

Product and Quotient of Independent Gauss Hypergeometric Variables

Produto e Quociente de Variáveis Independentes Gauss Hipergeométrica

Producto y Cociente de Variables Independientes Hipergeométrica de Gauss

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Abstract

In this article, we have derived the probability density functions of the product and the quotient of two independent random variables having Gauss hypergeometric distribution. These densities have been expressed in terms of Appell's first hypergeometric function F_1 . Further, Rényi and Shannon entropies have also been derived for the Gauss hypergeometric distribution.

Key words: Appell's first hypergeometric function, beta distribution, Gauss hypergeometric distribution, quotient, transformation.

Resumo

Neste artigo, vamos derivar as funções de densidade de probabilidade do produto e o quociente de duas variáveis aleatórias independentes com distribuição

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hipergeométrica de Gauss. Estas densidades foram expressas em termos da primeira função hipergeométrica de Appell F_1 . Além disso, entropias de Rényi e Shannon também foram derivadas para a distribuição hipergeométrica de Gauss.

Palavras chaves: Primeira função hipergeométrica de Appell, distribuição beta, função hipergeométrica de Gauss, quociente, produto, transformação.

Resumen

En este artículo, hemos derivado las funciones de densidad de probabilidad del producto y el cociente de dos variables aleatorias independientes que tienen una distribución hipergeométrica de Gauss. Estas densidades se han expresado en términos de la primera función hipergeométrica de Appell F_1 . Además, entropías Rényi y Shannon también se han derivado de la distribución hipergeométrica de Gauss.

Palabras claves: Primera función hipergeométrica de Appell, beta distribución, distribución hipergeométrica de Gauss, cociente, transformación.

1 Introduction

A random variable X is said to have a beta (type 1) distribution with parameters α and β if its probability density function (pdf) is given by

$$B1(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad (1)$$

where $\alpha > 0$ and $\beta > 0$, and

$$\begin{aligned} B(\alpha, \beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \quad \operatorname{Re}(\beta) > 0, \end{aligned}$$

denotes the beta function. The beta distribution is very versatile and a variety of uncertainties can be usefully modeled by it. Many of the finite range distributions encountered in practice can easily be transformed into the standard beta distribution. Several univariate and matrix variate generalizations of this distribution are given in Gordy [1], Gupta and Nagar [2], Johnson, Kotz and Balakrishnan [3], McDonald and Xu [4], and Nagar and Zarrazola [5]. A natural univariate generalization of the beta distribution is the Gauss hypergeometric distribution defined by Armero and Bayarri [6]. The random

variable X is said to have a Gauss hypergeometric distribution, denoted by $X \sim \text{GH}(\alpha, \beta, \gamma, \xi)$, if its density function is given by

$$f_{\text{GH}}(x; \alpha, \beta, \gamma, \xi) = C(\alpha, \beta, \gamma, \xi) \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1+\xi x)^\gamma}, \quad 0 < x < 1, \quad (2)$$

where $\alpha > 0$, $\beta > 0$, $-\infty < \gamma < \infty$ and $\xi > -1$. The normalizing constant $C(\alpha, \beta, \gamma, \xi)$ is given by

$$\{C(\alpha, \beta, \gamma, \xi)\}^{-1} = B(\alpha, \beta) {}_2F_1(\alpha, \gamma; \alpha + \beta; -\xi), \quad (3)$$

where ${}_2F_1$ represents the Gauss hypergeometric function (Luke [7]). Note that the Gauss hypergeometric function ${}_2F_1$ in (3) can be expanded in series form if $-1 < \xi < 1$. However, if $\xi > 1$, then we use suitably (7) to rewrite ${}_2F_1$ to have absolute value of the argument less than one.

The above distribution was suggested by Armero and Bayarri [6] in connection with the prior distribution of the parameter ρ , $0 < \rho < 1$, which represents the traffic intensity in a $M/M/1$ queueing system. A brief introduction of this distribution is given in the encyclopedic work of Johnson, Kotz and Balakrishnan [3, p. 253]. In the context of Bayesian analysis of unreported Poisson count data, while deriving the marginal posterior distribution of the reporting probability p , Fader and Hardie [8] have shown that $q = 1 - p$ has a Gauss hypergeometric distribution. The Gauss hypergeometric distribution has also been used by Dauxois [9] to introduce conjugate priors in the Bayesian inference for linear growth birth and death processes. Sarabia and Castillo [10] have pointed out that this distribution is conjugate prior for the binomial distribution.

Gauss hypergeometric distribution reduces to a beta type 1 distribution when either γ or ξ equals to zero. Further, for $\gamma = \alpha + \beta$ and $\xi = 1$, the Gauss hypergeometric distribution simplifies to a beta type 3 distribution given by the density (Cardaño, Nagar and Sánchez [11], Sánchez and Nagar [12]),

$$\text{B3}(x; \alpha, \beta) = \frac{2^\alpha x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)(1+x)^{\alpha+\beta}}, \quad 0 < x < 1, \quad (4)$$

where $\alpha > 0$ and $\beta > 0$. For $\gamma = \alpha + \beta$ and $\xi = -(1 - \lambda)$ it slides to a three parameter generalized beta distribution defined by the density

$$\text{B1}(x; \alpha, \beta; \lambda) = \frac{\lambda^\alpha x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)[1 - (1 - \lambda)x]^{\alpha+\beta}}, \quad 0 < x < 1, \quad (5)$$

where $\alpha > 0$ and $\beta > 0$. This distribution was defined and used by Libby and Novic [13] for utility function fitting. The beta distribution sometimes does not provide sufficient flexibility for a prior for probability of success in a binomial distribution. Among various properties, the distribution defined by the density (5) can more flexibly account for heavy tails or skewness, and it reduces to the ordinary beta (type 1) distribution for certain parameter choices. The resulting posterior distribution in this case is a four-parameter type of beta. Chen and Novic [14] provided tables as evidence for its usefulness. Several properties and special cases of this distribution are given in Johnson, Kotz and Balakrishnan [3, p. 251]. For further results and properties, the reader is referred to Aryal and Nadarajah [15], Nadarajah [16], Nagar and Rada-Mora [17], Pham-Gia and Duong [18], and Sarabia and Castillo [10].

In this article, we derive distributions of the product and the ratio of two independent random variables when at least one of them is Gauss hypergeometric. We also study several properties of this distribution including Rényi and Shannon entropies.

2 Some Known Definitions and Results

In this section, we give definitions and results that are used in subsequent sections. Throughout this work we will use the Pochhammer coefficient $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \dots$, and $(a)_0 = 1$. The integral representation of the Gauss hypergeometric function is given as

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt, \\ \operatorname{Re}(c) > \operatorname{Re}(a) > 0, |\arg(1-z)| < \pi. \quad (6)$$

Note that, by expanding $(1-zt)^{-b}$, $|zt| < 1$, in (6) and integrating t , the series expansion for ${}_2F_1$ can be obtained as

$${}_2F_1(a, b; c; z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{z^r}{r!}, \quad |z| < 1.$$

The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ satisfies Euler's relation

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right) \\ &= (1 - z)^{-b} {}_2F_1\left(c - a, b; c; \frac{z}{z - 1}\right) \\ &= (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z). \end{aligned} \tag{7}$$

For properties and further results the reader is referred to Luke [7].

The Appell's first hypergeometric function F_1 is defined by

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{r_1, r_2=0}^{\infty} \frac{(a)_{r_1+r_2} (b_1)_{r_1} (b_2)_{r_2} z_1^{r_1} z_2^{r_2}}{(c)_{r_1+r_2} r_1! r_2!}, \tag{8}$$

where $|z_1| < 1$ and $|z_2| < 1$. It is straightforward to show that

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= \sum_{r_1=0}^{\infty} \frac{(a)_{r_1} (b_1)_{r_1} z_1^{r_1}}{(c)_{r_1} r_1!} {}_2F_1(a + r_1, b_2; c + r_1; z_2) \\ &= \sum_{r_2=0}^{\infty} \frac{(a)_{r_2} (b_2)_{r_2} z_2^{r_2}}{(c)_{r_2} r_2!} {}_2F_1(a + r_2, b_1; c + r_2; z_1), \end{aligned} \tag{9}$$

where ${}_2F_1$ is the Gauss hypergeometric series. Using the results

$$\frac{(a)_{r_1+r_2}}{(c)_{r_1+r_2}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a+r_1+r_2-1} (1-v)^{c-a-1} dv,$$

where $\text{Re}(c) > \text{Re}(a) > 0$,

$$\sum_{r_i=0}^{\infty} \frac{(b_i)_{r_i} (vz_i)^{r_i}}{r_i!} = (1 - vz_i)^{-b_i}, \quad |vz_i| < 1, \quad i = 1, 2,$$

in (8), one obtains

$$F_1(a, b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1} (1-v)^{c-a-1} dv}{(1-vz_1)^{b_1} (1-vz_2)^{b_2}}, \tag{10}$$

$|z_1| < 1, |z_2| < 1,$

where $\text{Re}(c) > \text{Re}(a) > 0$. Note that for $b_1 = 0$, F_1 reduces to a ${}_2F_1$ function. For properties and further results of these function the reader is referred to Srivastava and Karlsson [19].

Lemma 2.1. *Let*

$$I(a, b, d, p, q; \theta_1, \theta_2, z) = \int_0^1 \frac{v^{a-1}(1-zv)^{b-1}(1-v)^{d-1} dv}{(1+\theta_1zv)^p(1+\theta_2v)^q}. \quad (11)$$

Then, for $a > 0, d > 0$ and $0 < z < 1$, we have

$$I(a, b, d, p, q; \theta_1, \theta_2, z) = \frac{B(a, d)}{(1+\theta_2)^q(1+\theta_1z)^p} \sum_{r=0}^{\infty} \frac{(1-b)_r(a)_r}{(a+d)_r r!} z^r \\ \times F_1 \left(d, p, q; a+d+r; \frac{\theta_1z}{1+\theta_1z}, \frac{\theta_2}{1+\theta_2} \right).$$

Proof. Writing

$$(1+\theta_1zv)^{-p} = (1+\theta_1z)^{-p} \left[1 - \frac{\theta_1z}{1+\theta_1z}(1-v) \right]^{-p},$$

$$(1+\theta_2v)^{-q} = (1+\theta_2)^{-q} \left[1 - \frac{\theta_2}{1+\theta_2}(1-v) \right]^{-q},$$

and

$$(1-zv)^{b-1} = \sum_{r=0}^{\infty} \frac{(1-b)_r}{r!} (zv)^r$$

in (11) and substituting $v = 1 - w$, we obtain

$$I(a, b, d, p, q; \theta_1, \theta_2, z) = \frac{1}{(1+\theta_2)^q(1+\theta_1z)^p} \sum_{r=0}^{\infty} \frac{(1-b)_r}{r!} z^r \\ \times \int_0^1 \frac{(1-w)^{a+r-1} w^{d-1} dw}{[1-\theta_1zw/(1+\theta_1z)]^p [1-\theta_2w/(1+\theta_2)]^q}. \quad (12)$$

Now, using the definition of the Appell's first hypergeometric function, we get the desired result. \square

3 Properties

The graph of the Gauss hypergeometric density for different values of the parameters is shown in the Figure 1.

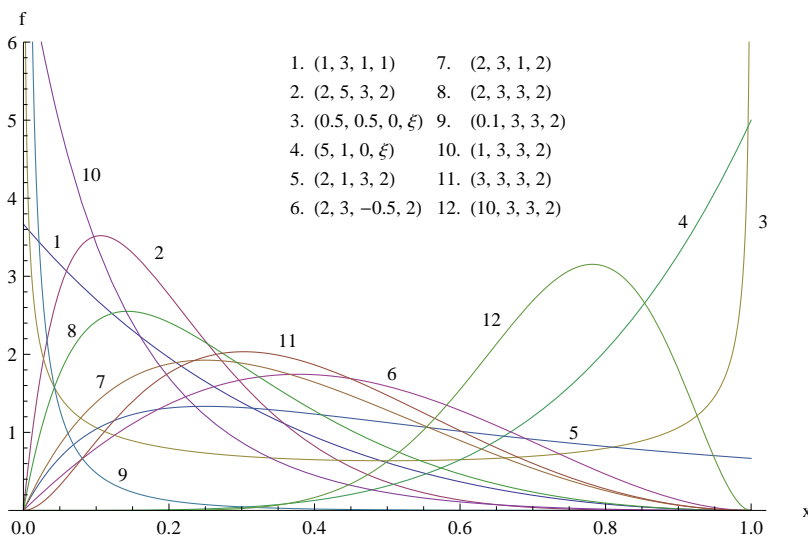


Figure 1: Graph of the Gauss hypergeometric density $f_{GH}(x; \alpha, \beta, \gamma, \xi)$ for different values of $(\alpha, \beta, \gamma, \xi)$.

The k -th moment of the variable X having Gauss hypergeometric distribution, obtained in Armero and Bayarri [6], can be calculated as

$$\begin{aligned}
 E(X^k) &= C(\alpha, \beta, \gamma, \xi) \int_0^1 \frac{x^{\alpha+k-1}(1-x)^{\beta-1}}{(1+\xi x)^\gamma} dx \\
 &= \frac{\Gamma(\alpha+k)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+k)} \frac{{}_2F_1(\gamma, \alpha+k; \alpha+\beta+k; -\xi)}{{}_2F_1(\gamma, \alpha; \alpha+\beta; -\xi)} \quad (13)
 \end{aligned}$$

from which the expected value and the variance of this distribution are obtained as

$$E(X) = \frac{\alpha}{\alpha+\beta} \frac{{}_2F_1(\gamma, \alpha+1; \alpha+\beta+1; -\xi)}{{}_2F_1(\gamma, \alpha; \alpha+\beta; -\xi)},$$

$$E(X^2) = \frac{\alpha(\alpha + 1)}{(\alpha + \beta)(\alpha + \beta + 1)} \frac{{}_2F_1(\gamma, \alpha + 2; \alpha + \beta + 2; -\xi)}{{}_2F_1(\gamma, \alpha; \alpha + \beta; -\xi)},$$

$$\begin{aligned} \text{Var}(X) = & \frac{\alpha}{\alpha + \beta} \left[\frac{\alpha + 1}{\alpha + \beta + 1} \frac{{}_2F_1(\gamma, \alpha + 2; \alpha + \beta + 2; -\xi)}{{}_2F_1(\gamma, \alpha; \alpha + \beta; -\xi)} \right. \\ & \left. - \frac{\alpha}{\alpha + \beta} \left\{ \frac{{}_2F_1(\gamma, \alpha + 1; \alpha + \beta + 1; -\xi)}{{}_2F_1(\gamma, \alpha; \alpha + \beta; -\xi)} \right\}^2 \right]. \end{aligned}$$

Moreover, the cumulative distribution function (CDF) can be derived in terms of special functions as shown in the following theorem.

Theorem 3.1. *Let $X \sim \text{GH}(\alpha, \beta, \gamma, \xi)$. Then, the CDF of X is given by*

$$\frac{x^\alpha}{\alpha B(\alpha, \beta)} \frac{F_1(\alpha, 1 - \beta, \gamma; \alpha + 1; x, -\xi x)}{{}_2F_1(\alpha, \gamma; \alpha + \beta; -\xi)}, \quad 0 < x < 1.$$

Proof. The CDF of X is evaluated as

$$P(X \leq x) = C(\alpha, \beta, \gamma, \xi) \int_0^x \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(1+\xi t)^\gamma} dt.$$

By making the substitution $u = t/x$, we can express the above integral as

$$C(\alpha, \beta, \gamma, \xi) x^\alpha \int_0^1 \frac{u^{\alpha-1}(1-u)^{(\alpha+1)-\alpha-1}}{(1-xu)^{1-\beta}(1+\xi xu)^\gamma} du.$$

Now, using the integral representation (10) of F_1 , substituting for $C(\alpha, \beta, \gamma, \xi)$ from (3) and simplifying, we obtain the desired result. \square

The following theorem suggests a generalized beta type 2 distribution, from the Gauss hypergeometric distribution.

Theorem 3.2. *Let $X \sim \text{GH}(\alpha, \beta, \gamma, \xi)$. Then, the pdf of the random variable $Y = X/(1 - X)$ is given by*

$$C(\alpha, \beta, \gamma, \xi) \frac{y^{\alpha-1}(1+y)^{\gamma-(\alpha+\beta)}}{[1+\xi(1+y)]^\gamma}, \quad y > 0, \quad (14)$$

where $C(\alpha, \beta, \gamma, \xi)$ is the normalizing constant given by (3).

Proof. Making the transformation $Y = X/(1 - X)$, with the Jacobian $J(x \rightarrow y) = (1 + y)^{-2}$ in (2), we get the desired result. \square

As expected, if in the density (14) we take $\gamma = 0$ or $\xi = 0$ with $\beta > \gamma$, then we obtain the beta type 2 density.

4 Entropies

In this section, exact forms of Rényi and Shannon entropies are obtained for the Gauss hypergeometric distribution defined in Section 1.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a pdf f associated with \mathcal{P} , dominated by σ -finite measure μ on \mathcal{X} . Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [20]. It is defined by

$$H_{SH}(f) = - \int_{\mathcal{X}} f(x) \log f(x) d\mu. \tag{15}$$

One of the main extensions of the Shannon entropy was defined by Rényi [21]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\log G(\eta)}{1 - \eta} \quad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \tag{16}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^\eta d\mu.$$

The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η , while Shannon entropy (15) is obtained from (16) for $\eta \uparrow 1$. For details see Nadarajah and Zografos [22], Zografos [23], and Zografos and Nadarajah [24].

First, we give the following lemma useful in deriving these entropies.

Lemma 4.1. *Let $g(\alpha, \beta, \gamma, \xi) = \lim_{\eta \rightarrow 1} h(\eta)$, where*

$$h(\eta) = \frac{d}{d\eta} {}_2F_1(\eta(\alpha - 1) + 1, \eta\gamma; \eta(\alpha + \beta - 2) + 2; -\xi). \tag{17}$$

Then, for $-1 < \xi < 1$, we have

$$g(\alpha, \beta, \gamma, \xi) = \sum_{j=1}^{\infty} \frac{\Gamma(\alpha + j)\Gamma(\gamma + j)\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\gamma)\Gamma(\alpha + \beta + j)} \frac{(-\xi)^j}{j!} [(\alpha - 1)\psi(\alpha + j) + \gamma\psi(\gamma + j) + (\alpha + \beta - 2)\psi(\alpha + \beta) - (\alpha - 1)\psi(\alpha) - \gamma\psi(\gamma) - (\alpha + \beta - 2)\psi(\alpha + \beta + j)], \quad (18)$$

and for $\xi \geq 1$,

$$g(\alpha, \beta, \gamma, \xi) = (1 + \xi)^{-\gamma} \sum_{j=1}^{\infty} \frac{\Gamma(\beta + j)\Gamma(\gamma + j)\Gamma(\alpha + \beta)}{\Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha + \beta + j)} \left(\frac{\xi}{1 + \xi}\right)^j \times [-\gamma \log(1 + \xi) + (\beta - 1)\psi(\beta + j) + \gamma\psi(\gamma + j) + (\alpha + \beta - 2)\psi(\alpha + \beta) - (\beta - 1)\psi(\beta) - \gamma\psi(\gamma) - (\alpha + \beta - 2)\psi(\alpha + \beta + j)], \quad (19)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.

Proof. Using the series expansion of ${}_2F_1$, for $-1 < \xi < 1$, we write

$$h(\eta) = \frac{d}{d\eta} \sum_{j=0}^{\infty} \Delta_j(\eta) \frac{(-\xi)^j}{j!} = \sum_{j=0}^{\infty} \left[\frac{d}{d\eta} \Delta_j(\eta) \right] \frac{(-\xi)^j}{j!}, \quad (20)$$

where

$$\Delta_j(\eta) = \frac{\Gamma[\eta(\alpha - 1) + 1 + j]\Gamma(\eta\gamma + j)\Gamma[\eta(\alpha + \beta - 2) + 2]}{\Gamma[\eta(\alpha - 1) + 1]\Gamma(\eta\gamma)\Gamma[\eta(\alpha + \beta - 2) + 2 + j]}.$$

Now, differentiating the logarithm of $\Delta_j(\eta)$ w.r.t. η , we arrive at

$$\begin{aligned} \frac{d}{d\eta} \Delta_j(\eta) &= \Delta_j(\eta) [(\alpha - 1)\psi(\eta(\alpha - 1) + 1 + j) + \gamma\psi(\eta\gamma + j) \\ &\quad + (\alpha + \beta - 2)\psi(\eta(\alpha + \beta - 2) + 2) - (\alpha - 1)\psi(\eta(\alpha - 1) + 1) \\ &\quad - \gamma\psi(\eta\gamma) - (\alpha + \beta - 2)\psi(\eta(\alpha + \beta - 2) + 2 + j)]. \end{aligned} \quad (21)$$

Finally, substituting (21) in (20) and taking $\eta \rightarrow 1$, we obtain (18). To obtain (19), we use (7) to write

$$\begin{aligned} &{}_2F_1(\eta(\alpha - 1) + 1, \eta\gamma; \eta(\alpha + \beta - 2) + 2; -\xi) \\ &= (1 + \xi)^{-\eta\gamma} {}_2F_1\left(\eta(\beta - 1) + 1, \eta\gamma; \eta(\alpha + \beta - 2) + 2; \frac{\xi}{1 + \xi}\right) \end{aligned} \quad (22)$$

and proceed similarly. □

Theorem 4.1. *For the Gauss hypergeometric distribution defined by the pdf (2), the Rényi and the Shannon entropies are given by*

$$\begin{aligned}
 H_R(\eta, f_{GH}) &= \frac{1}{1-\eta} [\eta \log C(\alpha, \beta, \gamma, \xi) + \log \Gamma[\eta(\alpha - 1) + 1] \\
 &\quad + \log \Gamma[\eta(\beta - 1) + 1] - \log \Gamma[\eta(\alpha + \beta - 2) + 2] \\
 &\quad + \log {}_2F_1(\eta(\alpha - 1) + 1, \eta\gamma; \eta(\alpha + \beta - 2) + 2; -\xi)] \quad (23)
 \end{aligned}$$

and

$$\begin{aligned}
 H_{SH}(f_{GH}) &= -\log C(\alpha, \beta, \gamma, \xi) - [(\alpha - 1)\psi(\alpha) + (\beta - 1)\psi(\beta) \\
 &\quad - (\alpha + \beta - 2)\psi(\alpha + \beta)] - \frac{g(\alpha, \beta, \gamma, \xi)}{{}_2F_1(\alpha, \gamma; \alpha + \beta; -\xi)}, \quad (24)
 \end{aligned}$$

respectively, where for $-1 < \xi < 1$, $g(\alpha, \beta, \gamma, \xi)$ is given by (18), and for $\xi \geq 1$ $g(\alpha, \beta, \gamma, \xi)$ is given by (19).

Proof. For $\eta > 0$ and $\eta \neq 1$, using the density of X given by (2), we have

$$\begin{aligned}
 G(\eta) &= \int_0^1 [f_{GH}(x; \alpha, \beta, \gamma, \xi)]^\eta dx \\
 &= [C(\alpha, \beta, \gamma, \xi)]^\eta \int_0^1 \frac{x^{\eta(\alpha-1)}(1-x)^{\eta(\beta-1)}}{(1+\xi x)^{\eta\gamma}} dx \\
 &= \frac{[C(\alpha, \beta, \gamma, \xi)]^\eta}{C(\eta(\alpha - 1) + 1, \eta(\beta - 1) + 1; \eta\gamma, \xi)} \\
 &= [C(\alpha, \beta, \gamma, \xi)]^\eta \frac{\Gamma[\eta(\alpha - 1) + 1]\Gamma[\eta(\beta - 1) + 1]}{\Gamma[\eta(\alpha + \beta - 2) + 2]} \\
 &\quad \times {}_2F_1(\eta(\alpha - 1) + 1, \eta\gamma; \eta(\alpha + \beta - 2) + 2; -\xi),
 \end{aligned}$$

where the last line has been obtained by using (3). Now, taking logarithm of $G(\eta)$ and using (16) we get (23). The Shannon entropy is obtained from (23) by taking $\eta \uparrow 1$ and using L'Hopital's rule. □

5 Distribution of The Product

In this section, we obtain distributional results for the product of two independent random variables involving Gauss hypergeometric distribution.

Theorem 5.1. *Let X_1 and X_2 be independent, $X_i \sim \text{GH}(\alpha_i, \beta_i, \gamma_i, \xi_i)$, $i = 1, 2$. Then, the pdf of $Z = X_1 X_2$ is given by*

$$K_1 B(\beta_1, \beta_2) \frac{z^{\alpha_1-1} (1-z)^{\beta_1+\beta_2-1}}{(1+\xi_2)^{\gamma_2} (1+\xi_1 z)^{\gamma_1}} \sum_{r=0}^{\infty} \frac{(\alpha_1 + \beta_1 - \gamma_1 - \alpha_2)_r (\beta_2)_r}{(\beta_1 + \beta_2)_r r!} \\ \times (1-z)^r F_1 \left(\beta_2 + r, \gamma_1, \gamma_2; \beta_1 + \beta_2 + r; \frac{1-z}{1+\xi_1 z}, \frac{(1-z)\xi_2}{1+\xi_2} \right), \quad 0 < z < 1,$$

where

$$K_1 = [B(\alpha_1, \beta_1) B(\alpha_2, \beta_2) {}_2F_1(\alpha_1, \gamma_1; \alpha_1 + \beta_1; -\xi_1) {}_2F_1(\alpha_2, \gamma_2; \alpha_2 + \beta_2; -\xi_2)]^{-1}.$$

Proof. Using the independence, the joint pdf of X_1 and X_2 is given by

$$K_1 \frac{x_1^{\alpha_1-1} (1-x_1)^{\beta_1-1} x_2^{\alpha_2-1} (1-x_2)^{\beta_2-1}}{(1+\xi_1 x_1)^{\gamma_1} (1+\xi_2 x_2)^{\gamma_2}}, \quad 0 < x_i < 1, \quad i = 1, 2. \quad (25)$$

Transforming $Z = X_1 X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \rightarrow z, x_2) = 1/x_2$, we obtain the joint pdf of Z and X_2 as

$$K_1 \frac{z^{\alpha_1-1} x_2^{\gamma_1+\alpha_2-\alpha_1-\beta_1} (x_2-z)^{\beta_1-1} (1-x_2)^{\beta_2-1}}{(x_2+\xi_1 z)^{\gamma_1} (1+\xi_2 x_2)^{\gamma_2}}, \quad 0 < z < x_2 < 1. \quad (26)$$

To find the marginal pdf of Z , we integrate (26) with respect to x_2 to get

$$K_1 z^{\alpha_1-1} \int_z^1 \frac{x_2^{\gamma_1+\alpha_2-\alpha_1-\beta_1} (x_2-z)^{\beta_1-1} (1-x_2)^{\beta_2-1} dx_2}{(x_2+\xi_1 z)^{\gamma_1} (1+\xi_2 x_2)^{\gamma_2}}. \quad (27)$$

In (27) change of variable $w = (1 - x_2)/(1 - z)$ yields

$$\begin{aligned} & K_1 \frac{z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{(1+\xi_2)^{\gamma_2}(1+\xi_1 z)^{\gamma_1}} \\ & \times \int_0^1 \frac{w^{\beta_2-1}(1-w)^{\beta_1-1}[1-(1-z)w]^{-(\alpha_1+\beta_1-\gamma_1-\alpha_2)} dw}{[1-(1-z)w/(1+\xi_1 z)]^{\gamma_1}[1-(1-z)\xi_2 w/(1+\xi_2)]^{\gamma_2}} \\ & = K_1 \frac{z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{(1+\xi_2)^{\gamma_2}(1+\xi_1 z)^{\gamma_1}} \sum_{r=0}^{\infty} \frac{(\alpha_1+\beta_1-\gamma_1-\alpha_2)_r}{r!} (1-z)^r \\ & \times \int_0^1 \frac{w^{\beta_2+r-1}(1-w)^{\beta_1-1} dw}{[1-(1-z)w/(1+\xi_1 z)]^{\gamma_1}[1-(1-z)\xi_2 w/(1+\xi_2)]^{\gamma_2}}, \end{aligned}$$

where the last step has been obtained by expanding $[1-(1-z)w]^{-(\alpha_1+\beta_1-\gamma_1-\alpha_2)}$ in power series. Finally, applying (10), we obtain the desired result. \square

Corollary 5.1.1. *Let $X_1 \sim \text{GH}(\alpha_1, \beta_1, \gamma_1, \xi_1)$ and $X_2 \sim \text{B1}(\alpha_2, \beta_2)$ be independent. Then, the pdf of $Z = X_1 X_2$ is*

$$\begin{aligned} & \frac{B(\beta_1, \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} {}_2F_1(\alpha_1, \gamma_1; \alpha_1 + \beta_1; -\xi_1) \frac{z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{(1+\xi_1 z)^{\gamma_1}} \\ & \times F_1\left(\beta_2, \alpha_1 + \beta_1 - \alpha_2 - \gamma_1, \gamma_1; \beta_1 + \beta_2; 1-z, \frac{1-z}{1+\xi_1 z}\right), \quad 0 < z < 1. \end{aligned}$$

Proof. Substituting $\gamma_2 = 0$ in Theorem 5.1 and using (9) we get the desired result. \square

Corollary 5.1.2. *Let the random variables X_1 and X_2 be independent, $X_1 \sim \text{B1}(\alpha_1, \beta_1)$ and $X_2 \sim \text{B3}(\alpha_2, \beta_2)$. If $\alpha_2 = \alpha_1 + \beta_1$, then the pdf of $Z = X_1 X_2$ is given by*

$$\frac{z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{2^{\beta_2} B(\alpha_1, \beta_1 + \beta_2)} {}_2F_1\left(\beta_2, \alpha_1 + \beta_1 + \beta_2; \beta_1 + \beta_2; \frac{1-z}{2}\right), \quad 0 < z < 1.$$

The graphs of the pdf of $Z = X_1 X_2$, where X_1 and X_2 are independent, $X_1 \sim \text{GH}(\alpha_1, \beta_1, \gamma_1, \xi_1)$ and $X_2 \sim \text{B1}(\alpha_2, \beta_2)$ for different values of $(\alpha_1, \beta_1, \gamma_1, \xi_1, \alpha_2, \beta_2)$ for different values of the parameters is shown in the Figure 2. Further, Figure 3 depicts graphs of the density of $Z = X_1 X_2$, where

X_1 and X_2 are independent, $X_1 \sim B1(\alpha_1, \beta_1)$ and $X_2 \sim B3(\alpha_1 + \beta_1, \beta_2)$ for different values of $(\alpha_1, \beta_1, \beta_2)$.

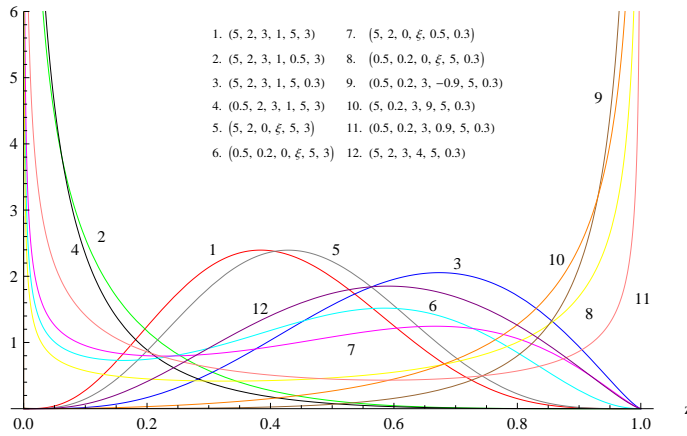


Figure 2: Graph of the density of $Z = X_1X_2$, where X_1 and X_2 are independent, $X_1 \sim GH(\alpha_1, \beta_1, \gamma_1, \xi_1)$ and $X_2 \sim B1(\alpha_2, \beta_2)$ for different values of $(\alpha_1, \beta_1, \gamma_1, \xi_1, \alpha_2, \beta_2)$.

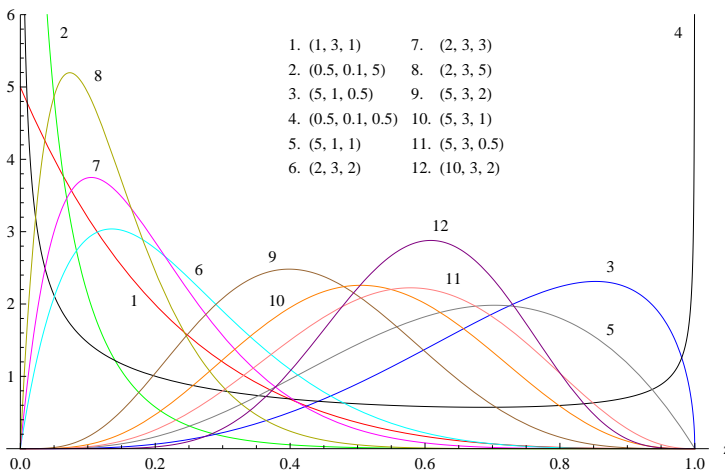


Figure 3: Graph of the density of $Z = X_1X_2$, where X_1 and X_2 are independent, $X_1 \sim B1(\alpha_1, \beta_1)$ and $X_2 \sim B3(\alpha_1 + \beta_1, \beta_2)$ for different values of $(\alpha_1, \beta_1, \beta_2)$.

Theorem 5.2. *Let the random variables X_1 and X_2 be independent, $X_1 \sim \text{GH}(\alpha_1, \beta_1, \gamma_1, \xi_1)$ and $X_2 \sim \text{B2}(\alpha_2, \beta_2)$. Then, the pdf of $Z = X_1X_2$ is given by*

$$K_2 \frac{B(\beta_1, \alpha_1 + \beta_2)}{(1 + \xi_1)^{\gamma_1}} \frac{z^{\alpha_2-1}}{(1 + z)^{\alpha_2+\beta_2}} \times F_1 \left(\beta_1, \gamma_1, \alpha_2 + \beta_2; \alpha_1 + \beta_1 + \beta_2; \frac{\xi_1}{1 + \xi_1}, \frac{1}{1 + z} \right), \quad z > 0,$$

where

$$K_2 = \{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)_2F_1(\alpha_1, \gamma_1; \alpha_1 + \beta_1; -\xi_1)\}^{-1}.$$

Proof. Since X_1 and X_2 are independent, their joint pdf is given by

$$K_2 \frac{x_1^{\alpha_1-1}(1 - x_1)^{\beta_1-1}x_2^{\alpha_2-1}}{(1 + \xi_1x_1)^{\gamma_1}(1 + x_2)^{\alpha_2+\beta_2}}, \quad 0 < x_1 < 1, \quad x_2 > 0.$$

Now consider the transformation $Z = X_1X_2$, $W = 1 - X_1$ whose Jacobian is $J(x_1, x_2 \rightarrow w, z) = 1/(1 - w)$. Thus, we obtain the joint pdf of W and Z as

$$K_2 \frac{z^{\alpha_2-1}}{(1 + \xi_1)^{\gamma_1}(1 + z)^{\alpha_2+\beta_2}} \frac{w^{\beta_1-1}(1 - w)^{\alpha_1+\beta_2-1}}{[1 - \xi_1w/(1 + \xi_1)]^{\gamma_1} [1 - w/(1 + z)]^{\alpha_2+\beta_2}}, \quad (28)$$

where $z > 0$ and $0 < w < 1$. Finally, integrating w using (10) and substituting for K_2 in (28), we obtain the desired result. \square

Corollary 5.2.1. *Let the random variables X_1 and X_2 be independent, $X_1 \sim \text{B1}(\alpha_1, \beta_1; \lambda_1)$ and $X_2 \sim \text{B2}(\alpha_2, \beta_2)$. Then, the pdf of $Z = X_1X_2$ is given by*

$$\frac{B(\beta_1, \alpha_1 + \beta_2)}{\lambda_1^{\beta_1} B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \frac{z^{\alpha_2-1}}{(1 + z)^{\alpha_2+\beta_2}} \times F_1 \left(\beta_1, \alpha_1 + \beta_1, \alpha_2 + \beta_2; \alpha_1 + \beta_1 + \beta_2; 1 - \frac{1}{\lambda_1}, \frac{1}{1 + z} \right), \quad z > 0.$$

Corollary 5.2.2. *Let the random variables X_1 and X_2 be independent, $X_1 \sim \text{B1}(\alpha_1, \beta_1)$ and $X_2 \sim \text{B2}(\alpha_2, \beta_2)$. Then, the pdf of $Z = X_1X_2$ is given by*

$$\frac{B(\beta_1, \alpha_1 + \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \frac{z^{\alpha_2-1}}{(1 + z)^{\alpha_2+\beta_2}} \times {}_2F_1 \left(\beta_1, \alpha_2 + \beta_2; \alpha_1 + \beta_1 + \beta_2; \frac{1}{1 + z} \right), \quad z > 0,$$

Corollary 5.2.3. *Let the random variables X_1 and X_2 be independent, $X_1 \sim B3(\alpha_1, \beta_1)$ and $X_2 \sim B2(\alpha_2, \beta_2)$. Then, the pdf of $Z = X_1 X_2$ is given by*

$$\frac{2^{-\beta_1} B(\beta_1, \alpha_1 + \beta_2)}{B(\alpha_1, \beta_1) B(\alpha_2, \beta_2)} \frac{z^{\alpha_2 - 1}}{(1 + z)^{\alpha_2 + \beta_2}} \times F_1 \left(\beta_1, \alpha_1 + \beta_1, \alpha_2 + \beta_2; \alpha_1 + \beta_1 + \beta_2; \frac{1}{2}, \frac{1}{1 + z} \right), \quad z > 0.$$

6 Distribution of The Quotient

In this section we obtain distributional results for the quotient of two independent random variables involving Gauss hypergeometric distribution.

In the following theorem, we consider the case where both the random variables are distributed as Gauss hypergeometric.

Theorem 6.1. *Let the random variables X_1 and X_2 be independent, $X_i \sim GH(\alpha_i, \beta_i, \gamma_i, \xi_i)$, $i = 1, 2$. Then, the pdf of $Z = X_1/X_2$ is given by*

$$K_1 \frac{B(\alpha_1 + \alpha_2, \beta_2) z^{\alpha_1 - 1}}{(1 + \xi_2)^{\gamma_2} (1 + \xi_1 z)^{\gamma_1}} \sum_{r=0}^{\infty} \frac{(1 - \beta_1)_r (\alpha_1 + \alpha_2)_r}{(\alpha_1 + \alpha_2 + \beta_2)_r r!} \times z^r F_1 \left(\beta_2, \gamma_1, \gamma_2; \alpha_1 + \alpha_2 + \beta_2 + r; \frac{\xi_1 z}{1 + \xi_1 z}, \frac{\xi_2}{1 + \xi_2} \right),$$

for $0 < z \leq 1$, and

$$K_1 \frac{B(\alpha_1 + \alpha_2, \beta_1) z^{-\alpha_2 - 1}}{(1 + \xi_1)^{\gamma_1} (1 + \xi_2/z)^{\gamma_2}} \sum_{r=0}^{\infty} \frac{(1 - \beta_2)_r (\alpha_1 + \alpha_2)_r}{(\alpha_1 + \alpha_2 + \beta_1)_r r!} \times z^{-r} F_1 \left(\beta_1, \gamma_1, \gamma_2; \alpha_1 + \alpha_2 + \beta_1 + r; \frac{\xi_1}{1 + \xi_1}, \frac{\xi_2}{\xi_2 + z} \right),$$

for $z > 1$.

Proof. The joint pdf of X_1 and X_2 is given by (25). Now, transforming $Z = X_1/X_2$, $V = X_2$ with the Jacobian $J(x_1, x_2 \rightarrow z, v) = v$, we obtain the joint pdf of Z and V as

$$K_1 \frac{z^{\alpha_1 - 1} v^{\alpha_1 + \alpha_2 - 1} (1 - zv)^{\beta_1 - 1} (1 - v)^{\beta_2 - 1}}{(1 + \xi_1 vz)^{\gamma_1} (1 + \xi_2 v)^{\gamma_2}}, \quad (29)$$

where $0 < v < 1$ for $0 < z \leq 1$, and $0 < v < 1/z$ for $z > 1$. For $0 < z \leq 1$, the marginal pdf of Z is obtained by integrating (29) over $0 < v < 1$. Thus, the pdf of Z , for $0 < z \leq 1$, is obtained as

$$\begin{aligned} & K_1 z^{\alpha_1-1} \int_0^1 \frac{v^{\alpha_1+\alpha_2-1}(1-zv)^{\beta_1-1}(1-v)^{\beta_2-1} dv}{(1+\xi_1 vz)^{\gamma_1}(1+\xi_2 v)^{\gamma_2}} \\ & = K_1 z^{\alpha_1-1} I(\alpha_1 + \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2; \xi_1, \xi_2, z). \end{aligned} \tag{30}$$

Now, using Lemma 2.1 and substituting K_1 in the density (30) we get the desired result. For $z > 1$, the density of Z is given by

$$\begin{aligned} & K_1 z^{\alpha_1-1} \int_0^{1/z} \frac{v^{\alpha_1+\alpha_2-1}(1-zv)^{\beta_1-1}(1-v)^{\beta_2-1} dv}{(1+\xi_1 vz)^{\gamma_1}(1+\xi_2 v)^{\gamma_2}} \\ & = K_1 z^{-\alpha_2-1} I\left(\alpha_1 + \alpha_2, \beta_2, \beta_1, \gamma_2, \gamma_1; \xi_2, \xi_1, \frac{1}{z}\right), \end{aligned} \tag{31}$$

where the last line has been obtained by substituting $w = vz$. Finally, using Lemma 2.1 and substituting for K_1 , we obtain the pdf of Z for $z > 1$. \square

Theorem 6.2. *Let the random variables X_1 and X_2 be independent, $X_i \sim \text{GH}(\alpha_i, \beta_i, \gamma_i, \xi_i)$, $i = 1, 2$. Then, the pdf of $T = X_1/(X_1 + X_2)$ is given by*

$$\begin{aligned} & K_1 \frac{B(\alpha_1 + \alpha_2, \beta_2)t^{\alpha_1-1}(1-t)^{\gamma_1-\alpha_1-1}}{(1+\xi_2)^{\gamma_2}[1-(1-\xi_1)t]^{\gamma_1}} \sum_{r=0}^{\infty} \frac{(1-\beta_1)_r(\alpha_1 + \alpha_2)_r}{(\alpha_1 + \alpha_2 + \beta_2)_r r!} \\ & \times \left(\frac{t}{1-t}\right)^r F_1\left(\beta_2, \gamma_1, \gamma_2; \alpha_1 + \alpha_2 + \beta_2 + r; \frac{\xi_1 t}{1-(1-\xi_1)t}, \frac{\xi_2}{1+\xi_2}\right), \end{aligned}$$

for $0 < t \leq 1/2$, and

$$\begin{aligned} & K_1 \frac{B(\alpha_1 + \alpha_2, \beta_1)t^{\gamma_2-\alpha_2-1}(1-t)^{\alpha_2-1}}{(1+\xi_1)^{\gamma_1}[t+\xi_2(1-t)]^{\gamma_2}} \sum_{r=0}^{\infty} \frac{(1-\beta_2)_r(\alpha_1 + \alpha_2)_r}{(\alpha_1 + \alpha_2 + \beta_1)_r r!} \\ & \times \left(\frac{1-t}{t}\right)^r F_1\left(\beta_1, \gamma_1, \gamma_2; \alpha_1 + \alpha_2 + \beta_1 + r; \frac{\xi_1}{1+\xi_2}, \frac{\xi_2(1-t)}{t+\xi_2(1-t)}\right), \end{aligned}$$

for $1/2 < t < 1$.

Proof. Making the transformation $T = Z/(1 + Z)$ with the Jacobian $J(z \rightarrow t) = (1 - t)^{-2}$ in Theorem 6.1 we get the desired result \square

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