On a fixed point theorem by Totik

Mohamed Akkouchi[1]

Faculté des Sciences–Semlalia
Département de Mathématiques
Université Cadi Ayyad, Avenue du Prince My. Abdellah, B.P. 2390
Marrakech – Maroc – Morocco

En este artículo se demuestra un teorema de punto fijo común (Teorema 3.1) en espacios métricos para dos auto–mapeos que satisfacen una relación implícita general que involucra el diámetro de conjuntos finitos sin exigir continuidad. Este teorema se puede considerar como una generalización de un resultado de Totik (1983). También, unifica y generaliza algunos otros resultados obtenidos por Fisher (1977), Akkouchi (2001) y Nova (1997).

Palabras claves: Teoremas de punto fijo común en espacios métricos

In this paper we prove a common fixed point theorem (see Theorem 3.1) in metric spaces for two self–mappings satisfying a general implicit relation involving the diameter of finite sets, without requiring continuity. This theorem may be considered as a generalization of a result by Totik (1983). Also, it unifies and generalizes some other results obtained by Fisher (1977), Akkouchi (2001) and Nova (1997).

Keywords: Common fixed point theorems in metric spaces.

MSC: 54H25, 47H10

1 Introduction

The common fixed point theory has seen a great developpement during the three last decades. One can say that its story has started with the well known result of Markov and Kakutani.

For two self–mappings S and T of a given metric space (X, d), many kind of contractive (exapnsive or nonexpansive) conditions may be considered. We list here some examples.

[1] akkouchimo@yahoo.fr

$$d(Sx,\, Sy) \le q\, d(Tx, Ty)\,, \quad \forall x,\, y \in X\,. \tag{1.1}$$

This condition generalizes the well known Banach contraction principle. It has been generalized by the following condition

$$\begin{aligned} d(Sx,\, Sy) \quad \le \quad & q \max\{d(Tx,\, Ty),\, d(Sx,\, Tx),\, d(Sy,\, Ty), \\ & d(Tx,\, Sy),\, d(Ty, Sx)\}\,. \end{aligned} \tag{1.2}$$

A more general condition may be defined by means of functions of five variables. That is contraction given by

$$\begin{aligned} d(Sx,\, Sy) \quad \le \quad & F(d(Tx,\, Ty),\, d(Sx,\, Tx),\, d(Sy,\, Ty), \\ & d(Tx,\, Sy),\, d(Ty,\, Sx)) \le 0\,, \end{aligned} \tag{1.3}$$

for all $x, y \in X$, where $F : [0, \infty)^5 \longrightarrow \mathbb{R}$ is a function.

The previous conditions do not involve compositions of the mappings S and T. In [3], Fisher has considered the following condition

$$\begin{aligned} d(Sx,\, TSy) \quad \le \quad & \alpha\, d(x,\, Sy) + \beta\, [d(x,\, Sx) + d(Sy,\, TSy)] \\ & + \gamma\, [d(x,\, TSy) + d(Sx,\, Sy)]\,, \end{aligned} \tag{1.4}$$

for all $x, y \in X$, where, $\alpha, \beta, \gamma \ge 0$, such that $\alpha + 2\beta + 2\gamma < 1$.

Fisher has proved in [3] that if S is continuous then S and T have a unique common fixed point.

This contraction was used by Nova [4] and Akkouchi [1] to establish some improvements to the previous result of Fisher.

In [2], Fisher made the following conjecture. Suppose S and T are self–mappings of the complete metric space X into itself, with either S or T continuous, satisfying the inequality

$$d(Sx,\, TSy) \le c\, \mathrm{diam}\{x,\, Sx,\, Sy,\, TSy\}\,, \tag{1.5}$$

for all $x, y \in X$, where $0 \le c < 1$. Then S and T have a unique common fixed point.

We observe that (1.5) is more general than the condition (1.4). Indeed, it can be shown that if the mappings S and T satisfy the condition (1.4), then they satisfy the condition (1.5) with $c := \alpha + 2\beta + 2\gamma$.

This conjecture has been solved by Totik who proved in [5] the following result.

Theorem 1.1. *If X is complete, $S : X \longrightarrow X$, $T : X \longrightarrow X$ with property (1.1), where $0 \leq c < \frac{1}{2}$, then S and T have a unique common fixed point. On the other hand, there is a four point set X and $S, T : X \longrightarrow X$, self–mappings of X without fixed point satisfying*

$$d(Sx, TSy) \leq \frac{1}{2} \operatorname{diam}\{x, Sx, Sy, TSy\}, \quad \forall x, y \in X. \tag{1.6}$$

Thus, if $0 \leq c < \frac{1}{2}$ we do not need any continuity assumption, and for $c \geq \frac{1}{2}$ even the simultaneous continuity of S and T and the compactness of X do not help.

The purpose of this paper is to give a generalization of this theorem (see Theorem 3.1 below) by using a general contractive condition defined by implicit relation of two variables (see section two and the condition (3.1) below). Implicit relations allow us to unify and generalize some results obtained in the papers [3], [1] and [4].

An example to support our result is given. We provide also two related common fixed point theorems for families of self–mappings in metric spaces (see Theorem 4.2 and Theorem 4.3). We point out that our results are established without making appeal to continuity.

The main result of this paper is established in the third section. In the fourth section, we have gathered some consequences and related results.

2 Implicit relations

We recall the a real valued function f defined on a topological space $(Y, \mathcal{T})$ is called lower semi–continuous on T if for every real number β, the set $\{y \in Y : f(y) > \beta\}$ is open in Y. This is equivalent to say that for every $\beta \in \mathbb{R}$, the set $\{y \in Y : f(y) \leq \beta\}$ is closed in Y.

In all this paper, (X, d) will be a metric space. If $A \subset X$ is a bounded subset of X, we shall denote the diameter of A by $\delta(A)$.

Let $\mathbb{R}_+$ be the set of all non–negative reals numbers and $\mathcal{F}_2$ the family of all lower semi–continuous mappings $F(t_1, t_2) : \mathbb{R}_+^2 \to \mathbb{R}$ satisfying the following conditions:

F1. F is non–increasing in the variable t_2.

F2. There exists $0 \le h < 1$ such that for all $u, v \ge 0$ with $F(u, u+v) \le 0$, we have $u \le hv$.

Remark 1.1. The condition **F2** implies the following property:

$$F(u,\, 2u) > 0\,, \quad \forall u \in (0,\, \infty) \quad \text{and} \quad \forall F \in \mathcal{F}_2\,.$$

Examples.

1. $F(t_1, t_2) = t_1 - c\,t_2$, with $0 \le c < \frac{1}{2}$.

2. $F(t_1, t_2) = \sqrt{t_1} - c\sqrt{\frac{t_2}{1+t_1^p}}$, with $0 \le c < \frac{1}{\sqrt{2}}$ and $p > 0$.

3. $F(t_1, t_2) = t_1^p - c\frac{t_2^p}{1+t_2}$, with $0 \le c < \frac{1}{2^p}$ and $p \ge 1$.

4. $F(t_1, t_2) = t_1^2 - \frac{1}{8}t_2^2 - \frac{1}{24}t_1 t_2$.

F1. It is clear that F is continuous and that F is non–increasing in the variable t_2.

F2. For all $u, v \ge 0$, we have $F(u, u+v) = u^2 - \frac{1}{8}(u+v)^2 - \frac{1}{24}u(u+v)$. Then, $F(u, u+v) \le 0$ is equivalent to say that

$$u^2 \le \frac{1}{8}(u+v)^2 + \frac{1}{24}u(u+v)\,,$$

which implies that

$$u^2 \le \left(\frac{1}{8} + \frac{1}{24}\right)(u+v)^2 = \frac{1}{6}\,(u+v)^2\,,$$

from which, we deduce that $u \le \frac{1+\sqrt{6}}{5}v = cv$, with $c = \frac{1+\sqrt{6}}{5} < 1$. So, F belongs to the set $\mathcal{F}_2$.

3 Main result

The following theorem is a generalization of Theorem 1.1.

Theorem 3.1. *Let S and T be two self–mappings of a metric space (X, d). Suppose that:*

i. *One of $S(X)$, $T(X)$ or X is complete, and*

ii. *there exists $F \in \mathcal{F}_2$ such that the following holds:*

$$F(d(Sx,\, TSy),\, \delta(\{x,\, Sx,\, Sy,\, TSy\}) \leq 0\,, \tag{3.1}$$

for all $x, y \in X$.

Then, S and T have a unique common fixed point z in X. Moreover, we have

$$S(X) \cap Fix(T) = Fix\{S,\, T\} = Fix(S) = \{z\}\,. \tag{3.2}$$

If in addition, S and T commute (i.e., $S \circ T = T \circ S$), then we have

$$S(Fix(T)) = Fix\{S,\, T\} = Fix(S) = \{z\}\,. \tag{3.3}$$

Proof. Let x_0 be an arbitrary point in X. We define inductively a sequence $\{x_n\}$ of points in X such that:

$$\begin{aligned} x_{2n+1} &= Sx_{2n}\,, \\ x_{2n+2} &= Tx_{2n+1}\,, \end{aligned} \tag{3.4}$$

for all $n = 0, 1, 2, \cdots$.

For each positive integer n, we set $t_n := d(x_n, x_{n+1})$.

For all positive integer n, by setting $x = x_{2n}$ and $y = x_{2n-2}$ in (3.1) we obtain

$$\begin{aligned} & F(d(Sx_{2n},\, TSx_{2n-2}),\, \delta(\{x_{2n},\, Sx_{2n},\, Sx_{2n-2},\, TS_{2n-2}\}) \\ = \;& F(d(x_{2n+1},\, x_{2n}),\, \delta(\{x_{2n-1},\, x_{2n},\, x_{2n+1}\}) \leq 0\,. \end{aligned}$$

As

$$\delta(\{x_{2n-1},\, x_{2n},\, x_{2n+1}\}) \leq t_{2n+1} + t_{2n}\,,$$

and since F is non–increasing in the second variable, we get

$$F(d(x_{2n},\, x_{2n+1}),\, d(x_{2n},\, x_{2n+1}) + d(x_{2n},\, x_{2n-1})) \leq 0\,, \quad \forall n \geq 1\,. \quad (3.5)$$

By virtue of property (F 2), we deduce that

$$d(x_{2n},\, x_{2n+1}) \leq hd(x_{2n-1},\, x_{2n})\,.$$

Similary, we obtain

$$d(x_{2n+1},\, x_{2n+2}) \leq hd(x_{2n},\, x_{2n+1})\,.$$

Therefore, for all positive integer n, we have

$$d(x_n,\, x_{n+1}) \leq hd(x_{n-1},\, x_n)\,.$$

Then, the sequence $\{x_n\}$ is a Cauchy sequence.

Assume that $S(X)$ is complete. Therefore, $\{x_{2n+1}\}$ converges to a point $z = Sv$ for some $v \in X$. Since $\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = 0$, we deduce that the whole sequence $\{x_n\}$ converges to the point z in X. Hence, we have

$$z = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1}\,. \quad (3.6)$$

Let us prove that $z = Tz$. By using the inequality (3.1) we have,

$$F(d(Sx_{2n},\, TSv),\, \delta(\{x_{2n},\, x_{2n+1},\, z,\, Tz\})) \leq 0\,. \quad (3.7)$$

As

$$\delta(\{x_{2n},\, x_{2n+1},\, z,\, Tz\}) \leq d(x_{2n},\, x_{2n+1}) + d(x_{2n+1},\, z) + d(z,\, Tz)\,,$$

and since F is decreasing in the second variable, from (3.7), we get

$$F(d(x_{2n+1},\, Tz),\, d(x_{2n},\, x_{2n+1}) + d(x_{2n+1},\, z) + d(z,\, Tz)) \leq 0\,. \quad (3.8)$$

Letting $n \longrightarrow \infty$, in (3.8), and using the lower semi–continuity of F, we obtain:

$$F(d(z, Tz), d(z, Tz)) \leq 0\,. \quad (3.9)$$

Suppose that $d(z,Tz) > 0$. Then by using the condition (F 2), we get

$$0 < d(z,\, Tz) \leq hd(z,\, Tz) < d(z,\, Tz)\,,$$

a contradiction. Hence, $d(z,Tz) = 0$. That is $z = Tz$.

By setting $x = z$ and $y = x_{2n}$ in (3.1), we get

$$F(d(Sz,\, x_{2n+2}),\, \delta(\{z,\, Sz,\, x_{2n+1},\, x_{2n+2}\})) \leq 0\,. \tag{3.10}$$

As

$$\delta(\{z,\, Sz,\, x_{2n+1},\, x_{2n+2}\}) \leq d(z,\, Sz) + d(Sz,\, x_{2n+1}) + d(x_{2n+1},\, x_{2n+2})\,,$$

and since F is decreasing in the second variable, from (3.10), we get

$$F(d(Sz,\, x_{2n+2}),\, d(z,Sz)+d(Sz,\, x_{2n+1})+d(x_{2n+1},\, x_{2n+2})) \leq 0\,. \tag{3.11}$$

Letting $n \longrightarrow \infty$ in (3.11), and using the lower semi–continuity of F, we obtain:

$$F(d(Sz,\, z),\, 2d(z,\, Sz)) \leq 0\,. \tag{3.12}$$

By using condition (F 2), we deduce that $d(Sz,z) = 0$, i.e., $Sz = z$. Thus z is a common fixed point of S and T.

The remainder of the proof is similar if we suppose that $T(X)$ or X is complete instead of $S(X)$. So we omit the details.

By using the assumption (F 2), it is easy to prove the uniqueness of z.

Let $w \in Fix(S)$ (i.e., $Sw = w$). We use the inequality (3.1) with $x = w$ and $y = z$. Then we obtain

$$F(d(Sw,\, TSz),\, \delta(\{w,\, Sw,\, Sz,\, TSz\}) = F(d(w,\, z),\, d(w,\, z)) \leq 0\,,$$

which implies that $w = z$. Thus we have proved that

$$Fix\{S,\, T\} = Fix(S) = \{z\}\,.$$

Let $w = Sv \in Fix(T)$. Then by using the inequality (3.1) we get

$$F(d(Sz,\, TSv),\, \delta(\{z,\, Sz,\, Sv,\, TSv\}) = F(d(z,\, w),\, d(z, w)) \leq 0\,,$$

which implies that $w = z$. Thus we have proved that

$$S(X) \cap Fix(T) = Fix\{S,\, T\} = Fix(S) = z\,.$$

Suppose that S and T commute. Let $y \in Fix(T)$ and set $w = Sy$. Then we have $Tw = TSy = STy = Sy = w$. Then by using the inequality (3.1) we get

$$F(d(Sz,\, TSy),\, \delta(\{z,\, Sz,\, Sy,\, TSy\}) = F(d(z,\, w),\, d(z,\, w)) \leq 0\,,$$

which implies that $w = z$. Thus we have proved that

$$S(Fix(T)) = Fix\{S,\, T\} = Fix(S) = z\,.$$

This ends the proof.

To support our result we give an example.

Example: We take $X = \{1, 2, 3, 4\}$ equipped with the metric d given by $d(1, 2) = d(3, 4) = \frac{3}{5}$, $d(1, 4) = d(2, 3) = \frac{2}{5}$, $d(1, 3) = \frac{1}{5}$, and $d(2, 4) = 1$. Let S and T be two mappings defined on X by setting $S(2) = 1$, $S(1) = S(3) = S(4) = 3$, $T(2) = 4$, and $T(1) = T(3) = T(4) = 3$.

It is easy to show that the self–mappings S and T are satisfying the following condition:

$$\begin{aligned} d(Sx,\, TSy)^2 \;\; &\leq \;\; \frac{1}{8}\,\delta(\{x,\, Sx,\, Sy,\, TSy\})^2 \\ &\quad + \frac{1}{24}\,d(Sx,\, TSy)\,\delta(\{x,\, Sx,\, Sy,\, TSy\})\,. \end{aligned}$$

The function F of Example 4 given by

$$F(t_1,\, t_2) = t_1^2 - \frac{1}{8}\,t_2^2 - \frac{1}{24}\,t_1\,t_2\,,$$

belongs to the set $\mathcal{F}_2$. So Theorem 3.1 may be applied to the mappings S and T. We see that S and T have 3 as unique common fixed point in X.

4 Related results

The following results are consequences of Theorem 3.1.

Theorem 4.1. *Let S be self–mappings of a metric space (X, d). Suppose that:*

(i) *One of $S(X)$ or X is complete, and*

(ii) *there exists $F \in \mathcal{F}_2$ such that the following holds:*

$$F(d(Sx, Sy), \delta(\{x, Sx, Sy\}) \leq 0, \tag{4.1}$$

for all $x, y \in X$.

Then, S is constant on X. That is, there exists a unique point z in X such that $Sx = z$ for all $x \in X$.

Proof. By applying Theorem (3.1) to the self–mappings S and $T = Id_X$ the identity mapping of X, there exists a unique common fixed point z of S and Id_X. Moreover, by (3.3), since S and Id_X commute, we have

$$S(X) = S(Fix(Id_X)) = Fix\{S, Id_X\} = Fix(S) = \{z\}\,.$$

We conclude that S must be constant.

Theorem 4.2. *Let $\{T_i : i \in I\}$ (I is a non–empty set) and S be self–mappings of a metric space (X, d) satisfying:*

(i) *For each $i \in I$, one of $T_i(X)$ or $S(X)$ or X is a complete.*

(ii) *There is some $F \in \mathcal{F}_2$ such that*

$$F(d(Sx, T_iSy), \delta(\{x, Sx, Sy, T_iSy\})) \leq 0\,, \tag{4.2}$$

for all $i \in I$ and for all $x, y \in X$.

Then the mappings $\{T_i : i \in I\}$, and S have a unique common fixed point in X. Moreover, for each $i \in I$, we have

$$S(X) \cap Fix(T_i) = Fix\{S, T_i\} = Fix(S) = \{z\}\,. \tag{4.3}$$

Proof. By Theorem 3.1, for each $i \in I$ there exists a unique point x_i such that

$$Sx_i = T_i x_i = x_i \,.$$

We may suppose that I contains more than two elements. Let i, j be two distinct elements in I. We set $x = x_i$ and $y = x_j$ in (4.2). Then we get

$$\begin{aligned} & F(d(Sx_i, T_j Sx_j), \delta(\{x_i, Sx_i, Sx_j, T_j Sx_j\})) \\ = \;& F(d(x_i, x_j), \delta(\{x_i, x_j\})) \leq 0 \,. \end{aligned}$$

The last inequality says that we have $F(d(x_i, x_j), d(x_i, x_j)) \leq 0$, which (by the property (F 2)) implies that $x_i = x_j$. The remainder is clear. This ends the proof.

Theorem 4.3. *Let $\{S_i : i \in I\}$ (I is a non–empty set) be a familly of self–mappings of a metric space (X, d) and let T be a self–mappings of X. We suppose that*

(i) $S_i \circ S_j = S_j \circ S_i$ *for all* $i, j \in I$.

(ii) $S_i \circ T = T \circ S_i$ *for all* $i \in I$.

(iii) *For each* $i \in I$, *one of* $S_i(X)$ *or* $T(X)$ *or* X *is complete.*

(iv) *There is some* $F \in \mathcal{F}_2$ *such that*

$$F(d(S_i x, TS_i y), \delta(\{x, S_i x, S_i y, TS_i y\})) \leq 0 \,, \tag{4.4}$$

for all $i \in I$ *and for all* $x, y \in X$.

Then the mappings $\{S_i : i \in I\}$, and T have a unique common fixed point z in X. Moreover, for each $i \in I$, we have

$$S_i(X) \cap Fix(T) = Fix\{S_i, T\} = \{z\} = S_i(Fix(T)) = Fix(S_i) \,. \tag{4.5}$$

Proof. By Theorem 3.1, for each $i \in I$ there exists a unique point x_i such that

$$S_i x_i = T x_i = x_i \,.$$

We may suppose that I contains more than two elements. Let i, j be two distinct elements in I. We set $x = x_i$ and $y = x_j$ in (4.4). Then, by using the condition (ii) above, we get

$$\begin{aligned} & F(d(S_i x_i, T S_i x_j), \delta(\{x_i, S_i x_i, S_i x_j, T S_i x_j\})) \\ = \ & F(d(x_i, S_i x_j), \delta(\{x_i, S_i x_j\})) \leq 0\,. \end{aligned}$$

The last inequality says that we have $F(d(x_i, S_i x_j), d(x_i, S_i x_j)) \leq 0$, which (by the property (F 2)) implies that $x_i = S_i x_j$. By using the condition (i), we have

$$S_j x_i = S_j S_i x_j = S_i S_j x_j = S_i x_j = x_i. \tag{4.6}$$

From the equalities (4.6) we deduce that x_i is common fixed point of S_j and T. By uniqueness, we deduce that $x_i = x_j$. The inequalities (4.5) are consequences from Theorem 3.1. This ends the proof.

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References

[1] M. Akkouchi, *A note on a common fixed point theorem of B. Fisher*, Boletín de Matemáticas, Nueva Serie **8**, 26–30 (2001).

[2] B. Fischer, *Results and a conjecture on fixed points*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **62**, 769–775 (1977).

[3] B. Fisher, *Results on common fixed points*, Math. Japonica **22**, 335–338 (1977).

[4] L. Nova, *Puntos fijos comunes*, Boletín de Matemáticas **4**, 43–47 (1997).

[5] V. Totik, *On two open problems of contractive mappings*, Publications de l'institut Mathématique, Nouvelle série **34**, 239–242 (1983).