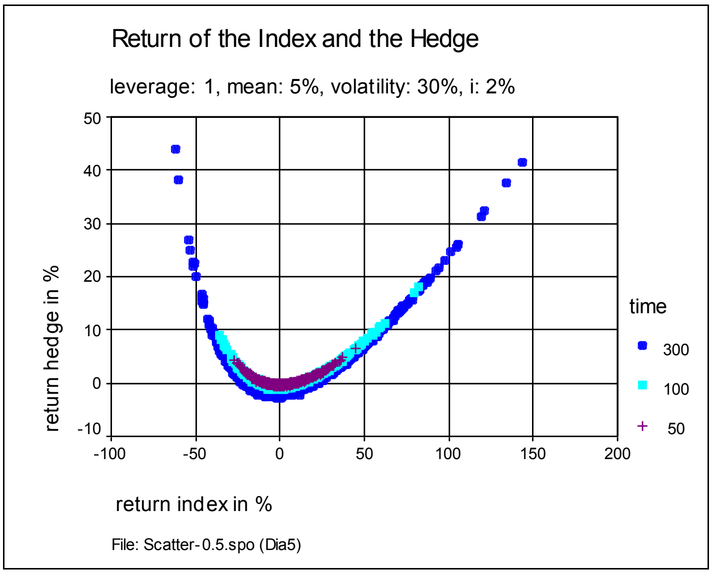

Figure 4.1-3: Return of an index (μ=5%, σ=30%) and the hedge (λ=1) with different time intervals T

Table 4.1-4 and Figure 4.1-4 show the effect on the hedge when different means are supposed. The minimal hedge return of each of the 1000 points in the scatter plot of Figure 4.1-4 seems to be similar. For the means of μ=-30% and μ=0% this minimal return is -1.12% and for μ=+30% this return is -1.24%. In this figure the paths of the 3 clusters are very similar. The only observable difference is logical. If the mean is high, more points will be on the right side of the scatter plot and the reverse.

Mean μ	**-30%**	**0%**	**+30%**
x_I	0.5000	0.5000	0.5000
x_S	0.5000	0.5000	0.5000
Correlation	-0.9756	-0.9756	-0.9755
r_{Hedge}	0.8480%	0.5593%	0.9642%
s_{Hedge}	2.14%	1.77%	2.28%

Table 4.1-4: Hedge for different means μ (T=100, λ=1, σ=30%) with correlations

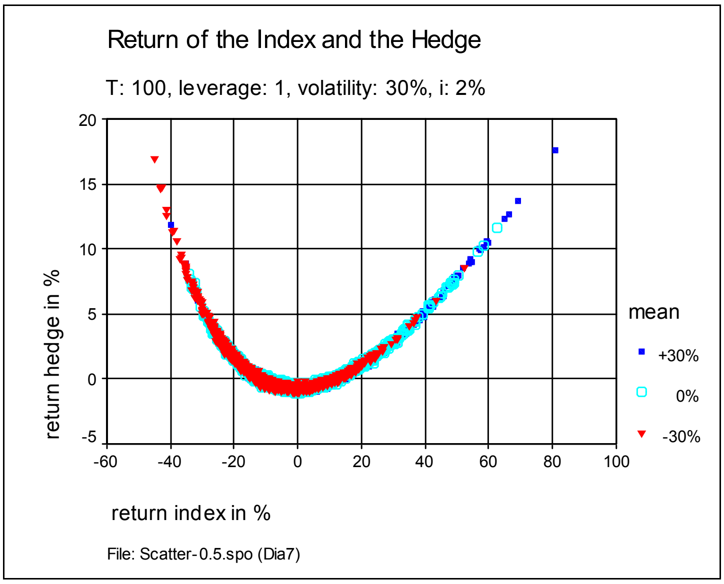

Figure 4.1-4: Return of an index with different means μ (T=100, σ=30%) and the hedge (λ=1)

4.2 Standard hedge of a portfolio with different correlations ρ

When a portfolio has to be hedged, the correlation ρ of the return of the index and this portfolio is an important determinant of the risk that remains in the hedge. The equations (2-6) to (2-8) offer the possibility to design such a ρ-correlated portfolio. Appendix A4 depicts, in the lines of Table A4-1, that in the case of a weak correlation the weighting for the short ETF becomes higher. The weighting x_S in the line of T=100 and correlation ρ=0.95 is x_S=0.510692 and for ρ=0.75 it is x_S=0.511962. Weakly increasing weightings can be observed in every line with T>10. For higher leverage factors, these weightings will decrease by the correlation (see Table A4-2).

For Table 4.2-1 the leverage factor λ=1 is used to simulate for T=100 days returns of a short ETF with an underlying index. The volatility of this index as well as of the portfolio is 30% and the mean 5%. Portfolio returns are generated, which are ρ-correlated with the return of the index (the correlation in the middle line of Table 4.2-1 refers to the correlation between the returns of the portfolio and the short ETF). While the mean return r_{Hedge} has small changes when the correlation ρ is reduced, the standard deviation s_{Hedge} rises to 8.08% in the case of ρ=0.50. Compared with the volatility of 30% of the portfolio return, the hedge has a reduced risk, although it is not riskless. For the correlation of ρ=0.95 and ρ=0.75 Figure 4.2-1 shows by a sample of 2·1000 points the return of the portfolio and its hedge. While for the correlation of 0.95 the sickle-shaped cloud can be recognized, for the lower correlation of 0.75 this cloud has lost this form. However, in the majority of cases the hedge return is between -10% and +10%, although the portfolio itself has higher losses and gains.

Correlation **ρ**	**1.00**	**0.95**	**0.85**	**0.75**	**0.50**
$x_{Portfolio}$	0.5000	0.5000	0.5000	0.5000	0.5000
x_S	0.5000	0.5000	0.5000	0.5000	0.5000
Correlation	-0.9755	-0.9273	-0.8308	-0.7339	-0.4909
r_{Hedge}	0.5784%	0.5788%	0.5790%	0.5797%	0.5738%
s_{Hedge}	1.80%	3.07%	4.67%	5.85%	8.08%

Table 4.2-1: Standard hedge for different portfolio–index correlations ρ (T=100, mean $\mu=\mu_P=5\%$, $\sigma=30\%$, $\lambda=1$) with correlations between portfolio and short ETF returns

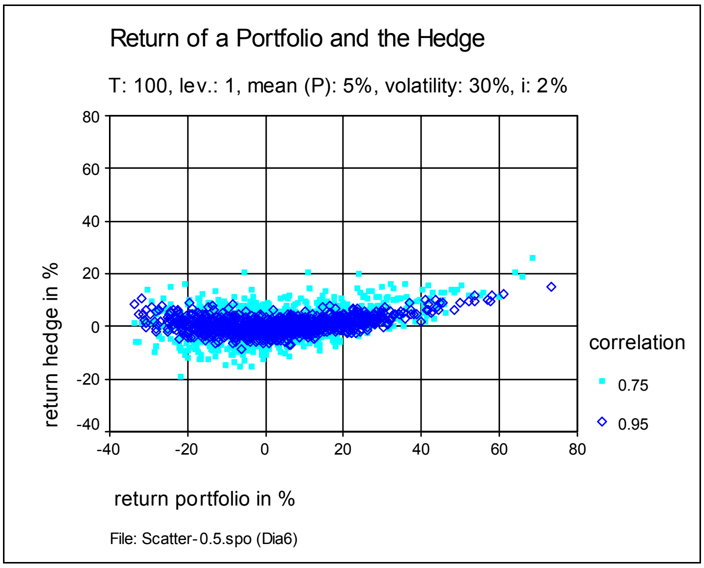

Figure 4.2-1: Return of 2 portfolios and the hedges (index: T=100, mean $\mu=\mu_P=5\%$, $\sigma=30\%$, $\lambda=1$)

In Table 4.2-2 an inverse ETF with a leverage factor of $\lambda=4$ is applied to hedge the return of a portfolio. The correlations ρ and the other parameters are as shown in Table 4.2-1. The higher leverage has the advantage of reducing the capital for the hedge ($x_S=0.2$). On the other side, the standard deviation of the hedge increases for higher leverage factors. For a portfolio with a correlation of only 50% and volatility of 30%, the risk reduction to 13.82% is not enough to refer to this situation as “hedged”. In the case of a correlation of ρ=0.75, the return of the hedge is in the majority of cases between -20% and +20% (see Figure 4.2-2). For a correlation ρ=0.95 the loss of the hedge is in most of the cases smaller than -10%. On the right lower side, the scatter plot (with 1000 points for each correlation) seems to have a linear border. The reason for this shape is the strong reduction in the value of the inverse ETF by the leverage factor 4 when the price of the index rises. In some extreme cases, the value of the inverse ETF is nearly zero. These small values are additionally multiplied with the weighting of $x_S=0.20$. Therefore, on the right side of the scatter plot, the value of the index alone represents nearly 100% of the hedge value. On the left side of this chart, a weak characteristic of hedging portfolios with highly leveraged ETFs can be seen. If the portfolio produces losses and the index profits, the loss of the leveraged ETF has to be added to the loss of the portfolio. This risk rises by a small correlation ρ. In Figure 4.2-2 for ρ=0.75 sometimes the hedge has higher losses than the portfolio itself. To call this mix of a portfolio and a leveraged ETF “hedged” would be an abuse of this word.

Correlation ρ	**1.00**	**0.95**	**0.85**	**0.75**	**0.50**
$x_{Portfolio}$	0.8000	0.8000	0.8000	0.8000	0.8000
x_S	0.2000	0.2000	0.2000	0.2000	0.2000
Correlation	-0.8538	-0.8132	-0.7315	-0.6486	-0.4374
r_{Hedge}	0.6463%	0.6402%	0.6412%	0.6411%	0.6429%
s_{Hedge}	7.05%	7.96%	9.54%	10.92%	13.82%

Table 4.2-2: Standard hedge for different portfolio–index correlations ρ (T=100, mean $\mu=\mu_P$=5%, σ=30%, λ=4) with correlations between portfolio and short ETF returns

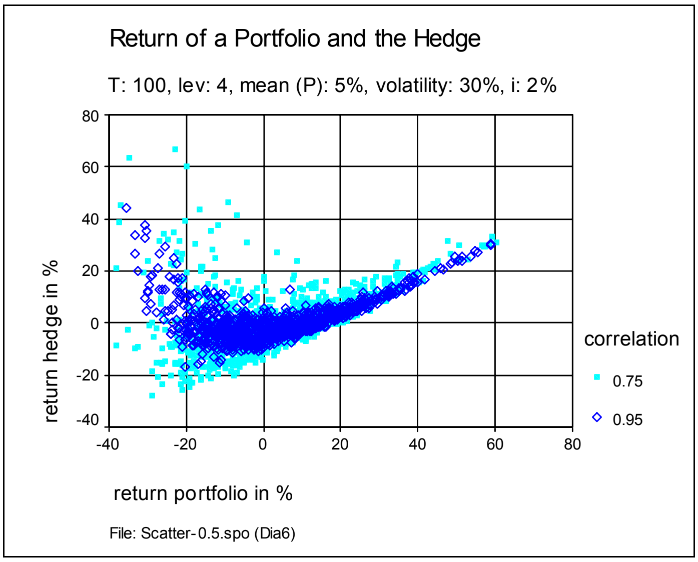

Figure 4.2-2: Return of 2 portfolios and the hedges (T=100, $\mu=\mu_P$=5%, σ=30%, λ=4)

4.3 Mean–variance hedge (MVH) versus standard hedge

Diversification is a principle of risk reduction in portfolio management. While the construction of a portfolio has to be successful for a longer time period, hedging normally has to avoid temporary losses when the market prices break down. Therefore, the expectation about the short-term development of the return is more important. In general, the weightings for the index $x_I=\lambda/(\lambda+1)$ and for the short ETF $x_S=1/(\lambda+1)$ (see equation (2-5)) produce a hedge with minimal variance without losses for very small time intervals T, e.g. T=1. For a higher T, this weighting has to be adapted to the expectation, to obtain an MVH. Due to a certain expected mean return of the index, the weights will be higher or lower compared with the standard weightings x_I and x_S, respectively. To judge the efficiency of the standard weights, the hedge returns and as a risk measure the standard deviations will be used in this chapter.

The MVH is computed for the leverage factor λ=1 and different expected mean μ. Table 4.3-1a shows the results. If an investor expects a bearish market, he should invest less than 50% of his budget in the short ETF and more in the index, to obtain an MVH. In the case of an annual continuous compounded loss of -30% the weightings of the MVH are: x_I=0.54 and x_S=0.46. If a bullish market with a return of +30% is supposed, the reverse should be carried out to reduce the variance (this situation may happen if a short ETF should be hedged buying an index). This recommendation to increase the "losing" part x_l is surprising. Its reason is founded on the return distribution and the variance as a risk measure. This problem will be discussed later.

Table 4.3-1b contains the hedge features when standard hedge weightings are applied. The data of the both tables depict that the standard results are efficient ($r_{Hedge} \geq r_{MVH}$ and $s_{Hedge} \geq s_{MVH}$). Figure 4.3-2 shows the efficient lines (tiny points) for the mean values μ=-30% and λ=1 and respectively λ=4 inclusive of the bold points of the MVH and standard hedge. As the data in Tables 4.3-2a and 4.3-2b depict, the difference between the MVH and the standard hedge grows with the leverage factor.

For high volatility (e.g. 50%) and high leverage factors, the efficiency of the standard hedge solutions becomes lost for smaller mean returns. As Tables 4.3-2a and 4.3-2b illustrate for a leverage factor of λ=4 and mean return between 0% and 10%, the standard hedge solution is not efficient. However, the differences between the MVH and the standard hedge are small in these inefficient constellations.

Figure 4.3-1 shows the scatter plots for the expected means μ=-30%, 0% and +30%. Obviously, the use of the MVH weightings shifts the curve of the plot to the left and right, respectively, when high negative and positive means, respectively, are used in the simulation.

Mean μ	**-30%**	**-20%**	**-10%**	**-5%**	**0%**	**+5%**	**+10%**	**+20%**	**+30%**
x_l	0.54	0.52	0.51	0.50	0.50	0.49	0.48	0.47	0.45
x_S	0.46	0.48	0.49	0.50	0.50	0.51	0.52	0.53	0.55
Correlation	-.98	-.98	-.98	-.98	-.98	-.98	-.98	-.98	-.98
r_{MVH}	0.26%	0.44%	0.53%	0.56%	0.55%	0.53%	0.50%	0.36%	0.14%
s_{MVH}	1,76%	1.76%	1.77%	1.77%	1.77%	1.77%	1.77%	1.77%	1.77%

Table 4.3-1a: MVH for different means (T=100, σ=30%, λ=1)

Mean μ	**-30%**	**-20%**	**-10%**	**-5%**	**0%**	**+5%**	**+10%**	**+20%**	**+30%**
x_l	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
x_S	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
Correlation	-.98	-.98	-.98	-.98	-.98	-.98	-.98	-.98	-.98
r_{Hedge}	0.85%	0.67%	0.58%	0.56%	0.56%	0.58%	0.62%	0.75%	0.96%
s_{Hedge}	2.14%	1.93%	1.80%	1.77%	1.77%	1.80%	1.85%	2.03%	2.28%

Table 4.3-1b: Standard hedge for different means (T=100, σ=30%, λ=1)

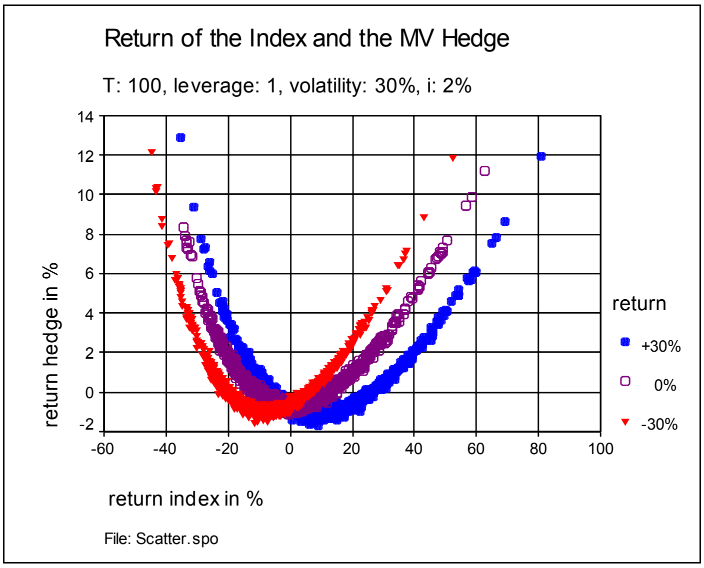

Figure 4.3-1: Return of an index (T=100, σ=30%) with different means μ and the MVH (λ=1)

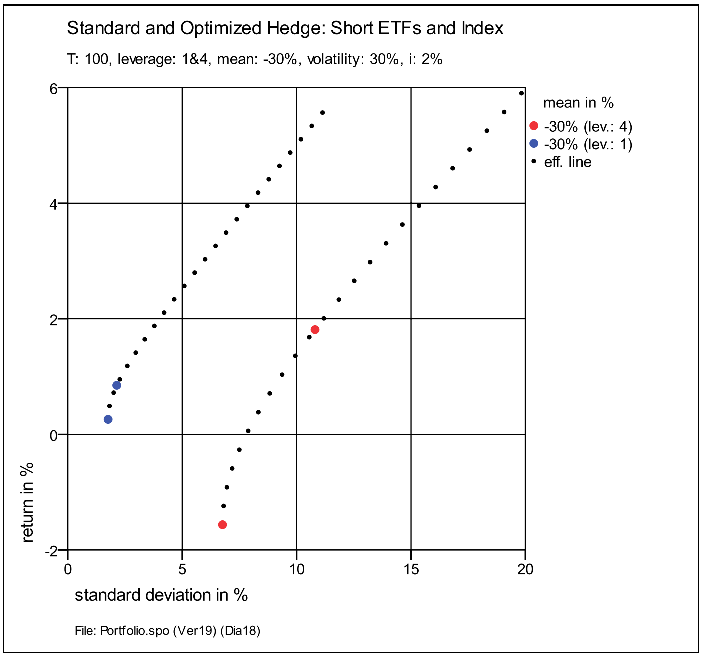

Figure 4.3-2: Portfolios of an index (T=100, σ=30%) with negative means and a short ETF (λ=1, 4)

Mean μ	**-30%**	**-20%**	**-10%**	**-5%**	**0%**	**+5%**	**+10%**	**+20%**	**+30%**
x_I	0.88	0.86	0.84	0.83	0.82	0.81	0.80	0.78	0.75
x_S	0.12	0.14	0.16	0.17	0.18	0.19	0.20	0.22	0.25
Correlation	-.86	-.85	-.85	-.85	-.85	-.85	-.85	-.85	-.85
r_{MVH}	-1.56%	-0.58%	0.17%	0.44%	0.65%	0.78%	0.83%	0.67%	0.14%
s_{MVH}	6.76%	6.84%	6.91%	6.92%	6.94%	6.96%	6.96%	6.96%	6.65%

Table 4.3-2a: MVH for different means (T=100, σ=30%, λ=4)

Mean μ	**-30%**	**-20%**	**-10%**	**-5%**	**0%**	**+5%**	**+10%**	**+20%**	**+30%**
x_I	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80	0.80
x_S	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
Correlation	-.86	-.85	-.85	-.85	-.85	-.85	-.85	-.85	-.85
r_{Hedge}	1.81%	1.05%	0.65%	0.56%	0.57%	0.65%	0.79%	1.29%	2.05%
s_{Hedge}	10.81%	9.17%	7.98%	7.53%	7.23%	7.05%	6.96%	7.12%	7.60%

Table 4.3-2b: Standard hedge for different means (T=100, σ=30%, λ=4)

As mentioned above, the volatility of the standard hedge can be reduced in bearish (bullish) markets by the selection of a smaller (higher) x_S as the standard weighting. In the following chapter, a more relevant risk measure will be discussed and applied to hedging.

4.4 Target–shortfall probability (TSP) versus mean–variance hedge (MVH)

The risk measure "variance" is only justified when symmetric return distributions exist. In portfolio optimization, this characteristic exists more or less due to the high number of independent return distributions of the different assets. Then the skewness is distributed away. In the case of a hedge with a short ETF, the two return distributions (underlying index and short ETF) are not independent. The return distribution of the hedge is very skewed. The examples in Figure 4.4-1 illustrate that the index has a weak positive skewness of 0.48 independent of the return. While the skewness of the short and leveraged ETF is small for low leverage factors, the skewness of the hedge return is always on a level between 2.28 and 4.91.

skewness	μ=5%			μ=-30%		
	index	ETF	hedge	index	ETF	hedge
λ=1	0.4812	0.4714	2.8453	0.4824	0.4715	2.2763
λ=2	0.4819	0.9903	2.8421	0.4807	0.9870	3.2409
λ=3	0.4809	1.6043	3.0895	0.4804	1.6023	3.9702
λ=4	0.4816	2.3945	3.6798	0.4804	2.3757	4.9096

Table 4.4-1: Skewness of an index (T=100, σ=30%, i=2%), the inverse ETF and the standard hedge

Skewed returns lead to the recommendation of other risk measures, e.g. the target-shortfall probability (TSP), which is the probability α that the return is lower than a given return target τ. In the case of hedging, the TSP can be described as the probability $\alpha_{Hedge}=P(r<\tau)$. For the standard hedge some TSPs are computed by simulation. In Table 4.4-2 the index is hedged by a leveraged ETF. The TSP α_{Hedge} is computed for 6 different targets. The simulation is applied for the time interval of T=100 days, the volatility of 30%, 3 different mean returns and leverage factors of 1 to 4. As expected, with the leverage factor, the TSP is rising, too. For the target τ=0% the mean return of 5% produces a higher TSP than high positive or negative means.[14]

[14] Although the results of the TSP simulation seem to be relatively steady, the repetition of the simulation may change the results on the right side of the decimal point.

To obtain a minimal probability hedge (MPH) with the 6 targets τ, as shown in Table 4.4-2, an algorithm[15] using only EXCEL is used that computes all the mean–TSP combinations for each target. In Figure 4.4-1a and respectively 4.4-1b these combination lines are depicted for the index mean=-30% and respectively +30%. In contrast to the MVH, the MPH is always built by weightings of the standard hedge (see bold points). In Figure 4.4-1a the TSPs for the targets 0%, -1% and -2% recommend buying only the leveraged ETF. As an index should be hedged, the targets below are more relevant to the hedging decision. The minimal TSP for the target -3% (-4% and respectively -5%) is α=38.13% (18.65% and respectively 3.20%) with a weighting x_S=0.1929 (0.2022 and respectively 0.1944). In Figure 4.4-1b, the minimal TSPs for the target -4% and respectively -5% are α=13.13% and respectively 1.97% and the weightings x_S=0.1911 and respectively 0.2003. In the two examples, the deviation of the weightings of the MPH from the standard hedge weightings is small. It must be noted that in the simulation of these figures only 3000 cases are used for each target. Using the standard weightings for hedging does not minimize the volatility when high losses are expected, but the risk criterion TSP, which reflects more what an investor is afraid of.

T	μ	σ	λ	**τ=0%**	**τ=-1%**	**τ=-2%**	**τ=-3%**	**τ=-4%**	**τ=-5%**
100	30%	30%	1	46.94	0.64	0.00	0.00	0.00	0.00
			2	53.28	37.29	6.89	0.03	0.00	0.00
			3	54.19	45.19	32.37	11.28	0.70	0.01
			4	53.94	47.48	39.60	28.88	13.12	2.24
100	5%	30%	1	53.32	0.75	0.00	0.00	0.00	0.00
			2	61.15	43.88	8.47	0.04	0.00	0.00
			3	63.02	53.61	39.28	14.03	0.89	0.01
			4	63.69	57.07	48.53	36.18	16.88	2.96
100	-30%	30%	1	48.75	0.69	0.00	0.00	0.00	0.00
			2	57.81	41.43	8.11	0.04	0.00	0.00
			3	61.14	52.24	38.54	14.01	0.91	0.01
			4	63.27	57.14	49.05	37.04	17.65	3.19

Table 4.4-2: Target-shortfall probabilities α (in %) of the return of the standard hedge of an index (i=2%)

[15] Michalik Th., Schubert L. (2009).

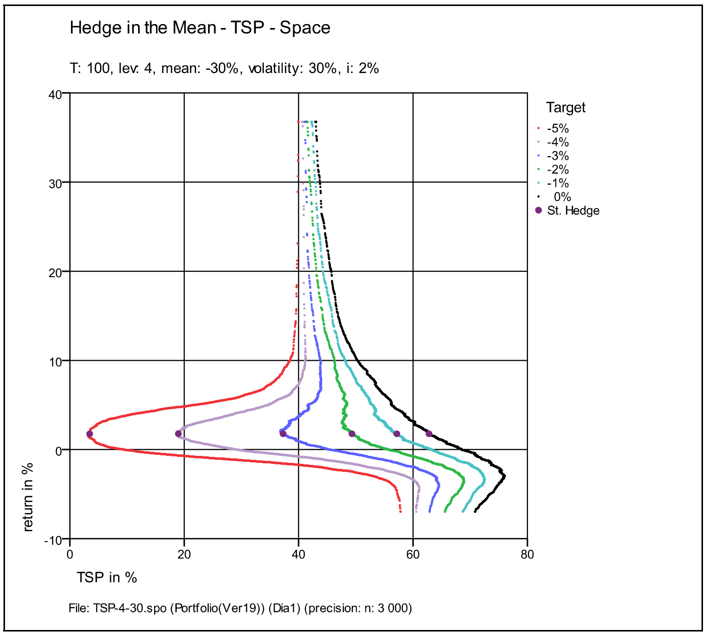

Figure 4.4-1a: Mean–TSP mix of an index (T=100, μ=-30%, σ=30%) and short ETF (λ=4)

$\mu=\mu_P$	ρ	λ	τ=0%	τ=-2%	τ=-4%	τ=-10%	τ=-15%	τ=-20%
30%	0.75	1	45.20	32.05	20.58	2.61	0.16	0.00
		2	45.67	35.75	26.46	7.17	1.36	0.13
		3	45.65	36.93	28.62	9.69	2.50	0.38
		4	45.61	37.57	29.82	11.33	3.41	0.65
5%	0.75	1	47.43	33.89	21.94	2.84	0.18	0.00
		2	49.27	39.10	29.39	8.37	1.66	0.17
		3	50.56	41.61	32.85	11.85	3.25	0.53
		4	51.71	43.46	35.23	14.36	4.62	0.95
-30%	0.75	1	45.86	32.58	20.95	2.67	0.17	0.00
		2	48.00	38.28	28.98	8.53	1.75	0.19
		3	49.92	41.56	33.25	12.65	3.67	0.64
		4	51.86	44.27	36.54	15.94	5.51	1.23

Table 4.4-3: TSPs α (in %) of the return of the standard hedge of a portfolio (T: 100, σ=30%, i=2%)

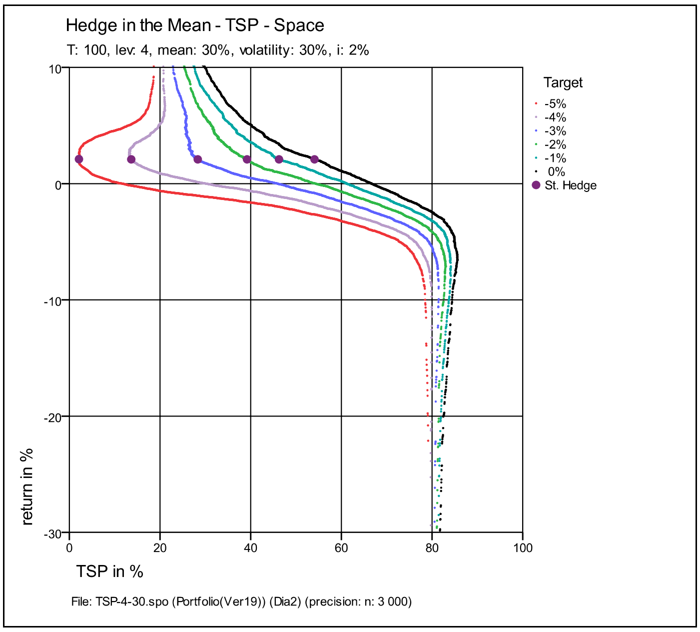

Figure 4.4-1b: Mean–TSP mix of an index (T=100, μ=+30%, σ=30%) and short ETF (λ=4)

In most cases, the underlying index of the inverse ETF is not identical to the assets that have to be hedged. Therefore, the correlation ρ between the return of the underlying index and the assets or portfolio again has to be taken into consideration. Table 4.4-3 depicts the TSP for different means μ, leverage factors λ and correlation ρ=0.75. In general, the probability of falling short of a very negative target becomes greater than in the case of ρ=1.00 (this ρ can be supposed to be the values of Table 4.4-2). However, the TSP for the target τ=0 is always smaller than that in Table 4.4-2. It must be pointed out that the two tables do not have the same targets in the columns.

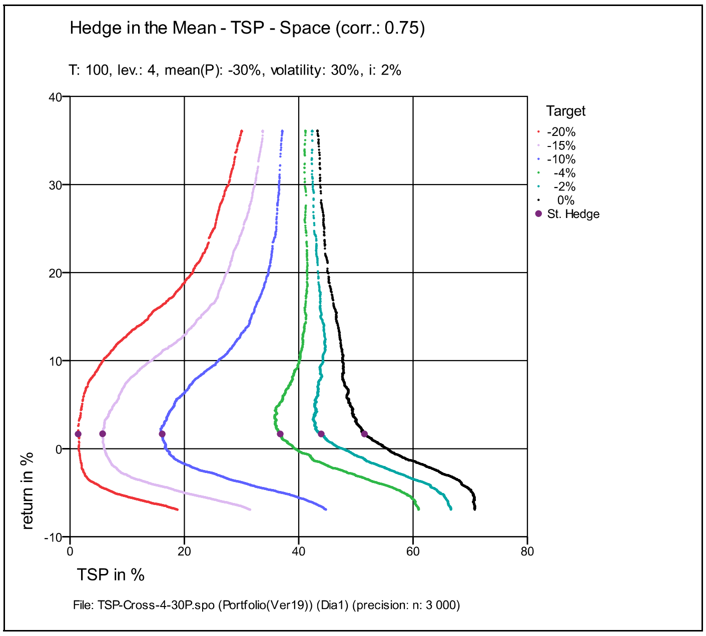

Figure 4.4-2a: Mean–TSP mix of a portfolio (ρ: 0.75; index: T=100, μ=-30%, σ=30%) and leveraged short ETF (λ=4)

To gain some insights into the TSP when portfolios have to be hedged by inverse ETFs, the mean–TSP space is used. In the examples again the leverage factor λ=4 is applied to show the difference between the standard hedge (with x_S=0.20) and the MPH. In Figure 4.4-2a the mean–TSP lines are plotted, when the mean $\mu=\mu_P$=-30%. Such negative expectations may be the reason to start hedging at the beginning of a bearish market. The minimal TSP mix in the chart is more or less where the mean–TSP mix of the standard hedge can be found (bold points). Similar results can be observed for other leverage factors, but for small or high positive returns, this characteristic of the standard hedge can not be found. Figure 4.4-2b shows for a mean $\mu=\mu_P$=+30% that the minimal mean–TSP point is not where the standard hedge is located. Especially the mean–TSP lines for τ=-10% to -20% show that the reduction of the TSP would be possible. This could be achieved if a smaller part of the budget (x_S=0.11 to 0.13) is invested in the inverse ETF. For the target τ=-10% the TSP_{Hedge}=11.33. However, the TSP of the MPH is only TSP_{MPH}=8.57. This feature of the standard hedge seems not to be very important due to the reason that positive expectations will in most cases not initiate hedging transactions. Table 4.4-4 contains for these 2 figures the minimal TSP and the weighting x_S.

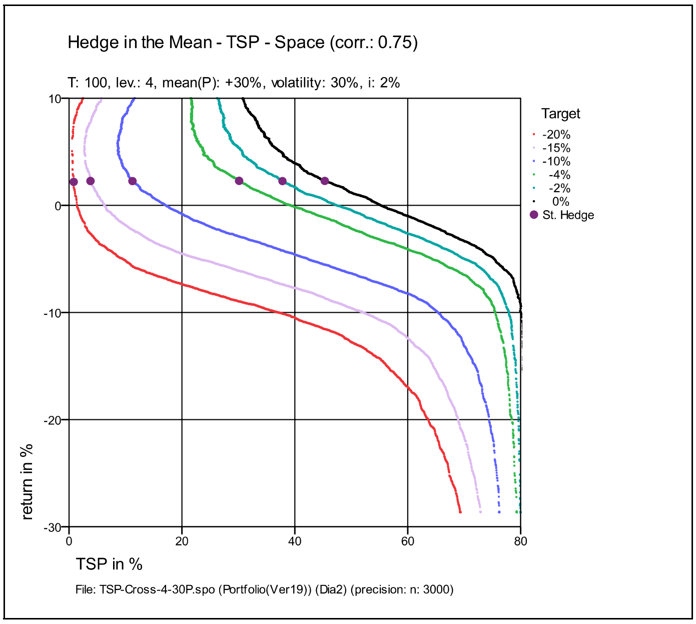

Figure 4.4-2b: Mean–TSP mix of a portfolio (ρ: 0.75; index: T=100, μ=+30%, σ=30%) and leveraged short ETF (λ=4)

	$\mu=\mu_P$=-30%			$\mu=\mu_P$=+30%		
Target	τ=-10%	τ=-15%	τ=-20%	τ=-10%	τ=-15%	τ=-20%
$x_{Portfolio}$	0.79	0.80	0.80	0.89	0.87	0.87
x_S	0.21	0.20	0.20	0.11	0.13	0.13
TSP_{MPH}	15.83	5.60	1.43	8.57	2.67	0.53
$x_{Portfolio}$	0.80	0.80	0.80	0.80	0.80	0.80
x_S	0.20	0.20	0.20	0.20	0.20	0.20
TSP_{Hedge}	15.94	5.51	1.23	11.33	3.41	0.65

Table 4.4-4: TSP of the MPH and standard hedge (ρ=0.75, T=100, lev.: 4, σ=30%, i=2%)[16]

5. Conclusion and preview

The standard hedge that uses weightings only determined by the leverage factor seems to be a good recommendation, even in cases when strong losses are expected. The hedge of the value of an index and a short or leveraged ETF causes positive skewness than cannot be ignored. Therefore, minimizing the variance of the hedge return leads to confusing results: when the underlying index has high losses, the minimal variance hedge (MVH) will elevate the part of this index in the hedge mix (due to the lower volatility) and not the part of the short or leveraged ETF as is usually expected. Furthermore, the standard hedge and the MVH mix their hedges in different ways. In contrast, the risk measure target-shortfall probability (TSP) seems to select in a similar way to the standard hedge. In the case when strong increasing or decreasing index prices are supposed, different leverage factors seem to lead to the same weightings as the standard hedge. This feature of minimizing the TSP can be observed even when a portfolio (which has a higher correlation e.g. ρ=0.75 with the return of the index) has to be hedged by an inverse ETF. The mean return of the underlying index of this ETF is supposed to have a negative development. Due to the skewness the risk measure TSP seems to be more appropriate to reduce the risk investors are afraid of: it is the probability of losing. The volatility is less appropriate for this type of hedging. Therefore, hedging should not be evaluated by this risk measure.

More dynamic hedge approaches – like rebalancing the weightings to those of the standard hedge[17] – have to be evaluated in the same way. Although this hedge approach was not investigated in this paper, some remarks may be added in this place. If the price of an index increases (and respectively decreases) very strongly, the hedge would produce profits that will be avoided by rebalancing. Certainly in these cases the profits will be reduced, but whether the losses will be too depends on the upper and lower price limits, where rebalancing will be performed. If these limits have too great a distance, low transaction costs have to be paid but the losses will remain. Losses occur

[16] Due to the small number of 3000 points in the scatter plot example, the TSP_{MPH} in not always smaller than the TSP_{Hedge}.
[17] See Hill, J., Teller, S. (2010).

when the index price goes to the side.[18] If the limits are close together, the rebalancing will cause higher transaction costs but then the losses can be reduced. The benefit of this dynamic hedging approach needs to be invested in detail.

Another proposal to reduce the losses of a hedge is the integration of a further instrument that is able to compensate for the losses of the hedge discussed in this paper. This instrument could be a short straddle or short strangle, having the index as the underlying feature. As these derivatives generate profit when the index goes aside, part of the losses can be reduced. On the other side, if strong increasing or decreasing index prices occur, the short straddle and the short strangle would have a negative return. Whether this hedging duo (e.g. short ETF and strangle) works well depends among other things on the offered exercise prices in the option market and respectively in the exchange boards. To gain more insights, further research is necessary.

[18] See Michalik Th., Schubert L. (2009).

6. References

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Appendix:

A1: Proof of the immunization of the index value by short ETFs

A2: Proof of formula (2-6): generation of a variable Y_P that has a correlation ρ with Y

A3: Simulated examples for a minimal variance hedge of an index with different means μ

A4: Simulated examples for a minimal variance hedge of a portfolio with different correlations

A1: Proof of the immunization of the index value by short ETFs

In the case of T=1, for leverage factor λ the weightings $x_I=\lambda/(\lambda+1)$ for the index and $x_S=1/(\lambda+1)$ for the short ETF (with $x_I+x_S=1$) immunized the index against changing prices.

Proof:

The value of the hedge H_1 in t=1 depends on the distribution of the budget B to the index (I_0) and to the short ETF (S_0) in t=0. These weightings are denominated by x_I and x_S with $x_I+x_S=1$. The budget B invested in the index is $I_0=B\cdot x_I$ and $S_0=B\cdot x_S$. In t=0 the investment would be $H_0=I_0+S_0=B\cdot x_I+B\cdot x_S=B$. In t=1 the hedge value is

$$H_1 = I_1 \cdot x_I + S_1 \cdot x_S \ . \qquad \text{(A1-1)}$$

The return of I_0 within one day is r_1 and respectively the development factor $I_1/I_0=1+r_1$. The value of the index after one day is $I_1=I_0\cdot(1+r_1)$. The value S_1 is the short ETF and can be determined by equation (2-3):

$$S_1 = S_0 \cdot \left((\lambda+1) - \lambda \cdot \frac{I_1}{I_0} \right) + (\lambda+1) \cdot S_0 \cdot \left(\frac{i}{360} \right) .$$

In I_1 and S_1, the values I_0 and S_0 can be replaced by $I_0=B\cdot x_I$ and $S_0=B\cdot x_S$. By this hedge equation (A1-1) can be transformed to

$$H_1 = B \cdot x_I \cdot (1+r_1) + \left(B \cdot x_S \cdot \left((\lambda+1) - \lambda \cdot (1+r_1) \right) + (\lambda+1) \cdot B \cdot x_S \cdot \left(\frac{i}{360} \right) \right) \qquad \text{(A1-2)}$$

and with $x_I=\lambda/(\lambda+1)$ and respectively $x_S=1/(\lambda+1)$ to

$$H_1 = (B + B \cdot r_1) \cdot \left(\frac{\lambda}{\lambda+1} \right) + \left(B \cdot \lambda + B - B \cdot \lambda - B \cdot \lambda \cdot r_1 + B \cdot \lambda \cdot \left(\frac{i}{360} \right) + B \cdot \left(\frac{i}{360} \right) \right) \cdot \left(\frac{1}{\lambda+1} \right) . \qquad \text{(A1-3)}$$

In equation (A1-3) the part $B\cdot\lambda-B\cdot\lambda$ is zero. Now the equation (A1-3) becomes

$$H_1 = \frac{B \cdot \lambda}{\lambda+1} + \frac{B \cdot \lambda}{\lambda+1} \cdot r_1 + \frac{B}{\lambda+1} - \frac{B \cdot \lambda}{\lambda+1} \cdot r_1 + \frac{B}{\lambda+1} \cdot \lambda \cdot \left(\frac{i}{360} \right) + \frac{B}{\lambda+1} \cdot \left(\frac{i}{360} \right) \text{ or} \qquad \text{(A1-4)}$$

$$H_1 = \frac{B \cdot \lambda}{\lambda+1} + \frac{B}{\lambda+1} + \frac{B}{\lambda+1} \cdot \lambda \cdot \left(\frac{i}{360} \right) + \frac{B}{\lambda+1} \cdot \left(\frac{i}{360} \right) . \qquad \text{(A1-5)}$$

Some transformations show that for T=1 only the interest rate i/360 can be earned by the perfect hedge:

$$H_1 = \frac{B}{\lambda+1} \cdot \left(\lambda + 1 + (\lambda+1) \cdot \left(\frac{i}{360} \right) \right) = B \cdot \left(1 + \frac{i}{360} \right) \quad \blacksquare \qquad \text{(A1-6)}$$

A2: Proof of formula (2-6): generation of a variable Y_P that has a correlation ρ with Y

(variables Y (Y_P) offer the random numbers for the index return (portfolio return)).

For the generation of a stochastic variable Y_P that has a correlation of ρ with another variable Y, a third variable Y' is used. The variables Y and Y' have the same mean μ and variance σ^2. Using normal distributed random numbers y, y', a realization of the variable y_P can be computed[19] by

$$\boxed{y_P = \rho \cdot y + \sqrt{1-\rho^2} \cdot y'} \qquad \text{(A2-1)}$$

<u>Proof</u>:

1.) The variable Y_P will have variance σ^2 like Y and Y':

$$\sigma_P^2 = \rho^2 \sigma^2 + (1 - \rho^2)\, \sigma^2 = \sigma^2. \qquad \text{(A2-2)}$$

The expected value μ_P of Y_P is:

$$\mu_P = \rho \cdot \mu + \sqrt{1-\rho^2} \cdot \mu = \left(\rho + \sqrt{1-\rho^2} \right) \cdot \mu \,. \qquad \text{(A2-3)}$$

[20]

2.) The parameter ρ is the correlation corr(y, y_P):

Equation (A2-1) can be transformed to

$$y' = \frac{1}{\sqrt{1-\rho^2}} \cdot y_P - \frac{\rho}{\sqrt{1-\rho^2}} \cdot y \,. \qquad \text{(A2-4)}$$

The variance σ^2 of Y' is

[19] The formula (A2-1) was mentioned by Siegfried Szeby (2002) without source or proof.

[20] The expected value μ_P is not necessary for the following proof. As Y and Y' are N(0,1) distributed, the expected value $\mu_P=0$ and respectively $Y_P \sim N(0,1)$.

$$\sigma^2 = \frac{1}{1-\rho^2} \cdot \sigma_P{}^2 + \frac{\rho^2}{1-\rho^2} \cdot \sigma^2 - 2 \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{\rho}{\sqrt{1-\rho^2}} \cdot \mathrm{cov}(y, y_P)\,. \qquad \text{(A2-5)}$$

As the variance of the variables Y, Y' and Y_P is σ^2 (see (A2-2)) equation (A2-5) becomes

$$\sigma^2 = \frac{1}{1-\rho^2} \cdot \sigma^2 + \frac{\rho^2}{1-\rho^2} \cdot \sigma^2 - 2 \cdot \frac{\rho}{1-\rho^2} \cdot \mathrm{cov}(y, y_P)\,. \qquad \text{(A2-6)}$$

The division by σ^2 results in

$$1 = \frac{1}{1-\rho^2} + \frac{\rho^2}{1-\rho^2} - \frac{2 \cdot \rho}{1-\rho^2} \cdot \mathrm{corr}(y, y_P)\,. \qquad \text{(A2-7)}$$

The following transformations of (A2-7) depict that $\mathrm{corr}(y,y_P)=\rho$:

$$\frac{2 \cdot \rho}{1-\rho^2} \cdot \mathrm{corr}(y, y_P) = \frac{1}{1-\rho^2} + \frac{\rho^2}{1-\rho^2} - 1, \qquad \text{(A2-7)}$$

$$\mathrm{corr}(y, y_P) = \frac{1-\rho^2}{2 \cdot \rho} \cdot \frac{1}{1-\rho^2} + \frac{1-\rho^2}{2 \cdot \rho} \cdot \frac{\rho^2}{1-\rho^2} - \frac{1-\rho^2}{2 \cdot \rho} \cdot \frac{1-\rho^2}{1-\rho^2}\,, \qquad \text{(A2-8)}$$

$$\mathrm{corr}(r, r_P) = \frac{1}{2 \cdot \rho} + \frac{\rho^2}{2 \cdot \rho} - \frac{1-\rho^2}{2 \cdot \rho} = \frac{1}{2 \cdot \rho} + \frac{\rho^2}{2 \cdot \rho} - \frac{1}{2 \cdot \rho} + \frac{\rho^2}{2 \cdot \rho} = \rho \quad \blacksquare \qquad \text{(A2-9)}$$

A3: Simulated examples for a minimal variance hedge of an index with different means μ

λ / μ	1	2	3	4
30%	x_I=0.454888 x_S=0.545112 corr: -0.975498 r_I: 0.100615 r_S: -0.081332 s_I^2: 0.030671 s_S^2: 0.021453	x_I=0.612047 x_S=0.387953 corr: -0.945504 r_I: 0.100582 r_S: -0.160764 s_I^2: 0.030660 s_S^2: 0.074467	x_I=0.696247 x_S=0.303753 corr: -0.904617 r_I: 0.100556 r_S: -0.233360 s_I^2: 0.030650 s_S^2: 0.149335	x_I=0.752358 x_S=0.247642 corr: -0.852972 r_I: 0.100646 r_S: -0.299933 s_I^2: 0.030675 s_S^2: 0.243411
20%	x_I=0.468730 x_S=0.531270 corr: -0.975524 r_I: 0.070420 r_S: -0.055400 s_I^2: 0.028989 s_S^2: 0.022635	x_I=0.631947 x_S=0.368053 corr: -0.945586 r_I: 0.070473 r_S: -0.112744 s_I^2: 0.029019 s_S^2: 0.083123	x_I=0.720033 x_S=0.279967 corr: -0.904587 r_I: 0.070405 r_S: -0.166436 s_I^2: 0.029013 s_S^2: 0.176312	x_I=0.778494 x_S=0.221506 corr: -0.853781 r_I: 0.070364 r_S: -0.217004 s_I^2: 0.028992 s_S^2: 0.303164
10%	x_I=0.482546 x_S=0.517454 corr: -0.975536 r_I: 0.041133 r_S: -0.028785 s_I^2: 0.027448 s_S^2: 0.023910	x_I=0.651427 x_S=0.348573 corr: -0.945721 r_I: 0.041032 r_S: -0.061817 s_I^2: 0.027431 s_S^2: 0.092699	x_I=0.742503 x_S=0.257497 corr: -0.904749 r_I: 0.041072 r_S: -0.093890 s_I^2: 0.027430 s_S^2: 0.207736	x_I=0.802352 x_S=0.197648 corr: -0.853962 r_I: 0.041068 r_S: -0.124863 s_I^2: 0.027438 s_S^2: 0.377164
5%	x_I=0.489570 x_S=0.510430 corr: -0.975527 r_I: 0.026746 r_S: -0.015179 s_I^2: 0.026679 s_S^2: 0.024568	x_I=0.661004 x_S=0.338996 corr: -0.945601 r_I: 0.026779 r_S: -0.035517 s_I^2: 0.026697 s_S^2: 0.098000	x_I=0.753210 x_S=0.246790 corr: -0.904846 r_I: 0.026696 r_S: -0.055207 s_I^2: 0.026693 s_S^2: 0.225536	x_I=0.813651 x_S=0.186349 corr: -0.853823 r_I: 0.026768 r_S: -0.074756 s_I^2: 0.026716 s_S^2: 0.421736
0%	x_I=0.496573 x_S=0.503427 corr: -0.975566 r_I: 0.012518 r_S: -0.001333 s_I^2: 0.025947 s_S^2: 0.025253	x_I=0.670361 x_S=0.329639 corr: -0.945625 r_I: 0.012619 r_S: -0.008355 s_I^2: 0.025968 s_S^2: 0.103476	x_I=0.763601 x_S=0.236399 corr: -0.904997 r_I: 0.012559 r_S: -0.015111 s_I^2: 0.025953 s_S^2: 0.244657	x_I=0.824362 x_S=0.175638 corr: -0.854101 r_I: 0.012573 r_S: -0.021915 s_I^2: 0.025952 s_S^2: 0.470333
-5%	x_I=0.503456 x_S=0.496544 corr: -0.975539 r_I: -0.001329 r_S: 0.012525 s_I^2: 0.025255 s_S^2: 0.025955	x_I=0.679681 x_S=0.320319 corr: -0.945722 r_I: -0.001411 r_S: 0.019659 s_I^2: 0.025234 s_S^2: 0.109252	x_I=0.773790 x_S=0.226210 corr: -0.904928 r_I: -0.001348 r_S: 0.026698 s_I^2: 0.025253 s_S^2: 0.265870	x_I=0.834341 x_S=0.165659 corr: -0.854289 r_I: -0.001371 r_S: 0.033726 s_I^2: 0.025245 s_S^2: 0.523604
-10%	x_I=0.510437 x_S=0.489563 corr: -0.975562 r_I: -0.015122 r_S: 0.026689 s_I^2: 0.024548 s_S^2: 0.026659	x_I=0.688701 x_S=0.311299 corr: -0.945754 r_I: -0.015114 r_S: 0.048249 s_I^2: 0.024560 s_S^2: 0.115369	x_I=0.783409 x_S=0.216591 corr: -0.905273 r_I: -0.015190 r_S: 0.070405 s_I^2: 0.024545 s_S^2: 0.287943	x_I=0.844223 x_S=0.155777 corr: -0.854200 r_I: -0.015200 r_S: 0.093138 s_I^2: 0.024551 s_S^2: 0.585772
-20%	x_I=0.524364 x_S=0.475636 corr: -0.975571 r_I: -0.042166 r_S: 0.055644 s_I^2: 0.023209 s_S^2: 0.028141	x_I=0.706563 x_S=0.293437 corr: -0.945825 r_I: -0.042163 r_S: 0.108214 s_I^2: 0.023224 s_S^2: 0.128733	x_I=0.802063 x_S=0.197937 corr: -0.905333 r_I: -0.042173 r_S: 0.163265 s_I^2: 0.023210 s_S^2: 0.339243	x_I=0.861931 x_S=0.138069 corr: -0.854764 r_I: -0.042136 r_S: 0.221097 s_I^2: 0.023228 s_S^2: 0.727582
-30%	x_I=0.538233 x_S=0.461767 corr: -0.975611 r_I: -0.068443 r_S: 0.085402 s_I^2: 0.021967 s_S^2: 0.029734	x_I=0.723670 x_S=0.276330 corr: -0.945869 r_I: -0.068367 r_S: 0.171364 s_I^2: 0.021987 s_S^2: 0.143627	x_I=0.819466 x_S=0.180534 corr: -0.905352 r_I: -0.068439 r_S: 0.264305 s_I^2: 0.021967 s_S^2: 0.400104	x_I=0.877992 x_S=0.122008 corr: -0.855066 r_I: -0.068447 r_S: 0.364380 s_I^2: 0.021961 s_S^2: 0.904905

Table A3: MVH: index & short ETF (T=100, σ=30%, i=2%)

A4: Simulated examples for a minimal variance hedge of a portfolio with different correlations

For the examples in appendix A4, a portfolio should be hedged by a short index ETF. The returns of the index and of the portfolio have a correlation ρ. For the following Table A4 the mean return of the index is supposed to be $\mu_P=\mu=5\%$.

ρ T	0.95	0.90	0.85	0.80	0.75
1	x_P=0.499977 x_S=0.500023 corr: -0.949970 r_P: 0.000264 r_S: -0.000155 s_P^2: 0.000250 s_S^2: 0.000250	x_P=0.499997 x_S=0.500003 corr: -0.900055 r_P: 0.000254 r_S: -0.000150 s_P^2: 0.000250 s_S^2: 0.000250	x_P=0.500045 x_S=0.499955 corr: -0.850011 r_P: 0.000273 r_S: -0.000154 s_P^2: 0.000250 s_S^2: 0.000250	x_P=0.499988 x_S=0.500012 corr: -0.799873 r_P: 0.000267 r_S: -0.000156 s_P^2: 0.000250 s_S^2: 0.000250	x_P=0.500085 x_S=0.499915 corr: -0.750050 r_P: 0.000272 r_S: -0.000157 s_P^2: 0.000250 s_S^2: 0.000250
5	x_P=0.499573 x_S=0.500427 corr: -0.949053 r_P: 0.001328 r_S: -0.000772 s_P^2: 0.001254 s_S^2: 0.001250	x_P=0.499549 x_S=0.500451 corr: -0.899077 r_P: 0.001308 r_S: -0.000755 s_P^2: 0.001254 s_S^2: 0.001250	x_P=0.499475 x_S=0.500525 corr: -0.849360 r_P: 0.001332 r_S: -0.000766 s_P^2: 0.001255 s_S^2: 0.001250	x_P=0.499619 x_S=0.500381 corr: -0.799378 r_P: 0.001339 r_S: -0.000771 s_P^2: 0.001255 s_S^2: 0.001251	x_P=0.499564 x_S=0.500436 corr: -0.749495 r_P: 0.001321 r_S: -0.000758 s_P^2: 0.001254 s_S^2: 0.001250
10	x_P=0.499048 x_S=0.500952 corr: -0.947869 r_P: 0.002643 r_S: -0.001521 s_P^2: 0.002516 s_S^2: 0.002497	x_P=0.498994 x_S=0.501006 corr: -0.898047 r_P: 0.002635 r_S: -0.001510 s_P^2: 0.002516 s_S^2: 0.002496	x_P=0.499025 x_S=0.500975 corr: -0.848212 r_P: 0.002655 r_S: -0.001545 s_P^2: 0.002516 s_S^2: 0.002498	x_P=0.498930 x_S=0.501070 corr: -0.798426 r_P: 0.002662 r_S: -0.001552 s_P^2: 0.002515 s_S^2: 0.002496	x_P=0.498985 x_S=0.501015 corr: -0.748629 r_P: 0.002630 r_S: -0.001500 s_P^2: 0.002517 s_S^2: 0.002499
50	x_P=0.494741 x_S=0.505259 corr: -0.938687 r_P: 0.013250 r_S: -0.007602 s_P^2: 0.012903 s_S^2: 0.012388	x_P=0.494503 x_S=0.505497 corr: -0.889540 r_P: 0.013317 r_S: -0.007658 s_P^2: 0.012921 s_S^2: 0.012395	x_P=0.494451 x_S=0.505549 corr: -0.840375 r_P: 0.013327 r_S: -0.007667 s_P^2: 0.012909 s_S^2: 0.012392	x_P=0.494216 x_S=0.505784 corr: -0.791169 r_P: 0.013312 r_S: -0.007611 s_P^2: 0.012918 s_S^2: 0.012394	x_P=0.494208 x_S=0.505792 corr: -0.741927 r_P: 0.013279 r_S: -0.007607 s_P^2: 0.012911 s_S^2: 0.012400
100	x_P=0.489308 x_S=0.510692 corr: -0.927308 r_P: 0.026690 r_S: -0.015115 s_P^2: 0.026670 s_S^2: 0.024560	x_P=0.489022 x_S=0.510978 corr: -0.879130 r_P: 0.026776 r_S: -0.015218 s_P^2: 0.026703 s_S^2: 0.024588	x_P=0.488623 x_S=0.511377 corr: -0.830804 r_P: 0.026769 r_S: -0.015189 s_P^2: 0.026705 s_S^2: 0.024570	x_P=0.488491 x_S=0.511509 corr: -0.782415 r_P: 0.026699 r_S: -0.015155 s_P^2: 0.026679 s_S^2: 0.024577	x_P=0.488038 x_S=0.511962 corr: -0.733946 r_P: 0.026755 r_S: -0.015175 s_P^2: 0.026693 s_S^2: 0.024567
300	x_P=0.466934 x_S=0.533066 corr: -0.883189 r_P: 0.082368 r_S: -0.044818 s_P^2: 0.091251 s_S^2: 0.071105	x_P=0.466112 x_S=0.533888 corr: -0.838286 r_P: 0.082320 r_S: -0.044785 s_P^2: 0.091218 s_S^2: 0.071070	x_P=0.465182 x_S=0.534818 corr: -0.793203 r_P: 0.082428 r_S: -0.044845 s_P^2: 0.091259 s_S^2: 0.071060	x_P=0.464536 x_S=0.535464 corr: -0.748087 r_P: 0.082397 r_S: -0.044863 s_P^2: 0.091251 s_S^2: 0.071178	x_P=0.463433 x_S=0.536567 corr: -0.702106 r_P: 0.082330 r_S: -0.044690 s_P^2: 0.091213 s_S^2: 0.071075

Table A4-1: MVH: portfolio & short index ETF ($\mu_P=\mu=5\%$, $\sigma=30\%$, i=2%, leverage factor: 1)

ρ / T	0.95	0.90	0.85	0.80	0.75
100	x_P=0.818042 x_S=0.181958 corr: -0.813167 r_P: 0.026695 r_S: -0.074771 s_P^2: 0.026671 s_S^2: 0.421193	x_P=0.822466 x_S=0.177534 corr: -0.772389 r_P: 0.026741 r_S: -0.074904 s_P^2: 0.026687 s_S^2: 0.421062	x_P=0.827010 x_S=0.172990 corr: -0.731472 r_P: 0.026723 r_S: -0.074830 s_P^2: 0.026687 s_S^2: 0.420565	x_P=0.831961 x_S=0.168039 corr: -0.689847 r_P: 0.026717 r_S: -0.074839 s_P^2: 0.026673 s_S^2: 0.420828	x_P=0.836880 x_S=0.163120 corr: -0.648594 r_P: 0.026771 r_S: -0.075030 s_P^2: 0.026684 s_S^2: 0.420772

Table A4-2: MVH: portfolio & short index ETF (μ_P=μ=5%, σ=30%, i=2%, leverage factor: 4)