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A characterization of structural Nikol'skiĭ–Besov spaces using fractional derivatives[1]

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The Nikol'skiĭ–Besov spaces $\mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$, $1 \leq p$, $q \leq \infty$, $\alpha > 0$, are defined in terms of modulus of continuity and they are characterized using the partial derivatives of the Poisson integrals. In this work we show a characterization of the functions of $\mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$ using the Caputo noninteger order derivative of the Poisson integral.

Keywords: Nikol'skiĭ–Besov spaces, fractional derivative, Poisson kernel, Poisson integral.

Los espacios de Nikol'skiĭ–Besov $\mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$, $1 \leq p$, $q \leq \infty$, $\alpha > 0$, están definidos en términos de módulos de continuidad y se caracterizan usando las derivadas parciales de las integrales de Poisson. En este trabajo mostramos una caracterización de las funciones de $\mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$ usando la derivada de orden no–entero de Caputo de la integral de Poisson.

Palabras claves: espacios de Nikol'skiĭ–Besov, derivada fraccional, kernel de Poisson, integral de Poisson.

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1 Introduction

The idea of generalizing the notion of differentiation $\frac{d^p f(x)}{dx^p}$ to noninteger orders of p appeared at the birth of the differential calculus itself. The first attempt to discuss such an idea recorded in history was contained in the correspondence of Leibniz. In one of his letters to Leibniz concerning the theorem on the differentiation of a product of functions, Bernoulli asked about the meaning of this theorem in the case of noninteger order of differentiation. Leibniz in his letters to L'Hôpital (1695) made some remarks on the possibility of considering differentials and derivatives of order 1/2. In 1820 Lacroix showed an exact formula for the evaluation of the derivative $\frac{d^{1/2}x^\alpha}{dx^{1/2}}$. At the present the fractional calculus is developing intensively because the study of derivatives and integrals of arbitrary order have to many applications to several fields of science and engineering such as fluid mechanics, probability and statistics, visco–elasticity and chemical–physics.

There exist different non equivalent definitions of noninteger derivatives, but in this work we are concerned in the Caputo noninteger derivatives because it preserves certain properties of the usuual derivatives for the convolution. Using the Caputo noninteger derivative we show a characterization of the Nikol'skiĭ–Besov spaces $\mathcal{B}^\alpha_{pq}(\mathbb{R}^n) \equiv \mathcal{B}^\alpha_{pq}$, $1 \leq p, q \leq \infty$, $\alpha > 0$.

We remark that the Nikol'skiĭ–Besov spaces $\mathcal{B}^{\vec{\gamma}}_{pq}(\mathbb{R}^n)$, introduced by O. V. Besov, coincide for $q = \infty$ with the Nikol'skiĭ spaces $H^{\vec{\gamma}}_p(\mathbb{R}^n)$ which possess continuous and anisotropical characteristics of smoothness. The $\mathcal{B}^{\vec{\gamma}}_{pq}(\mathbb{R}^n)$ spaces were important in order to find the solution of the trace problem in the Sobolev spaces, which were studied in the works by Aranzajin, Bavich, Besov, Gagliardo, Lizorkin, Stein, Uspensky and others. The scale of Besov spaces arises also in a natural way in the approximation theory and the Fourier series study.

For the case $0 < \alpha < 1$ the Nikol'skiĭ–Besov spaces are defined in terms of the modulus of continuity.

Definition 1.1. *Let* $1 \leq p, q \leq \infty$, $0 < \alpha < 1$. *We say that* $f \in \mathcal{B}^\alpha_{pq}$ *if* $f \in L_p(\mathbb{R}^n)$ *and* $\|f\|_{B^\alpha_{pq}} < \infty$, *where*

$$|f|_{B^{\alpha}_{pq}} := |f|_p + \left[\int_{\mathbb{R}^n} \left(\frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}}\right)^q dt\right]^{1/q}, \quad (1)$$

$$|f|_{B^{\alpha}_{p\infty}} := |f|_p + \sup_{|t|>0} \left(\frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\alpha}}\right). \quad (2)$$

The spaces $\mathcal{B}^{\alpha}_{pq}$ are Banach spaces and has been described in terms of the partial derivatives of the Poisson integral of the function f:

$$u(x,\, y) := (P_y * f)(x) = \int_{\mathbb{R}^n} P_y(x-t)\, f(t)\, dt\,, \quad (3)$$

where

$$P_y(x) := \int_{\mathbb{R}^n} \exp\left[-2\,\pi\left(i\sum_{k=1}^{n} x_k\, t_k + y|t|\right)\right] dt\,,$$

for $y > o$, is the Poisson kernel whose explicit form is

$$P_y(x) = c_n\, y\, \left(|x|^2 + y^2\right)^{-(n+1)/2},$$

for $y > 0$, with $c_n = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)$; $\Gamma(\tau) = \int_0^\infty s^{\tau-1}e^{-s}ds$, $\tau > 0$.

Specifically, in [5][Chapter V, section 5] the following is proved.

Theorem 1.1. *Let $0 < \alpha < 1$, $1 \le p, q \le \infty$, $f \in L_p(\mathbb{R}^n)$. Then $f \in \mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if*

$$\left\| y^{1-\alpha-\frac{1}{q}}\, \|\partial_y u(\cdot,\, y)\|_p \right\|_q < \infty\,, \quad (4)$$

taking the q–norm with respect to $y \in \mathbb{R}_+$.

In this work we establish a generalization of (4) for the case of fractional derivatives.

Let $0 < \alpha < \beta \le 1$, $1 \le p, q \le \infty$, $f \in L_p(\mathbb{R}^n)$. Then $f \in \mathcal{B}^{\alpha}_{pq}$ if and only if

$$\left\| y^{\beta-\alpha-\frac{1}{q}}\, \left\|\mathcal{D}^{\beta}_y u(\cdot,\, y)\right\|_p \right\|_q < \infty\,.$$

Here $\mathcal{D}_y^\beta$ denotes the fractional derivative of order β with respect to y.

According to theorem 1.1, if $\alpha \geq 1$ the space $\mathcal{B}_{pq}^\alpha(\mathbb{R}^n)$ is defined by

$$\mathcal{B}_{pq}^\alpha(\mathbb{R}^n) = \left\{ f \in L_p(\mathbb{R}^n) : \left\| y^{k-\alpha-\frac{1}{q}} \left\| \partial_y^k u(\cdot, y) \right\|_p \right\|_q < \infty \right\},$$

where k be is the smallest integer greater than α. An equivalent definition can be obtained by replacing k for any other integer $l > \alpha$. In this case (see [5][Chapter V, section 5] the following result holds.

Theorem 1.2. *Suppose $f \in L_p(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$ and $\alpha > 0$. Let k and l be two integers, both greater than α. Then the two conditions*

$$\left\| y^{k-\alpha-\frac{1}{q}} \left\| \partial_y^k u(\cdot, y) \right\|_p \right\|_q < \infty,$$

and

$$\left\| y^{l-\alpha-\frac{1}{q}} \left\| \partial_y^l u(\cdot, y) \right\|_p \right\|_q < \infty,$$

are equivalent.

So that, if $\alpha \geq 1$, $f \in L_p(\mathbb{R}^n)$ and k is any integer greater than α, then $f \in \mathcal{B}_{pq}^\alpha(\mathbb{R}^n)$ if and only if

$$\left\| y^{k-\alpha-\frac{1}{q}} \left\| \partial_y^k u(\cdot, y) \right\|_p \right\|_q < \infty.$$

In this work we also show a similar characterization for the noninteger case.

If $\alpha \geq 1$, $f \in L_p(\mathbb{R}^n)$ and β is any real number greater than α, then $f \in \mathcal{B}_{pq}^\alpha(\mathbb{R}^n)$ if and only if

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot, y) \right\|_p \right\|_q < \infty.$$

Similar results were obtained for the Lipschitz spaces in the works [1, 2, 3].

2 Preliminaries

In this section we recall the Caputo fractional derivative definition, some properties of the Hardy operator, of the Poisson kernel and the Poisson integral. We will use the same notation as in [4] and [5].

2.1 The Hardy inequality

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. We define the Hardy operators by

$$\begin{aligned} (H_1 f)(x) &:= \frac{1}{x} \int_0^x f(y)\,dy\,, \\ (H_2 f)(x) &:= \frac{1}{x} \int_x^\infty f(y)\,dy\,. \end{aligned}$$

Then we have the following theorem (see [5]).

Theorem 2.1. *Suppose* $1 \le p \le \infty$ *and* p' *such that* $\frac{1}{p} + \frac{1}{p'} = 1$.

1. *If* $\alpha < \frac{1}{p'}$ *then*

$$\|x^\alpha H_1 f(x)\|_p \le \left(\frac{1}{p'} - \alpha\right)^{-1} \|x^\alpha f(x)\|_p\,. \tag{5}$$

2. *If* $\alpha > \frac{1}{p'}$ *then*

$$\|x^\alpha H_2 f(x)\|_p \le \left(\alpha - \frac{1}{p'}\right)^{-1} \|x^\alpha f(x)\|_p\,. \tag{6}$$

2.2 The Poisson kernel and the Poisson integral

The Poisson kernel and the Poisson integral have the following properties (see [5]):

1. $\int_{\mathbb{R}^n} P_y(x)\,dx = 1$, $\int_{\mathbb{R}^n} \frac{\partial P_y(x)}{\partial x}\,dx = 0$.

2. $P_y(x) \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$ (considering the L_p–norm with respect to the x–variable).

3. $P_y(x)$ is a harmonic function on $\mathbb{R}^{n+1}_+ := \big\{(x,\, y)/x \in \mathbb{R}^n,\, y > 0\big\}$.

4. Semigroup property:

$$\forall y_1 > 0,\, \forall y_2 > 0,\; P_{y_1}(x) * P_{y_2}(x) = P_{y_1+y_2}(x)\,.$$

5. $\frac{\partial^k}{\partial x^k} P_y(x) \in L_p(\mathbb{R}^n)$, for $1 \le p \le \infty$ and $k \in \mathbb{N}$.

6. $\frac{\partial^{k+m}}{\partial y^{k+m}} P_y(x) = \frac{\partial^k}{\partial y^k} P_{y/2}(x) * \frac{\partial^m}{\partial y^m} P_{y/2}(x)$, $k, m \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$.

7. For $y > 0$, $1 \le q \le \infty$, $j = 1, \cdots, n$ and $k \in \mathbb{N}_0$ the following inequalities are satisfied:

a. $\left|\frac{\partial^k P_y(x)}{\partial y^k}\right| \ll y^{-n-k}$, $\left|\frac{\partial^k P_y(x)}{\partial y^k}\right| \ll |x|^{-n-k}$,

b. $\left|\frac{\partial^k P_y(x)}{\partial x_j^k}\right| \ll y^{-n-k}$, $\left|\frac{\partial^k P_y(x)}{\partial x_j^k}\right| \ll |x|^{-n-k}$,

c. $\left\|\frac{\partial^k P_y(\cdot)}{\partial y^k}\right\|_q \ll y^{-\frac{n}{q'}-k}$, $\left\|\frac{\partial^k P_y(\cdot)}{\partial x_j^k}\right\|_q \ll y^{-\frac{n}{q'}-k}$, $\left(\frac{1}{q} + \frac{1}{q'} = 1\right)$.

We use the notation $A \ll B$ if there exists a constant $c > 0$ such that $A \le cB$.

If $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, then for $u(x, y)$ the following statements holds:

1. $u(x, y) \in L_p(\mathbb{R}^n)$.

2. $u(x, y)$ is a harmonic function on $\mathbb{R}^{n+1}_+$.

3. $\lim\limits_{y \to +0} u(x, y) = f(x)$ almost everywhere on $\mathbb{R}^n$.

4. $\frac{\partial^k}{\partial y^k} u(x, y) = \frac{\partial^k}{\partial y^k} P_y(x) * f(x)$.

5. $\frac{\partial^{k+m}}{\partial y^{k+m}} u(x, y) = \frac{\partial^k}{\partial y^k} u(x, \frac{y}{2}) * \frac{\partial^m}{\partial y^m} P_{\frac{y}{2}}(x)$.

2.3 Fractional derivatives

If β is a real number we will denote the integer part of β by $[\beta]$ and the fractional part of β by $\{\beta\}$, so that $0 \le \{\beta\} < 1$ and $\beta = [\beta] + \{\beta\}$.

Definition 2.1. *Let $\beta > 0$ and f be a function defined on $\mathbb{R}$. Then the Liouville integral of order β of the function f is defined by*

$$(I^{\beta} f)(x) := \frac{1}{\Gamma(\beta)} \int_x^{\infty} (t-x)^{\beta-1} f(t)\, dt\,,$$

whenever the expression is well defined.

Definition 2.2. *Let f be a function defined on $\mathbb{R}$, β be a noninteger positive real number and $m = [\beta]+1$. Then the Caputo fractional derivative of order β of the function f is defined by*

$$(\mathcal{D}^{\beta} f)(x) := \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^{\infty} (t-x)^{-\{\beta\}} f^{(m)}(t)\, dt\,,$$

whenever the expression is well defined.

For $\beta \in \mathbb{N}_0$, the fractional derivative is defined by

$$(\mathcal{D}^{\beta} f)(x) := (-1)^{\beta} \frac{d^{\beta}}{dx^{\beta}} f(x)\,, \quad (\mathcal{D}^0 f \equiv f)\,.$$

Example 2.1.

1. Let $f(x) = e^{-ax}$, $a > 0$, $0 < \beta < 1$. Since

$$\left(\mathcal{D}^{\beta} f\right)(x) = -\frac{1}{\Gamma(1-\beta)} \int_0^{\infty} s^{-\beta} f'(x+s)\, ds\,,$$

and

$$f'(x) = -a\, e^{-ax}\,,$$

then

$$\begin{aligned}\mathcal{D}^{\beta}\left(e^{-ax}\right) &= \frac{a}{\Gamma(1-\beta)}\int_0^{\infty} s^{-\beta}\, e^{-a(x+s)}\, ds \\ &= \frac{a}{\Gamma(1-\beta)}\left(e^{-ax}\right) a^{\beta-1}\int_0^{\infty} v^{-\beta}\, e^{-v}\, dv \\ &= \frac{a^{\beta}}{\Gamma(1-\beta)}\left(e^{-ax}\right)\Gamma(1-\beta) = a^{\beta}\, e^{-ax}\,.\end{aligned}$$

Therefore

$$\mathcal{D}^{\beta}\left(e^{-ax}\right) = a^{\beta}\, e^{-ax}\,,$$

for $a > 0$, and $0 < \beta < 1$. In the special case $a = 1$ we have $\mathcal{D}^{\beta}\left(e^{-x}\right) = e^{-x}$.

2. Let $f(x) = x^{-\mu}$, $0 < \beta < 1$. Since $f'(x) = -\mu x^{-\mu-1}$, then

$$\begin{aligned}\mathcal{D}^{\beta}\left(x^{-\mu}\right) &= \frac{\mu}{\Gamma(1-\beta)}\int_0^{\infty} s^{-\beta}\,(x+s)^{-\mu-1}\, ds \\ &= \frac{\mu}{\Gamma(1-\beta)}\left(x^{-\mu-1}\right) x^{1-\beta}\int_0^{\infty} v^{-\beta}\,(1+v)^{-\mu-1}\, dv \\ &= \frac{\mu\,\Gamma(\mu+\beta)}{\Gamma(\mu+1)}\, x^{-\mu-\beta}\,.\end{aligned}$$

Therefore

$$\mathcal{D}^{\beta}\left(x^{-\mu}\right) = \begin{cases} \frac{\mu\Gamma(\mu+\beta)}{\Gamma(\mu+1)} x^{-\mu-\beta}, & \mu > -\beta \\ \frac{\Gamma(\mu+\beta)}{\Gamma(\mu)} x^{-\mu-\beta}, & \mu > 0 \end{cases}.$$

In the special case $\mu = 0$ we have $D^{\beta}(c) = 0$, $c \in \mathbb{R}$.

More examples can be found in [4].

Definition 2.3. *Let $\beta > 0$ and f be a function defined on $\mathbb{R}^n$. Then the partial integral of order β of f with respect to x_k–variable ($k = 1, \cdots, n$) is defined by*

$$(I^{\beta}_{x_k} f)(x) := \frac{1}{\Gamma(\beta)}\int_{x_k}^{\infty} (t - x_k)^{\beta-1}\, f(x + (t - x_k)\,\vec{e}_k)\, dt\,.$$

where $\vec{e}_k = (0, \cdots, 0, 1, 0, \cdots, 0)$.

Definition 2.4. *Let f be a function defined on $\mathbb{R}$, β be a noninteger real number and $m = [\beta] + 1$. Then the Caputo partial fractional derivative of order β of f with respect to x_k–variable is defined by*

$$(\mathcal{D}^{\beta}_{x_k} f)(x) := \frac{(-1)^m}{\Gamma(m-\beta)} \frac{\partial^m}{\partial x_k^m} \int_{x_k}^{\infty} (t - x_k)^{-\{\beta\}} f(x + (t - x_k)\,\vec{e}_k)\,dt\,.$$

For $\beta \in \mathbb{N}_0$ give, the partial fractional derivative is defined by

$$(\mathcal{D}^{\beta}_{x_k} f)(x) := (-1)^{\beta} \frac{\partial^{\beta}}{\partial x_k^{\beta}} f(x)\,.$$

In [1], [2] and [3] the authors proved that $\mathcal{D}^{\beta}_{y} P_y(x)$ exists for any $\beta > 0$ and established the following results:

1. Let $\beta > 0$ and f be a function such that $\mathcal{D}^{\beta}_{y} f(y)$ exists. If c is a constant and $y = z + c$ then $\mathcal{D}^{\beta}_{z} f(y) = \mathcal{D}^{\beta}_{y} f(y)$.

2. For all $\beta, \lambda > 0$

$$\begin{aligned}
\mathcal{D}^{\beta+\lambda}_{y} P_y(x) &= \mathcal{D}^{\beta}_{y} P_{\frac{y}{2}}(x) * \mathcal{D}^{\lambda}_{y} P_{\frac{y}{2}}(x)\,,\\
\mathcal{D}^{\beta+\lambda}_{y} P_y(x) &= \mathcal{D}^{\beta}_{y} \mathcal{D}^{\lambda}_{y} P_y(x)\,,\\
\mathcal{D}^{\beta+\lambda}_{x_j} P_y(x) &= \mathcal{D}^{\beta}_{x_j} \mathcal{D}^{\lambda}_{x_j} P_y(x)\,,
\end{aligned}$$

for $j = 1, \cdots, n$.

3. If $0 < \beta < 1$ then

$$\begin{aligned}
\left| D^{\beta}_{y} P_y(x) \right| &\ll y^{-n-\beta}\,,\\
\left| D^{\beta}_{y} P_y(x) \right| &\ll |x|^{-n-\beta}\,,\\
\left| \mathcal{D}^{\beta}_{x_j} P_y(x) \right| &\ll y^{-n-\beta}\,,\\
\left| \mathcal{D}^{\beta}_{x_j} P_y(x) \right| &\ll |x|^{-n-\beta}\,,
\end{aligned}$$

for $j = 1, \cdots, n$.

We use theses estimates to obtain estimates for L_q–norms.

Lemma 2.1. *Let* $0 < \beta < 1$, $1 \leq q \leq \infty$, $y > 0$ *and* $j = 1, \cdots, n$. *Then we have*

$$\begin{aligned} \left\| \mathcal{D}_y^{\beta} P_y(\cdot) \right\|_q &\ll y^{-\frac{n}{q'}-\beta}, \\ \left\| \mathcal{D}_{x_j}^{\beta} P_y(\cdot) \right\|_q &\ll y^{-\frac{n}{q'}-\beta}. \end{aligned}$$

Proof. We will prove the first inequality. For the second one can proceed similarly. If $1 \leq q < \infty$

$$\begin{aligned} \left\| \mathcal{D}_y^{\beta} P_y(\cdot) \right\|_q^q &= \int_{|x|\leq y} \left| \mathcal{D}_y^{\beta} P_y(x) \right|^q dx + \int_{|x|>y} \left| \mathcal{D}_y^{\beta} P_y(x) \right|^q dx \\ &\ll \int_{|x|\leq y} y^{-q(n+\beta)}\, dx + \int_{|x|>y} |x|^{-q(n+\beta)}\, dx\,. \end{aligned}$$

By using spherical coordinates in $\mathbb{R}^n$ we obtain

$$\begin{aligned} \left\| \mathcal{D}_y^{\beta} P_y(\cdot) \right\|_q^q &\ll y^{-q(n+\beta)} \int_0^y r^{n-1}\, dr + \int_y^{+\infty} r^{-q(n+\beta)}\, r^{n-1}\, dr \\ &\ll y^{-q(n+\beta)}\, y^n + y^{-q(n+\beta)}\, y^n \ll y^{n-q(n+\beta)}, \end{aligned}$$

thus

$$\left\| \mathcal{D}_y^{\beta} P_y(\cdot) \right\|_q \ll y^{\frac{n}{q}-(n+\beta)} = y^{\frac{n}{q'}-\beta}.$$

For $q = \infty$ we proceed in similar way.

□

Remark 2.1.

1. For any $\beta > 0$

$$\mathcal{D}_y^{\beta} P_y(x) = \mathcal{D}_y^{[\beta]} P_{\frac{y}{2}}(x) * \mathcal{D}_y^{\{\beta\}} P_{\frac{y}{2}}(x)\,,$$

then using the property 7 of the Poisson kernel and the lemma 2.1 we obtain

$$\left\| \mathcal{D}_y^{\beta} P_y(\cdot) \right\|_q \ll y^{-\frac{n}{q'}-\beta}. \tag{7}$$

2. Let us observe that these results (of the Poisson kernel) were obtained for the Liouville fractional derivative, but in [2] is proved that this derivative coincides with the Caputo derivative of the Poisson kernel. Moreover, if $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, a direct application of the Fubini's theorem implies that for $\beta > 0$ and $j = 1, \cdots, n$

$$\begin{aligned} \mathcal{D}_y^\beta u(x,\, y) &= \mathcal{D}_y^\beta P_y(x) * f(x)\,, \\ \mathcal{D}_{x_j}^\beta u(x,\, y) &= \mathcal{D}_{x_j}^\beta P_y(x) * f(x)\,. \end{aligned}$$

3 Characterization of $\boldsymbol{\mathcal{B}}_{pq}^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$, in terms of $\boldsymbol{\mathcal{D}}_y^\beta$

We will use the following lemma in order to obtain our main result.

Lemma 3.1. *Let* $f \in L_p(\mathbb{R}^n)$, $0 < \alpha < \beta < 1$, $1 \le p, q \le \infty$. *Then*

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot,\, y) \right\|_p \right\|_q < \infty\,,$$

if and only if

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_j^\beta u(\cdot,\, y) \right\|_p \right\|_q < \infty\,,$$

for $\mathcal{D}_j^\beta \equiv \mathcal{D}_{x_j}^\beta$, $j = 1, \cdots, n$.

Proof. We have that

$$\mathcal{D}_y^\gamma \mathcal{D}_j^\beta u(x,\, y) = \mathcal{D}_j^\gamma P_{y/2}(x) * \mathcal{D}_y^\beta u(x,\, y/2)\,,$$

$\forall \gamma > 0$. By using lemma 2.1 with $q = 1$ we conclude that

$$\begin{aligned} \left\| \mathcal{D}_y^\gamma \mathcal{D}_j^\beta u(\cdot,\, y) \right\|_p &\le \left\| \mathcal{D}_j^\gamma P_{y/2}(\cdot) \right\|_1 \left\| \mathcal{D}_y^\beta u(\cdot,\, y/2) \right\|_p \\ &\ll y^{-\gamma} \left\| \mathcal{D}_y^\beta u(\cdot,\, y/2) \right\|_p\,. \end{aligned} \tag{8}$$

Therefore

$$y^{\gamma}\left\|\mathcal{D}_y^{\gamma}\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p \ll \left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\,,$$

$\forall\gamma>0$. So, for all $\gamma>0$,

$$\left\|y^{\beta-\alpha-\frac{1}{q}+\gamma}\left\|\mathcal{D}_y^{\gamma}\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right\|_q \ll \left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,.$$

In particular, for $\gamma=\beta$ we have that

$$\left\|y^{2\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,,$$

for $j=1,\cdots,n$. Furthermore, the lemma 2.1 implies that $\mathcal{D}_j^{\beta}u(\cdot,y)\to +0$, as $y\to\infty$; for this reason

$$\mathcal{D}_j^{\beta}u(\cdot,\,y) = -\int_y^{\infty}\frac{\partial}{\partial y}\mathcal{D}_j^{\beta}u(\cdot,\,y')\,dy'\,,$$

$$\frac{\partial}{\partial y}\mathcal{D}_j^{\beta}u(\cdot,\,y) = \mathcal{D}_y^{1-\beta}\mathcal{D}_y^{\beta}D_j^{\beta}u(\cdot,\,y)\,.$$

Then, using (8),

$$\begin{aligned}
\left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right\|_q &= \left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\int_y^{\infty}\mathcal{D}_y^{1-\beta}\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y')\,dy'\right\|_p\right\|_q\\
&\le \left\|y^{\beta-\alpha-\frac{1}{q}}\int_y^{\infty}\left\|\mathcal{D}_y^{1-\beta}\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y')\right\|_p dy'\right\|_q\\
&\ll \left\|y^{\beta-\alpha-\frac{1}{q}}\int_y^{\infty}y'^{\beta-1}\left\|\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y')\right\|_p dy'\right\|_q\\
&= \left\|y^{1+\beta-\alpha-\frac{1}{q}}H_2\left(y^{\beta-1}\left\|\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right)\right\|_q\,.
\end{aligned}$$

Due to the fact that $1+\beta-\alpha-\frac{1}{q}>\dfrac{1}{q'}$, the Hardy inequality implies

$$\left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right\|_q \ll \left\|y^{2\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,.$$

For the converse we use that for all $\gamma>0$,

$$\left\| \mathcal{D}_j^{\beta+\gamma} u(\cdot,\, y) \right\|_p \ll \left\| \mathcal{D}_j^{\gamma} P_{y/2}(\cdot) \right\|_1 \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y/2) \right\|_p \ll y^{-\gamma} \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y) \right\|_p ,$$

thus

$$y^{\beta-\alpha+\gamma} \left\| \mathcal{D}_j^{\beta+\gamma} u(\cdot,\, y) \right\|_p \ll y^{\beta-\alpha} \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y) \right\|_p .$$

Then for all $\gamma > 0$

$$\left\| y^{\beta-\alpha-\frac{1}{q}+\gamma} \left\| \mathcal{D}_j^{\beta+\gamma} u(\cdot,\, y) \right\|_p \right\|_q \ll \left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y) \right\|_p \right\|_q < \infty.$$

In particular. if $\gamma = 2-\beta$ we have

$$\left\| y^{2-\alpha-\frac{1}{q}} \left\| \frac{\partial^2 u(\cdot,\, y)}{\partial x_j^2} \right\|_p \right\|_q < \infty .$$

Using the fact that u is a harmonic function, we will prove that

$$\left\| y^{2-\alpha-\frac{1}{q}} \left\| \frac{\partial^2 u(\cdot,\, y)}{\partial y^2} \right\|_p \right\|_q < \infty .$$

Indeed, since $\mathcal{D}_y^{\beta} u(\cdot, y) = \mathcal{I}_y^{2-\beta} \frac{\partial^2}{\partial y^2} u(\cdot, y)$, then

$$\begin{aligned} \left\| \mathcal{D}_y^{\beta} u(\cdot,\, y) \right\|_p &= \left\| \frac{1}{\Gamma(2-\beta)} \int_y^\infty (\tau - y)^{1-\beta} \frac{\partial^2}{\partial \tau^2} u(\cdot,\, \tau)\, d\tau \right\|_p \\ &\ll \int_y^\infty (\tau - y)^{1-\beta} \left\| \frac{\partial^2}{\partial \tau^2} u(\cdot,\, \tau) \right\|_p d\tau \\ &\le \int_y^\infty \tau^{1-\beta} \left\| \frac{\partial^2}{\partial \tau^2} u(\cdot,\, \tau) \right\|_p d\tau . \end{aligned}$$

This implies

$$\begin{aligned} \left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^{\beta} u(\cdot,\, y) \right\|_p \right\|_q &\le \left\| y^{\beta-\alpha-\frac{1}{q}} \int_y^\infty \tau^{1-\beta} \left\| \frac{\partial^2}{\partial y \tau^2} u(\cdot,\, \tau) \right\|_p d\tau \right\|_q \\ &\le \left\| y^{1+\beta-\alpha-\frac{1}{q}} H_2 \left(y^{1-\beta} \left\| \frac{\partial^2}{\partial y^2} u(\cdot,\, y) \right\|_p \right) \right\|_q . \end{aligned}$$

Because $1+\beta-\alpha-\frac{1}{q}>\frac{1}{q'}$, applying the Hardy inequality, we get

$$\left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right\|_q \ll \left\|y^{2-\alpha-\frac{1}{q}}\left\|\frac{\partial^2}{\partial y^2}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,.$$

□

Now we show a characterization of $\mathcal{B}_{pq}^{\alpha}$, $0<\alpha<11$, using fractional derivative.

Theorem 3.1. *Let* $f\in L_p(\mathbb{R}^n)$, $0<\alpha<\beta\leq 1$, $1\leq p,q\leq\infty$. *Then* $f\in\mathcal{B}_{pq}^{\alpha}$ *if and only if*

$$\left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,.$$

Proof. If $\beta=1$, this case coincides with the theorem 1.1. If $0<\beta<1$ we will show that

$$\left\|\frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}}\right\|_q < \infty\,,$$

implies

$$\left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right\|_q < \infty\,.$$

Since $\mathcal{D}_y^{\beta}u(x,y)=\mathcal{D}_y^{\beta}P_y(x)*f(x)=\int\limits_{\mathbb{R}^n}\mathcal{D}_y^{\beta}P_y(t)f(x-t)dt$ and, according to the property 1 of the Poisson kernel, $\int_{\mathbb{R}^n}\mathcal{D}_y^{\beta}P_y(t)dt=0$, then

$$\mathcal{D}_y^{\beta}u(x,\,y)=\int_{\mathbb{R}^n}\mathcal{D}_y^{\beta}P_y(t)\,[f(x-t)-f(x)]\,dt\,,$$

therefore

$$\begin{aligned}\left\|\mathcal{D}_y^\beta u(\cdot,\,y)\right\|_p &\leq \int_{\mathbb{R}^n} \left|\mathcal{D}_y^\beta P_y(t)\right| \|f(\cdot-t)-f(\cdot)\|_p \, dt \\ &\ll \int_{|t|\leq y} \left|\mathcal{D}_y^\beta P_y(t)\right| \|f(\cdot-t)-f(\cdot)\|_p \, dt \\ &\quad + \int_{|t|>y} \left|\mathcal{D}_y^\beta P_y(t)\right| \|f(\cdot-t)-f(\cdot)\|_p \, dt \\ &\ll \int_{|t|\leq y} y^{-n-\beta} \|f(\cdot-t)-f(\cdot)\|_p \, dt \\ &\quad + \int_{|t|>y} |t|^{-n-\beta} \|f(\cdot-t)-f(\cdot)\|_p \, dt \,.\end{aligned}$$

Hence, using the notation $W(t) := \|f(\cdot-t)-f(\cdot)\|_p$, we obtain

$$\left\|\mathcal{D}_y^\beta u(\cdot,\,y)\right\|_p \ll y^{-n-\beta} \int_{|t|\leq y} W(t)\,dt + \int_{|t|>y} |t|^{-n-\beta}\, W(t)\,dt \,.$$

Now, using spherical coordinates in $\mathbb{R}^n$ we get

$$\left\|\mathcal{D}_y^\beta u(x,\,y)\right\|_p \ll y^{-n-\beta} \int_0^y r^{n-1}\Omega(r)\,dr + \int_y^\infty r^{-1-\beta}\,\Omega(r)\,dr \,,$$

where $\Omega(r) := \int_{S^{n-1}} W(\varphi_1,\cdots,\varphi_{n-1},r)d\varphi_1\cdots d\varphi_{n-1}$, and

$$S^{n-1} := \left\{(x_1,\,\cdots\,,\,x_n) : \sum_{i=1}^n x_i^2 = r^2\right\} .$$

Therefore

$$\begin{aligned}y^{\beta-\alpha-\frac{1}{q}} \left\|\mathcal{D}_y^\beta u(\cdot,\,y)\right\|_p &\ll y^{-n-\alpha-\frac{1}{q}} \int_0^y r^{n-1}\,\Omega(r)\,dr \\ &\quad + y^{\beta-\alpha-\frac{1}{q}} \int_y^\infty r^{-1-\beta}\,\Omega(r)\,dr \,.\end{aligned}$$

So that

$$\begin{aligned}y^{\beta-\alpha-\frac{1}{q}} \left\|\mathcal{D}_y^\beta u(\cdot,\,y)\right\|_p &\ll y^{1-n-\alpha-\frac{1}{q}}\, H_1\left(y^{n-1}\,\Omega(y)\right) \\ &\quad + y^{1+\beta-\alpha-\frac{1}{q}}\, H_2\left(y^{-1-\beta}\,\Omega(y)\right) \,.\end{aligned}$$

Using the Hardy inequality we have

$$\begin{aligned}
\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot,\, y) \right\|_p \right\|_q &\ll \left\| y^{1-n-\alpha-\frac{1}{q}}\, H_1\left(y^{n-1}\,\Omega(y)\right) \right\|_q \\
&\quad + \left\| y^{1+\beta-\alpha-\frac{1}{q}}\, H_2\left(y^{-1-\beta}\,\Omega(y)\right) \right\|_q \\
&\ll \left\| y^{-\alpha-\frac{1}{q}}\,\Omega(y) \right\|_q + \left\| y^{-\alpha-\frac{1}{q}}\,\Omega(y) \right\|_q \\
&\ll \left\| y^{-\alpha-\frac{1}{q}}\,\Omega(y) \right\|_q \\
&= \left(\int_0^{+\infty} \left| \Omega(r)\, r^{-\alpha-\frac{1}{q}} \right|^q dr \right)^{1/q},
\end{aligned}$$

for $1 \leq q < \infty$.

Since $\Omega(r) = \int_{S^{n-1}} W(\varphi, r) d\varphi$, then

$$\begin{aligned}
|\Omega(r)|^q &\leq \left(\int_{S^{n-1}} |W(\varphi,\, r)|\, d\varphi \right)^q \\
&\leq \left(\|W(\varphi,\, r)\|_{L_q(S^{n-1})}\, |1|_{L_{q'}(S^{n-1})} \right)^q \\
&\ll \int_{S^{n-1}} |W(\varphi,\, r)|^q\, d\varphi\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot,\, y) \right\|_p \right\|_q &\ll \left[\int_0^\infty r^{-\alpha q-1} \left(\int_{S^{n-1}} W^q(\varphi,\, r)\, d\varphi \right) dr \right]^{1/q} \\
&= \left(\int_{\mathbb{R}^n} |t|^{-\alpha q-n}\, W^q(t)\, dt \right)^{1/q} \\
&= \left[\int_{\mathbb{R}^n} \left(\frac{W(t)}{|t|^{\frac{n}{q}+\alpha}} \right)^q dt \right]^{1/q}.
\end{aligned}$$

Then

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot,\, y) \right\|_p \right\|_q \ll \left\| \frac{\|f(\cdot+t) - f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}} \right\|_q < \infty\,.$$

Now, we will show that

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^{\beta} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,,$$

implies

$$\left\| \frac{\|f(\cdot + t) - f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}} \right\|_q < \infty\,.$$

Since

$$\begin{aligned} f(x+t) - f(x) &= u(x+t,\, y) - u(x,\, y) + f(x+t) \\ &\quad -u(x+t,\, y) + u(x,\, y) - f(x)\,, \end{aligned} \tag{9}$$

and

$$u(x+t,\, y) - u(x,\, y) = \int_0^1 \nabla_x u(x+s\,t,\, y)\cdot t\, ds\,,$$

then

$$\begin{aligned} |u(x+t,\, y) - u(x,\, y)| &\leq \int_0^1 |\nabla_x u(x+s\,t,\, y)|\, |t|\, ds \\ &\ll |t| \int_0^1 \sum_{j=1}^n \left| \frac{\partial}{\partial x_j} u(x+s\,t,\, y) \right| ds\,, \end{aligned}$$

therefore

$$\|u(\cdot + t,\, y) - u(\cdot,\, y)\|_p \leq |t| \sum_{j=1}^n \left\| \frac{\partial}{\partial x_j} u(\cdot,\, y) \right\|_p .$$

Since

$$\begin{aligned} \left\| \frac{\partial}{\partial x_j} u(\cdot,\, y) \right\|_p &= \left\| \mathcal{D}_j^{1-\beta} \mathcal{D}_j^{\beta} u(\cdot,\, y) \right\|_p \leq \left\| \mathcal{D}_j^{1-\beta} P_{y/2}(\cdot) \right\|_1 \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y/2) \right\|_p \\ &\ll y^{\beta-1} \left\| \mathcal{D}_j^{\beta} u(\cdot,\, y) \right\|_p\,, \end{aligned}$$

then

$$\|u(\cdot+t,\,y)-u(\cdot,\,y)\|_p \ll |t|\,y^{\beta-1}\sum_{j=1}^{n}\left\|\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p . \tag{10}$$

Now, by using the property **3** of the Poisson integral we obtain

$$f(\cdot)-u(\cdot,\,y)=-\int_0^y \frac{\partial}{\partial\tau}u(\cdot,\,\tau)\,d\tau\,,$$

then

$$\begin{aligned}\|f(\cdot)-u(\cdot,\,y)\|_p &\leq \int_0^y \left\|\frac{\partial}{\partial\tau}u(\cdot,\,\tau)\right\|_p d\tau=\int_0^y \left\|\mathcal{D}_\tau^{1-\beta}\mathcal{D}_\tau^{\beta}u(\cdot,\,\tau)\right\|_p d\tau\\ &\ll \int_0^y \tau^{\beta-1}\left\|\mathcal{D}_\tau^{\beta}u(\cdot,\,\tau)\right\|_p d\tau\,.\end{aligned}$$

Hence

$$\begin{aligned}\left\|y^{-\alpha-\frac{1}{q}}\,\|f(\cdot)-u(\cdot,\,y)\|_p\right\|_q &\leq \left\|y^{-\alpha-\frac{1}{q}}\int_0^y \tau^{\beta-1}\left\|\mathcal{D}_\tau^{\beta}u(\cdot,\,\tau)\right\|_p d\tau\right\|_q\\ &\leq \left\|y^{1-\alpha-\frac{1}{q}}\,H_1\left(y^{\beta-1}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right)\right\|_q .\end{aligned}$$

Since $1-\alpha-\frac{1}{q}<\frac{1}{q'}$, and using the Hardy inequality,

$$\left\|y^{-\alpha-\frac{1}{q}}\,\|f(\cdot)-u(\cdot,\,y)\|_p\right\|_q \ll \left\|y^{\beta-\alpha-\frac{1}{q}}\left\|\mathcal{D}_y^{\beta}u(\cdot,\,y)\right\|_p\right\|_q<\infty\,. \tag{11}$$

By using (9) y (10) one can easily deduce that

$$\begin{aligned}\frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}} &\ll |t|^{1-\alpha-\frac{n}{q}}\,y^{\beta-1}\sum_{j=1}^{n}\left\|\mathcal{D}_j^{\beta}u(\cdot,\,y)\right\|_p\\ &\quad+|t|^{-\alpha-\frac{n}{q}}\,\|f(\cdot)-u(\cdot,\,y)\|_p\,.\end{aligned}$$

Let us choose $y=|t|$,

$$
\begin{aligned}
\left\| \frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}} \right\|_q &\ll \sum_{j=1}^{n} \left\| |t|^{\beta-\alpha-\frac{n}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\,|t|) \right\|_p \right\|_q \\
&+ \left\| |t|^{-\alpha-\frac{n}{q}} \, \|f(\cdot)-u(\cdot,\,|t|)\|_p \right\|_q .
\end{aligned}
$$

If we show that the left hand side of the last estimate is finite then the proof is complete. In order to do this we use spherical coordinates in $\mathbb{R}^n$ and calculate the respective L_p–norm, $1 \leq q \leq \infty$,

$$
\begin{aligned}
\left\| |t|^{\beta-\alpha-\frac{n}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\,|t|) \right\|_p \right\|_q^q &= \int_{\mathbb{R}^n} \left(|t|^{\beta-\alpha-\frac{n}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\,|t|) \right\|_p \right)^q dt \\
&\ll \int_0^{\infty} \left(r^{\beta-\alpha-\frac{n}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\,r) \right\|_p \right)^q r^{n-1}\, dr \\
&= \int_0^{\infty} \left(y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_j^{\beta} u(\cdot,\,y) \right\|_p \right)^q dy < \infty .
\end{aligned}
$$

Also

$$
\begin{aligned}
\left\| |t|^{-\alpha-\frac{n}{q}} \, \|f(\cdot)-u(\cdot,\,|t|)\|_p \right\|_q^q &= \int_{\mathbb{R}^n} \left(|t|^{-\alpha-\frac{n}{q}} \, \|f(\cdot)-u(\cdot,\,|t|)\|_p \right)^q dt \\
&\ll \int_0^{\infty} \left(r^{-\alpha-\frac{n}{q}} \, \|f(\cdot)-u(\cdot,\,r)\|_p \right)^q r^{n-1}\, dr \\
&= \int_0^{\infty} \left(y^{-\alpha-\frac{1}{q}} \, \|f(\cdot)-u(\cdot,\,y)\|_p \right)^q dy < \infty .
\end{aligned}
$$

Therefore

$$
\left\| \frac{\|f(\cdot+t)-f(\cdot)\|_p}{|t|^{\frac{n}{q}+\alpha}} \right\|_q < \infty .
$$

The case $q=\infty$ is proven similarly.

□

4 Characterization of $\boldsymbol{\mathcal{B}}^{\alpha}_{pq}(\mathbb{R}^n)$, $\alpha \geq 1$

We show the following theorem.

Theorem 4.1. *Let $\alpha \geq 1$, $1 \leq p, q \leq \infty$, $f \in L_p(\mathbb{R}^n)$. If β is any real number greater than α, then $f \in \mathcal{B}^{\alpha}_{pq}(\mathbb{R}^n)$ if and only if*

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,.$$

Proof. The theorem proof is a straightforward consequence from theorem 1.2 and the following lemma.

□

Lemma 4.1. *Suppose $f \in L_p(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$ and $\alpha > 0$. If β is any real number greater than α and $k = [\beta] + 1$, then the conditions*

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,,$$

and

$$\left\| y^{k-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{k}_{y} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,.$$

are equivalent.

Proof. Let $\gamma = \{\beta\}$. If

$$\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y) \right\|_p \right\|_q < \infty,$$

then

$$\begin{aligned}
\left\| y^{k-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{k}_{y} u(\cdot,\, y) \right\|_p \right\|_q &= \left\| y^{k-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y/2) * \mathcal{D}^{1-\gamma}_{y} P_{y/2}(\cdot) \right\|_p \right\|_q \\
&\ll \left\| y^{k-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y/2) \right\|_p \left\| \mathcal{D}^{1-\gamma}_{y} P_{y/2}(\cdot) \right\|_1 \right\|_q \\
&\ll \left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}^{\beta}_{y} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,.
\end{aligned}$$

For the converse, we first assume that $[\beta] > [\alpha]$. If $m \in \mathbb{Z}^+$ then

$$\begin{aligned} \left\| \frac{\partial^m u(x,\, y)}{\partial y^m} \right\|_\infty &= \left\| \frac{\partial^m P_y(x)}{\partial y^m} * f(x) \right\|_\infty \leq \left\| \frac{\partial^m P_y(x)}{\partial y^m} \right\|_{p'} \|f\|_p \\ &\ll y^{(\frac{-n}{p} - m)} \|f\|_p \,, \end{aligned}$$

therefore $\dfrac{\partial^m u(x,\, y)}{\partial y^m} \longrightarrow 0$ if $y \longrightarrow \infty$. Hence

$$\frac{\partial^{m-1} u(x,\, y)}{\partial y^{m-1}} = -\int_y^\infty \frac{\partial^m u(x,\, z)}{\partial z^m}\, dz \,.$$

So that

$$\begin{aligned} \left\| y^{\beta - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^\beta u(\cdot,\, y) \right\|_p \right\|_q &\leq \left\| y^{\beta - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^{k-1} u(\cdot,\, y/2) \right\|_p \left\| \mathcal{D}_y^\gamma P_{y/2}(\cdot) \right\|_1 \right\|_q \\ &\ll \left\| y^{[\beta] - \alpha - \frac{1}{q}} \left\| \int_y^\infty \mathcal{D}_\tau^k u(\cdot,\, \tau/2)\, d\tau \right\|_p \right\|_q \\ &\leq \left\| y^{k - \alpha - \frac{1}{q}}\, y^{-1} \int_y^\infty \left\| \mathcal{D}_\tau^k u(\cdot,\, \tau/2) \right\|_p d\tau \right\|_q \\ &= \left\| y^{k - \alpha - \frac{1}{q}}\, H_2\Big(\left\| \mathcal{D}_y^k u(\cdot,\, y/2) \right\|_p \Big) \right\|_q \\ &\leq (k - \alpha - 1) \left\| y^{k - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^k u(\cdot,\, y/2) \right\|_p \right\|_q < \infty \,. \end{aligned}$$

Now, we assume that $[\beta] = [\alpha]$. If

$$\left\| y^{k - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^k u(\cdot,\, y) \right\|_p \right\|_q < \infty \,.$$

Then, using the theorem 1.2,

$$\left\| y^{k + 1 - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^{k+1} u(\cdot,\, y) \right\|_p \right\|_q < \infty \,.$$

Hence, using the previous case,

$$\left\| y^{\beta + 1 - \alpha - \frac{1}{q}} \left\| \mathcal{D}_y^{\beta+1} u(\cdot,\, y) \right\|_p \right\|_q < \infty \,.$$

Therefore

$$\begin{aligned}
\left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^{\beta} u(\cdot,\, y) \right\|_p \right\|_q &= \left\| y^{\beta-\alpha-\frac{1}{q}} \left\| \int_y^{\infty} \mathcal{D}_\tau^{\beta+1} u(\cdot,\, \tau)\, d\tau \right\|_p \right\|_q \\
&\leq \left\| y^{\beta+1-\alpha-\frac{1}{q}}\, y^{-1} \int_y^{\infty} \left\| \mathcal{D}_\tau^{\beta+1} u(\cdot,\, \tau) \right\|_p d\tau \right\|_q \\
&= \left\| y^{\beta+1-\alpha-\frac{1}{q}}\, H_2\Big(\left\| \mathcal{D}_y^{\beta+1} u(\cdot,\, y) \right\|_p \Big) \right\|_q \\
&\ll \left\| y^{\beta+1-\alpha-\frac{1}{q}} \left\| \mathcal{D}_y^{\beta+1} u(\cdot,\, y) \right\|_p \right\|_q < \infty\,.
\end{aligned}$$

□

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