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Openness of the induced map $C_n(f)$

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Dada una función continua entre espacios métricos compactos $f : X \to Y$, es posible definir la función inducida entre los hiperespacios de dimensión n $C_n(f) : C_n(X) \to C_n(Y)$ para cada entero positivo n. Sean $\mathcal{A}$ y $\mathcal{B}$ clases de funciones continuas. Un problema general es encontrar las relaciones entre las siguientes dos afirmaciones: **1.** $f \in \mathcal{A}$; **2.** $C_n(f) \in \mathcal{B}$. Se sabe que **1** y **2** son condiciones equivalentes si $\mathcal{A}$ y $\mathcal{B}$ son la clase de homeomorfismos. Si $\mathcal{A}$ y $\mathcal{B}$ son la clase de funciones abiertas, entonces **2** implica **1**. Además, existe una función abierta f tal que $C_n(f)$ no es abierta. Además, si $C_n(f)$ es abierta y $n \geq 3$, entonces f es abierta y monótona. Nuestro principal resultado es el Teorema 3.2, en el cual se demuestra que si la función inducida $C_n(f)$ es abierta para algún $n \geq 2$, entonces f es un homeomorfismo.

Palabras Claves: continuos, hiperespacios de continuos, funciones inducidas, funciones abiertas.

Given a map between compact metric spaces $f : X \to Y$, it is possible to induce a map between the n–fold hyperspaces $C_n(f) : C_n(X) \to C_n(Y)$ for each positive integer n. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of maps. A general problem is to find the interrelations between the following two statements: **1.** $f \in \mathcal{A}$; **2.** $C_n(f) \in \mathcal{B}$. It is known that **1** and **2** are equivalent conditions if both $\mathcal{A}$ and $\mathcal{B}$ are the class of homeomorphisms. If $\mathcal{A}$ and $\mathcal{B}$ are the class of open maps, then the openness of $C_n(f)$ implies the openness of f. Furthermore, there exists an open map f such that $C_n(f)$ is not open. Moreover, if $C_n(f)$ is open and $n \geq 3$, then f is both open and monotone. Our main result is Theorem 3.2, where we prove that if the induced map $C_n(f)$ is an open map, for $n \geq 2$, then f is a homeomorphism.

Keywords: continua, hyperspaces of continua, induced maps, open maps.

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1 Introduction

A continuum is a nonempty, compact, connected and metric space. For a continuum X and for a positive integer n we denote by $C_n(X)$ the hyperspace of all nonempty closed subsets of X with at most n components. Given a map $f : X \to Y$ between continua X and Y, we define the induced map $C_n(f) : C_n(X) \to C_n(Y)$ by $C_n(f)(A) = f(A)$ [4, p. 783].

A map $f : X \to Y$ is called open if f maps every open set in X onto an open set in Y. It is known that if $C_n(f)$ is open, for some positive integer n, then f is also an open map [4, Theorem 8, p. 786].

There are some works in which the openness of $C_n(f)$ has been studied ([1], [2], [3], [5] and [6] for $n = 1$, and [4] for $n \geq 1$). In [3, Question 4, p. 68] the following question is asked:

Question 1.1. *What (locally connected) continua X have the property that if f is a map of X onto a continuum Y such that the induced map $C_1(f) : C_1(X) \to C_1(Y)$ is open, then f is a homeomorphism?*

There are some partial answers to Question 1.1: In [5, Theorem 1, p. 3729] it is proved that if $C_1(f)$ is open, where f is defined between locally connected continua, then f is monotone. Thus, if f is defined between hereditarily locally connected continua such that $C_1(f)$ is open, then f is a homeomorphism [5, Corollary 2, p. 3730]. It is known that if X is a fan, then we have a positive answer to Question 1.1 [3, Theorem 9, p. 70]. Recently, we expanded [3, Theorem 9, p. 70] proving that if f is defined between dendroids and $C_1(f)$ is open, then f is a homeomorphism [2, Theorem 3.4, p. 233]. It is important to emphasize that there are maps f defined between continua such that $C_1(f)$ is open and f is not a homeomorphism [5, Example 3, p. 3730], even when f is defined between locally connected continua [3, Corollary 19, p. 73].

If $n \geq 1$, then we know that [4, Theorem 10, p. 786] generalizes [5, Theorem 1, p. 3729] proving that f is monotone, if $C_n(f)$ is open and f is defined between locally connected continua, for any $n \in \mathbb{N}$.

It is very natural to ask, what does it happen if we change 1 by any positive integer n in Question 1.1?

Our goal is to prove that if $C_n(f)$ is an open map and $n \geq 2$, then f is a homeomorphism (see Theorem 3.2).

2 Definitions

If (X,d) is a metric space, then given $A \subset X$, the interior, the closure and the boundary of A are denoted by $\mathrm{Int}_X(A)$, $\mathrm{Cl}_X(A)$ and $\mathrm{Bd}_X(A)$, respectively. The cardinality of A is denoted by $|A|$. The symbol $\mathbb{N}$ denotes the set of positive integers. A *map* is assumed to be a continuous function.

Remark 2.1. Here every map will be assumed surjective and defined between nondegenerate continua.

Let X be a continuum and let A and B be closed subsets of X. We say that $C \subset X$ is *irreducible between A and B* provided that $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$, and no proper subcontinuum of C intersects both A and B.

Definition 2.2. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of X. We define the *inferior limit of* $\{A_i\}_{i=1}^{\infty}$, denoted by $\liminf_{i\to\infty} A_i$, and the *superior limit of* $\{A_i\}_{i=1}^{\infty}$, denoted by $\limsup_{i\to\infty} A_i$, as follows:

1. $\liminf_{i\to\infty} A_i = \{x \in X :$ for any open U in X such that $x \in U, U \cap A_i \neq \emptyset$ for all but finitely many $i\}$;
2. $\limsup_{i\to\infty} A_i = \{x \in X :$ for any open U in X such that $x \in U, U \cap A_i \neq \emptyset$ for infinitely many $i\}$;

We say that $\{A_i\}_{i=1}^{\infty}$ is convergent to A in X, which we denote by $\lim_{i\to\infty} A_i = A$, provided that $\liminf_{i\to\infty} A_i = A = \limsup_{i\to\infty} A_i$.

A proof of the following result may be found in [11, Theorem 4.32, p. 130].

Theorem 2.3 (Eilenberg). *Let $f : X \to Y$ be a map between continua. Then f is an open map if and only if $\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y)$ for each sequence $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = y$.*

Eilenberg's theorem is a characterization of open maps and may be written in the following way:

Theorem 2.4. *Let $f : X \to Y$ be a map between continua. Then f is open if and only if for each sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $\lim_{n\to\infty} y_n = y$, for some point $y \in Y$, and for any $x \in f^{-1}(y)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} x_n = x$ and $x_n \in f^{-1}(y_n)$, for each $n \in \mathbb{N}$.*

Proof. Suppose that $f : X \to Y$ is an open map between continua. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in Y such that $\lim_{n\to\infty} y_n = y$ for some $y \in Y$. Let $x \in f^{-1}(y)$. We know that $\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y)$, by Theorem 2.3.

Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of open subsets of X such that $x \in U_n$, $U_{n+1} \subset U_n$, for each $n \in \mathbb{N}$, and $\cap_{n=1}^{\infty} U_n = \{x\}$. Since $x \in \liminf_{n\to\infty} f^{-1}(y_n)$, for each $m \in \mathbb{N}$, there exists $k_m \in \mathbb{N}$ such that $U_m \cap f^{-1}(y_n) \neq \emptyset$ for each $n \geq k_m$. Without loss of generality, we may assume that $k_j < k_\ell$, if $j < \ell$. We define the sequence $\{x_n\}_{n=1}^{\infty}$ in X as follows:

1. $x_n \in f^{-1}(y_n)$ if $n < k_1$;

2. $x_n \in f^{-1}(y_n) \cap U_m$ if $k_m \leq n < k_{m+1}$.

Clearly, $\lim_{n\to\infty} x_n = x$ and $f(x_n) = y_n$ for all $n \in \mathbb{N}$. The converse implication follows from Theorem 2.3.

□

Given a continuum X, we consider the following hyperspaces of X:

1. $2^X = \{A \subset X : A \text{ is closed and nonempty}\}$;

2. $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}, n \in \mathbb{N}$.

2^X is topologized with the Vietoris topology [7, p. 3], which is generated by the collection of sets $\langle U_1, U_2, \cdots, U_\ell\rangle$, where $U_1, U_2, \cdots, U_\ell$ are open sets in X and

$$\langle U_1,\, U_2,\, \cdots,\, U_\ell\rangle = \left\{A \in 2^X : A \subset \cup_{i=1}^{\ell} U_i \text{and} A \cap U_i \neq \emptyset \text{ for each } i\right\}.$$

Clearly, $C_n(X)$ is a subspace of 2^X. The space $C_n(X)$ is called the *n–fold hyperspace of X*. It is well known that if X is a continuum, then both 2^X and $C_n(X)$ are continua. The reader may consult [7] and [8] for general information about hyperspaces.

Notation 2.5. We denote $\langle U_1, U_2, \cdots, U_\ell\rangle \cap C_n(X)$ by $\langle U_1, U_2, \cdots, U_\ell\rangle_n$ and write $C(X)$ instead of $C_1(X)$.

Let X be a continuum. A *Whitney map* for $C(X)$ is a map $\mu : C(X) \to [0, 1]$ that satisfies the following two conditions:

1. for each $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$;

2. $\mu(A) = 0$ if and only if $|A| = 1$.

In [7, Theorem 13.4, p. 107] the following important theorem is proved.

Theorem 2.6. *If X is a continuum, then there exists a Whitney map for the hyperspace $C(X)$.*

Let μ be a Whitney map for $C(X)$. Let $A_0, A_1 \in C(X)$. A map $\sigma : [0,1] \to C(X)$ is said to be a *segment in $C(X)$ with respect to μ from A_0 to A_1* provided that σ has the following three properties:

1. $\sigma(0) = A_0$ and $\sigma(1) = A_1$;

2. $\mu(\sigma(t)) = (1-t)\mu(\sigma(0)) + t\mu(\sigma(1))$ for each $t \in [0,1]$;

3. $\sigma(t) \subset \sigma(s)$ whenever $0 \leq t \leq s \leq 1$.

The following result may be found in [7, Theorem 16.9, p. 131].

Theorem 2.7. *Let X be a continuum and let μ be a Whitney map for $C(X)$. Let $A_0, A_1 \in C(X)$. Then there is a segment with respect to μ from A_0 to A_1 if and only if $A_0 \subset A_1$.*

Let $f : X \to Y$ be a map between continua. Then the function $2^f : 2^X \to 2^Y$ given by $2^f(A) = f(A)$ for each $A \in 2^X$, is called the *induced map between 2^X and 2^Y*. The function $2^f|_{C_n(X)}$ is denoted by $C_n(f)$ and it is called the *induced map between the hyperspaces $C_n(X)$ and $C_n(Y)$*. In [7, Lemma 13.3, p. 106] it is shown that 2^f is a map. Since $2^f(C_n(X)) \subset C_n(Y)$, $C_n(f)$ is a map between $C_n(X)$ and $C_n(Y)$, for each $n \in \mathbb{N}$.

3 Openness of $C_n(f)$ for $n \geq 2$

We begin this section with a theorem that will be used in the proof of our main result.

Theorem 3.1. *Let $f : X \to Y$ be a map between continua and let $n \in \mathbb{N}$, such that $C_n(f) : C_n(X) \to C_n(Y)$ is open. If there is a point $y \in Y$ such that $|f^{-1}(y)| > 1$, then there exist two subcontinua D and E of X such that $D \cap E = \emptyset$ and $f(D) = f(E) = L$, where L is a proper and irreducible subcontinuum between y and some point $y_0 \neq y$ of Y.*

Proof. Let $f : X \to Y$ be a map between continua and let $n \in \mathbb{N}$ such that $C_n(f) : C_n(X) \to C_n(Y)$ is open. Let $b_1, b_2, \cdots, b_{n-1}$ and y be different points in Y such that $|f^{-1}(y)| > 1$. Let $a_1, a_2, \cdots, a_{n-1}, x_1$ and x_2 be points in X such that $f(a_i) = b_i$, for each $i \in \{1, 2, \cdots, n-1\}$, and $x_1, x_2 \in f^{-1}(y)$, where $x_1 \neq x_2$. We denote by B_m the open ball in Y about y of radius $\frac{1}{m}$. Let C_m be the component of $\mathrm{Cl}_Y(B_m)$ such that $y \in C_m$. Notice that $C_m \cap \mathrm{Bd}_Y(B_m) \neq \emptyset$, by [10, Theorem 5.4, p. 73]. Let D_m be an irreducible subcontinuum of C_m between y and some point of $\mathrm{Bd}_Y(B_m)$ [10, Proposition 11.30, p. 212]. Observe that $\lim_{m\to\infty} D_m = \{y\}$. Hence, we have that:

$$\lim_{m\to\infty} (D_m \cup \{b_1, b_2, \cdots, b_{n-1}\}) = \{y, b_1, b_2, \cdots, b_{n-1}\}.$$

Since $\{x_1, a_1, \cdots, a_{n-1}\}$ and $\{x_2, a_1, \cdots, a_{n-1}\}$ both belong to $C_n(f)^{-1}(\{y, b_1, \cdots, b_{n-1}\})$, by Theorem 2.4, there are two sequences $\{E_m\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of $C_n(X)$ such that $\lim_{m\to\infty} E_m = \{x_1, a_1, \cdots, a_{n-1}\}$, $\lim_{m\to\infty} F_m = \{x_2, a_1, \cdots, a_{n-1}\}$ and:

$$E_m,\, F_m \in C_n(f)^{-1}(D_m \cup \{b_1, b_2, \cdots, b_{n-1}\}),\ \text{for each } m \in \mathbb{N}.$$

Let $U_1, U_2, V_1, \cdots, V_{n-2}$ and V_{n-1} be open and pairwise disjoint subsets of X such that $x_1 \in U_1, x_2 \in U_2$ and $a_i \in V_i$ for each $i \in \{1, 2, \cdots, n-1\}$. Thus, there is $\ell \in \mathbb{N}$ such that $E_k \subset U_1 \cup V_1 \cup \cdots \cup V_{n-1}$ and $F_k \subset U_2 \cup V_1 \cup \cdots \cup V_{n-1}$, for each $k \geq \ell$ [11, Theorem 7.2, p. 12]. Hence, both E_k and F_k have exactly n components, for each $k \geq \ell$. Let k_0 be a sufficiently large number such that if E and F are the components of E_{k_0} and F_{k_0}, respectively, such that $E \subset U_1$ and $F \subset U_2$, then $f(E) = f(F) = D_{k_0}$. Note that $E \cap F = \emptyset$, D_{k_0} is irreducible between y and some point $y_0 \neq y$ and we may suppose that $D_{k_0} \neq Y$. The proof is complete.

□

In [6, Theorem 4.3, p. 243] it is proved that 2^f is open if and only if f is open. Furthermore, there is an open map f such that $C_n(f)$ is not open, for any $n \in \mathbb{N}$ [4, Remark 9, p. 786].

Theorem 3.2. *Let $f : X \to Y$ be a map between continua and let $n \geq 2$. If $C_n(f) : C_n(X) \to C_n(Y)$ is open, then f is a homeomorphism.*

Proof. Let $f : X \to Y$ be a map between continua and let $n \geq 2$, such that $C_n(f) : C_n(X) \to C_n(Y)$ is open. Suppose that f is not a homeomorphism and take a point $y \in Y$ such that $|f^{-1}(y)| > 1$, for some point $y \in Y$. Hence, there are two subcontinua D and E of X such that $D \cap E = \emptyset$ and $f(D) = f(E) = L$, where L is an irreducible continuum between y and y_0, where $y \neq y_0$, by Theorem 3.1. Notice that L is a nondegenerate subcontinuum of Y and $D \setminus (f^{-1}(y) \cup f^{-1}(y_0)) \neq \emptyset$. Let $d \in D \setminus (f^{-1}(y) \cup f^{-1}(y_0))$. We consider two cases:

1. $n = 2$. Observe that $\{d\}$ and $f^{-1}(y) \cup E \cup f^{-1}(y_0)$ are disjoint and closed subsets of X. Thus, there are two open and disjoint sets U and V of X such that $d \in U$ and $f^{-1}(y) \cup E \cup f^{-1}(y_0) \subset V$. We show that $C_2(f)(\langle U, V\rangle_2)$ is not open. Note that $\{d\} \cup E \in \langle U, V\rangle_2$. Therefore, $C_2(f)(\{d\} \cup E) = L \in C_2(f)(\langle U, V\rangle_2)$.

Let μ be a Whitney map in $C(X)$ (see Theorem 2.6). Since $\{y\}$ and $\{y_0\}$ are subsets of L, there are two segments $\sigma_1, \sigma_2 : [0, 1] \to C(X)$, with respect to μ, from $\{y\}$ to L and from $\{y_0\}$ to L, respectively, by Theorem 2.7. Since $\sigma_1(1) = \sigma_2(1) = L$, it is not difficult to prove that there exists a point $s \in [0, 1]$ such that $\sigma_1(s) \cap \sigma_2(s) \neq \emptyset$ and $\sigma_1(t) \cap \sigma_2(t) = \emptyset$, for each $t < s$. Observe that $\sigma_1(0) \cup \sigma_2(0) = \{y, y_0\}$. Hence, $s > 0$. Since L is irreducible between y and y_0, and $y, y_0 \in \sigma_1(s) \cup \sigma_2(s)$, we have that $\sigma_1(s) \cup \sigma_2(s) = L$. Let $\{t_m\}_{m\in\mathbb{N}}$ be an increasing sequence in $[0, 1]$ such that $\lim_{m\to\infty} t_m = s$. Clearly, $\sigma_1(t_m) \cup \sigma_2(t_m) \in C_2(Y) \setminus C(Y)$, for each $m \in \mathbb{N}$. We denote $L_m = \sigma_1(t_m) \cup \sigma_2(t_m)$. Since σ_1 and σ_2 are both continuous function, and $\lim_{m\to\infty} t_m = s$, we have that $\lim_{m\to\infty} L_m = L$.

We show that $L_m \notin C_2(f)(\langle U, V\rangle_2)$, for any $m \in \mathbb{N}$. Suppose that there exists $D \in \langle U, V\rangle_2$ such that $C_2(f)(D) = L_k$, for some $k \in \mathbb{N}$. Since $U \cap V = \emptyset$, $D \cap U \neq \emptyset$ and $D \cap V \neq \emptyset$, D has two components. Let E and F be the components of D such that $E \subset U$ and $F \subset V$. Note that y and y_0 belong to L_k. Thus, $D \cap f^{-1}(y) \neq \emptyset$ and $D \cap f^{-1}(y_0) \neq \emptyset$. Since $U \cap (f^{-1}(y) \cup f^{-1}(y_0)) = \emptyset$ and $E \subset U$, we have that $F \cap f^{-1}(y) \neq \emptyset$ and $F \cap f^{-1}(y_0) \neq \emptyset$. Hence, $f(F)$ is connected such that $y, y_0 \in f(F) \subset L_k$, but this contradicts the fact that y and y_0 belong to different components of L_k. Thus, $L_m \notin C_2(f)(\langle U, V\rangle_2)$, for any $m \in \mathbb{N}$. Since $\lim_{m\to\infty} L_m = L$ and $L \in C_2(f)(\langle U, V\rangle_2)$, we have that L is not an interior point of $C_2(f)(\langle U, V\rangle_2)$.

2. $n > 2$. Observe that $L \neq Y$ by Theorem 3.1. Hence, $X \setminus f^{-1}(L) \neq$

$\emptyset$. Let $x_1, x_2, \cdots, x_{n-3}$ and x_{n-2} be different points in $X \setminus f^{-1}(L)$. Let $U, V, W_1, \cdots, W_{n-3}$ and W_{n-2} be open and disjoint subsets of X such that $d \in U, f^{-1}(y) \cup E \cup f^{-1}(y_0) \subset V$ and $x_i \in W_i$, for each $i \in \{1, 2, \cdots, n-2\}$. Let us remind that each point in $\langle U, V, W_1, \cdots, W_{n-2}\rangle_n$ has n components. Therefore, using an argument similar to that in the case $n = 2$ in **1**, we may conclude that $C_n(f)(\langle U, V, W_1, \cdots, W_{n-2}\rangle_n)$ is not an open set.

Thus $C_n(f)$ is not open, for any $n \geq 2$, by **1** and **2**. Hence, $|f^{-1}(y)| = 1$ for each $y \in Y$. Since f is defined between continua, f is closed. Therefore, f is a homeomorphism.

□

Corollary 3.3. *Let $f : X \to Y$ be a map between continua and let $n \geq 2$. The following conditions are equivalent:*

1. *$C_n(f)$ is open;*

2. *f is a homeomorphism;*

3. *$C_n(f)$ is a homeomorphism.*

Proof. That **1** implies that **2** follows from Theorem 3.2. Using [4, Theorem 46, p. 801], we have that **2** implies that **3**. Finally, since every homeomorphism is open, we have that **3** implies that **1**.

□

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