On a New Relative Invariant Covering Dimension

D.N. Georgiou, A.C. Megaritis

Department of Mathematics, University of Patras, 26500 Patras, Greece
giorgiou@math.upatras.gr, megariti@master.math.upatras.gr

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Abstract: In [7] (see also [2, p. 35]) two relative covering dimensions, denoted by dim and \(\text{dim}^*\), defined and studied. In [3] and [4] we studied these dimensions and we gave some properties including subspace, sum, partition, compactification, and product theorems. Also, we gave partial answers for the questions which are given in [7]. Here we give and study a new relative covering dimension, denoted by \(r\text{-dim}\), which is different from \(\text{dim}\) and \(\text{dim}^*\). Finally, we give some questions concerning the new relative dimension \(r\text{-dim}\).

Key words: Covering dimension, relative dimension.


1. Introduction and preliminaries

The first infinite cardinal is denoted by \(\omega\). We also consider two symbols, \(\text{"−1" and } \text{"∞"}\), for which we suppose that:

(i) \(-1 < n < \infty\) for every \(n \in \omega\);
(ii) \(\infty + n = n + \infty = \infty\) and \(-1 + n = n + (-1) = n\) for every \(n \in \omega \cup \{-1, \infty\}\).

By a class of subsets we mean a class consisting of pairs \((Q, X)\), where \(Q\) is a subset of a topological space \(X\).

Let \(A\) and \(B\) be two disjoint subsets of a topological space \(X\). We say that a subset \(L\) of \(X\) is a partition between \(A\) and \(B\) if there exist two open subsets \(U\) and \(W\) of \(X\) such that (1) \(A \subseteq U, B \subseteq W\), (2) \(U \cap W = \emptyset\), and (3) \(X \setminus L = U \cup W\).

Let \(X\) be a topological space. A cover of \(X\) is a non-empty set of subsets of \(X\), whose union is \(X\). A cover \(c\) of \(X\) is said to be open (respectively, closed) if all elements of \(c\) is open (respectively, closed). A family \(r = \{R_t : t \in T\}\) of subsets of \(X\) is said to be a refinement of a family \(c = \{C_s : s \in S\}\) of

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subsets of $X$ if each element of $r$ is contained in an element of $c$, that is for every $t \in T$ there exists $s(t) \in S$ such that $R_t \subseteq C_{s(t)}$.

Define the order of a family $r$ of subsets of a space $X$ as follows:

(a) $\text{ord}(r) = -1$ if and only if $r$ consists of the empty set only;

(b) $\text{ord}(r) = n$, where $n \in \omega$, if and only if the intersection of any $n + 2$ distinct elements of $r$ is empty and there exist $n + 1$ distinct elements of $r$, whose intersection is not empty;

(c) $\text{ord}(r) = \infty$, if and only if for every $n \in \omega$ there exist $n$ distinct elements of $r$, whose intersection is not empty.

The given below definitions are actually the definitions of dimensions $\dim$ and $\dim^*$ given in [7] (see also [2]) for regular spaces.

**Definition 1.1.** We denote by $\dim$ the (unique) function with as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition:

$$\dim(Q, X) \leq n,$$

where $n \in \{-1\} \cup \omega$,

if and only if for every finite open cover $c$ of the space $X$ there exists a finite family $r_Q$ of open subsets of $Q$ refinement of $c$ which is a cover of $Q$ and $\text{ord}(r_Q) \leq n$.

**Definition 1.2.** We denote by $\dim^*$ the (unique) function with as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition:

$$\dim^*(Q, X) \leq n,$$

where $n \in \{-1\} \cup \omega$,

if and only if for every finite open cover $c$ of the space $X$ there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup\{V : V \in r\}$ and $\text{ord}(r) \leq n$.

In [3] and [4] we studied the above dimensions and we gave some properties including subspace, sum, partition, compactification, and product theorems. Also, we gave partial answers for the questions which are given in [7]. In this paper, we give and study a new relative covering dimension.
2. The new relative covering dimension

**Definition 2.1.** We denote by $r\text{-dim}$ the (unique) function that has as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$ satisfying the following condition

$$r\text{-dim}(Q, X) \leq n, \quad \text{where } n \in \{-1\} \cup \omega,$$

if and only if for every finite family $c$ of open subsets of $X$ such that

$$Q \subseteq \bigcup\{U : U \in c\},$$

there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that

$$Q \subseteq \bigcup\{V : V \in r\}$$

and $\text{ord}(r) \leq n$.

**Remark.** We observe that if $r\text{-dim}(Q, X) \leq n$, where $n \in \omega$, then for every finite family $c$ of open subsets of $X$ such that $Q \subseteq \bigcup\{U : U \in c\}$ there exists a finite family $r_Q$ of open subsets of $Q$ refinement of $c$ which is a cover of $Q$ and $\text{ord}(r_Q) \leq n$.

**Proposition 2.2.** Let $Q$ be a subset of a topological space $X$. The following statements are true:

(a) $$\dim(Q) \leq r\text{-dim}(Q, X),$$

where $\dim(Q)$ is the covering dimension of the subset $Q$ of $X$. Moreover, if the subset $Q$ of $X$ is open, then

$$\dim(Q) = r\text{-dim}(Q, X).$$

(b) $$\dim(Q, X) \leq \dim^*(Q, X) \leq r\text{-dim}(Q, X).$$

(c) If the subset $Q$ of $X$ is closed, then

$$\dim^*(Q, X) = r\text{-dim}(Q, X) \leq \dim(X),$$

where $\dim(X)$ is the covering dimension of $X$. 

Proof. (a) Let $r\text{-dim}(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $\dim(Q) \leq n$. Let $c_Q = \{U_1^Q, \ldots, U_m^Q\}$ be a finite open cover of the space $Q$. For every $i = 1, \ldots, m$ there exists an open subset $U_i$ of $X$ such that $U_i^Q = Q \cap U_i$. We consider the family $c = \{U_1, \ldots, U_m\}$. Then, the family $c$ is a finite family of open subsets of $X$ such that $Q \subseteq \bigcup_{i=1}^{m} U_i$. Since $r\text{-dim}(Q, X) = n$, there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup\{V : V \in r\}$ and $\text{ord}(r) \leq n$. We set $r_Q = \{Q \cap V : V \in r\}$. Then, the family $r_Q$ is a finite open cover of $Q$ refinement of $c_Q$ such that $\text{ord}(r_Q) \leq n$. Thus, $\dim(Q) \leq n$.

Now, we suppose that the subset $Q$ of $X$ is open. Clearly, it suffices to prove the inequality

$$r\text{-dim}(Q, X) \leq \dim(Q). \tag{1}$$

Let $\dim(Q) = n \in \omega \cup \{-1, \infty\}$. The inequality (1) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $r\text{-dim}(Q, X) \leq n$. Let $c$ be a finite family of open subsets of $X$ such that $Q \subseteq \bigcup\{U : U \in c\}$. Then, the family $c_Q = \{Q \cap U : U \in c\}$ is a finite open cover of the space $Q$. Since $\dim(Q) = n$, there exists a finite open cover $r_Q$ of $Q$ refinement of $c_Q$ such that $\text{ord}(r_Q) \leq n$. Obviously, the family $r_Q$ is a refinement of $c$. Also, since the subspace $Q$ of $X$ is open, every element of the family $r_Q$ is open subset of $X$. Thus, $r\text{-dim}(Q, X) \leq n$.

(b) It is known that $\dim^*(Q, X) \leq \dim^*(Q, X)$ (see [7]). So it suffices to prove the inequality

$$\dim^*(Q, X) \leq r\text{-dim}(Q, X). \tag{2}$$

Let $r\text{-dim}(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality (2) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $\dim^*(Q, X) \leq n$. Let $c$ be a finite open cover of the space $X$. Obviously, $Q \subseteq \bigcup\{U : U \in c\}$. Since $r\text{-dim}(Q, X) = n$ there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup\{V : V \in r\}$ and $\text{ord}(r) \leq n$. Thus, $\dim^*(Q, X) \leq n$.

(c) Suppose that the subset $Q$ of $X$ is closed. By (b) it suffices to prove the inequality

$$r\text{-dim}(Q, X) \leq \dim^*(Q, X). \tag{3}$$

Let $\dim^*(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality (3) is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $r\text{-dim}(Q, X) \leq n$. Let $c$ be a finite family of open subsets of $X$ such that $Q \subseteq \bigcup\{U : U \in c\}$. Since the subspace $Q$ of $X$ is closed, the family $c \cup \{X \setminus Q\}$ is a finite open cover of the space $X$. Also, since $\dim^*(Q, X) = n$, there exists a finite family $r$ of open subsets of $X$ refinement of $c \cup \{X \setminus Q\}$ such that $Q \subseteq \bigcup\{V : V \in r\}$ and $\text{ord}(r) \leq n$. 

Then, the family $r' = r \setminus \{ V \in r : V \subseteq X \setminus Q \}$ is a refinement of $c$ such that $Q \subseteq \bigcup \{ V : V \in r' \}$ and $\text{ord}(r') \leq n$. Thus, $\text{r-dim}(Q, X) \leq n$.

Also, it is clear that $\text{r-dim}(Q, X) \leq \dim(X)$. 

**Examples.**

(1) Let $(X, \tau)$ be a topological space, where $X = \{a, b, c, d\}$ and

\[
\tau = \{ \emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X \}.
\]

Let $Q = \{a, c\}$. We observe that $\text{r-dim}(Q, X) = 1$ and

\[
\dim(Q, X) = \dim^*(Q, X) = \dim(X) = 0.
\]

(2) Let $X$ be the space of the real numbers and $Q = \{0\}$. Then, $\text{r-dim}(Q, X) = 0$ and $\dim(X) = 1$.

(3) Let $X = [-1, 1]$ and $Q = \{-1, 1\}$. The family consisting of all sets of the form $[-1, b)$ for $b > 0$, $(a, 1]$ for $a < 0$, and $(a, b)$ is a basis for some topology in $X$. It is easy to see that $\dim(Q) = 0$ and $\text{r-dim}(Q, X) = 1$.

The relations between the dimension-like functions of the type $\dim$ are summarized in the following diagram, where “→” means “≤” and “⇒” means that “in general”:

\[
\begin{array}{ccc}
\dim(X) & \downarrow & \dim^*(Q, X) \\
\uparrow & \nearrow & \uparrow \\
\text{r-dim}(Q, X) & \Rightarrow & \dim(Q)
\end{array}
\]

It is known that (see [3], [4] and [7]) there exist examples such that in the above diagram the invariants $\dim(X)$, $\dim(Q, X)$, $\dim^*(Q, X)$, and $\dim(Q)$ to be different.

**Proposition 2.3.** For every subset $Q$ of a space $X$ the following conditions are equivalent:

1. $\text{r-dim}(Q, X) \leq n$.

2. For every finite family $c$ of open subsets of $X$ with $Q \subseteq \bigcup \{ U : U \in c \}$ there exists a family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup \{ V : V \in r \}$ and $\text{ord}(r) \leq n$. 


For every finite family \( \{U_1, U_2, \ldots, U_m\} \) of open subsets of \( X \) with \( Q \subseteq \bigcup_{i=1}^m U_i \) there exists a family \( \{V_1, V_2, \ldots, V_m\} \) of open subsets of \( X \) such that

\[
V_i \subseteq U_i \quad \text{for} \quad i = 1, \ldots, m,
\]

\[
Q \subseteq \bigcup_{i=1}^m V_i \quad \text{and} \quad \text{ord}(\{V_1, V_2, \ldots, V_m\}) \leq n.
\]

(4) For every family \( \{U_1, U_2, \ldots, U_{n+2}\} \) of open subsets of \( X \) with \( Q \subseteq \bigcup_{i=1}^{n+2} U_i \) there exists a family \( \{V_1, V_2, \ldots, V_{n+2}\} \) of open subsets of \( X \) such that

\[
V_i \subseteq U_i \quad \text{for} \quad i = 1, \ldots, n+2, \quad Q \subseteq \bigcup_{i=1}^{n+2} V_i \quad \text{and} \quad \bigcap_{i=1}^{n+2} V_i = \emptyset.
\]

**Proof.** (1) \( \Rightarrow \) (2) It is obvious.

(2) \( \Rightarrow \) (3) Let \( c = \{U_1, \ldots, U_m\} \) be a finite family of open subsets of \( X \) such that \( Q \subseteq \bigcup_{i=1}^m U_i \). By assumption there exists a family \( r \) of open subsets of \( X \) refinement of \( c \) such that \( Q \subseteq \bigcup \{V : V \in r\} \) and \( \text{ord}(r) \leq n \). For every \( V \in r \) we choose an element \( l(V) \) of \( \{1, \ldots, m\} \) such that \( V \subseteq U_{l(V)} \) and we set

\[
V_i = \bigcup \{V \in r : l(V) = i\}, \quad i = 1, \ldots, m.
\]

It is clear that \( \{V_1, \ldots, V_m\} \) is a family of open subsets of \( X \) such that \( V_i \subseteq U_i \) for \( i = 1, \ldots, m, \) \( Q \subseteq \bigcup_{i=1}^m V_i \), and \( \text{ord}(\{V_1, \ldots, V_m\}) \leq n \).

(3) \( \Rightarrow \) (4) It is obvious.

(4) \( \Rightarrow \) (1) Let \( c = \{U_1, \ldots, U_m\} \) be a finite family of open subsets of \( X \) such that \( Q \subseteq \bigcup_{i=1}^m U_i \). We prove that there exists a family \( \{V_1, \ldots, V_m\} \) of open subsets of \( X \) such that

\[
V_i \subseteq U_i \quad \text{for} \quad i = 1, \ldots, m, \quad Q \subseteq \bigcup_{i=1}^m V_i \quad \text{and} \quad \text{ord}(\{V_1, \ldots, V_m\}) \leq n.
\]

If \( m \leq n + 1 \), then the required family \( \{V_1, \ldots, V_m\} \) of open subsets of \( X \) is the family \( \{U_1, \ldots, U_m\} \). Let us suppose that \( m \geq n + 2 \). We consider the family \( g = \{G_1, \ldots, G_{n+2}\} \), where \( G_i = U_i \) for \( i = 1, \ldots, n + 1 \) and \( G_{n+2} = \bigcup_{i=n+3}^m U_i \). Obviously, \( Q \subseteq \bigcup_{i=1}^{n+2} G_i \). Therefore, by assumption there exists a family \( \{H_1, \ldots, H_{n+2}\} \) of open subsets of \( X \) such that \( H_i \subseteq G_i \) for \( i = 1, \ldots, n + 2 \), \( Q \subseteq \bigcup_{i=1}^{n+2} H_i \), and \( \bigcap_{i=1}^{n+2} H_i = \emptyset \). We consider the family \( w = \{W_1, \ldots, W_m\} \), where \( W_i = H_i \) for \( i = 1, \ldots, n + 1 \) and \( W_i = U_i \cap H_{n+2} \) for \( i = n + 2, \ldots, m \). It is clear that \( w \) is a family of open subsets of \( X \) such that

\[
W_i \subseteq U_i \quad \text{for} \quad i = 1, \ldots, m, \quad Q \subseteq \bigcup_{i=1}^m W_i \quad \text{and} \quad \bigcap_{i=1}^{n+2} W_i = \emptyset.
\]
If the intersection of any $n + 2$ distinct elements of $w$ is empty, then the required family $\{V_1, \ldots, V_m\}$ of open subsets of $X$ is the family $w$. We suppose that there exists a subset $A = \{i_1, \ldots, i_{n+2}\}$ of $\{1, \ldots, m\}$ such that $\cap_{i \in A} W_i \neq \emptyset$ and let $p = \{P_1, \ldots, P_m\}$ be the same family $w$ reordered so that

$$P_1 = W_{i_1}, \ldots, P_{n+2} = W_{i_{n+2}}.$$ 

Since $p = w$ and $\cap_{i=1}^{n+2} W_i = \emptyset$, there exists a subset $B_1 \neq \{1, \ldots, n + 2\}$ of $\{1, \ldots, m\}$ with $n + 2$ elements such that $\cap_{i \in B_1} P_i = \emptyset$. Applying the above construction to $p$ we find a family $w' = \{W'_1, \ldots, W'_m\}$ of open subsets of $X$ such that

$$W'_i \subseteq P_i \quad \text{for} \ i = 1, \ldots, m, \quad Q \subseteq \cup_{i=1}^{m} W'_i \quad \text{and} \quad \cap_{i=1}^{n+2} W'_i = \emptyset.$$ 

We observe that

$$W'_1 \subseteq W_{i_1}, \ldots, W'_{n+2} \subseteq W_{i_{n+2}}.$$ 

If the intersection of any $n + 2$ distinct elements of $w'$ is empty, then the required family $\{V_1, \ldots, V_m\}$ of open subsets of $X$ is the family $w'$. We suppose that there exists a subset $A' = \{i'_1, \ldots, i'_{n+2}\}$ of $\{1, \ldots, m\}$ such that $\cap_{i \in A'} W'_i \neq \emptyset$ and let $p' = \{P'_1, \ldots, P'_m\}$ be the same family $w'$ reordered so that

$$P'_1 = W'_{i'_1}, \ldots, P'_{n+2} = W'_{i'_{n+2}}.$$ 

Since $p' = w'$ and $\cap_{i=1}^{n+2} W'_i = \emptyset$, there exists a subset $B_2 \neq \{1, \ldots, n + 2\}$ of $\{1, \ldots, m\}$ with $n + 2$ elements such that $\cap_{i \in B_2} P'_i = \emptyset$. Applying the above construction to $p'$ we find a family $w'' = \{W''_1, \ldots, W''_m\}$ of open subsets of $X$ such that

$$W''_i \subseteq P'_i \quad \text{for} \ i = 1, \ldots, m, \quad Q \subseteq \cup_{i=1}^{m} W''_i \quad \text{and} \quad \cap_{i=1}^{n+2} W''_i = \emptyset.$$ 

We observe that

$$W''_1 \subseteq W'_{i'_1}, \ldots, W''_{n+2} \subseteq W'_{i'_{n+2}}.$$ 

Since the family $\{U_1, \ldots, U_m\}$ is finite, after a finite number of repetitions of the above process we find a family $\{V_1, \ldots, V_m\}$ of open subsets of $X$ such that

$$V_i \subseteq U_i \quad \text{for} \ i = 1, \ldots, m, \quad Q \subseteq \cup_{i=1}^{m} V_i \quad \text{and} \quad \text{ord}(\{V_1, \ldots, V_m\}) \leq n.$$ 

Thus, $r\dim(Q, X) \leq n$. 

3. Subspace theorems

In this section we give subspace theorems for the dimension $r$-dim.

**Proposition 3.1.** Let $K$ and $Q$ be two subspaces of a space $X$ with $K \subseteq Q$. If $K$ is a closed subspace of $X$ or $Q \setminus K$ is an open subspace of $X$, then

$$r \text{-dim}(K, X) \leq r \text{-dim}(Q, X).$$

**Proof.** Suppose that the subset $Q \setminus K$ of $X$ is open. Let

$$r \text{-dim}(Q, X) = n \in \omega \cup \{-1, \infty\}.$$  

The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $r \text{-dim}(K, X) \leq n$. Let $c$ be a finite family of open subsets of $X$ such that $K \subseteq \bigcup \{U : U \in c\}$. Since the subspace $Q \setminus K$ of $X$ is open, the family $c \cup \{Q \setminus K\}$ consists of open subsets of $X$ such that

$$Q \subseteq \bigcup \{U : U \in c\} \cup \{Q \setminus K\}.$$  

Also, since $r \text{-dim}(Q, X) = n$, there exists a finite family $r$ of open subsets of $X$ refinement of $c \cup \{Q \setminus K\}$ such that $Q \subseteq \bigcup \{V : V \in r\}$ and $\text{ord}(r) \leq n$. Thus,

$$r \text{-dim}(K, X) \leq n.$$  

**Proposition 3.2.** Let $Y$ be a subspace of a space $X$ and $Q \subseteq Y$. Then,

$$r \text{-dim}(Q, Y) \leq r \text{-dim}(Q, X).$$

**Proof.** Let $r \text{-dim}(Q, X) = n \in \omega \cup \{-1, \infty\}$. The inequality is clear if $n = -1$ or $n = \infty$. Let $n \in \omega$. We prove that $r \text{-dim}(Q, Y) \leq n$. Let $c_Y = \{U_1^Y, \ldots, U_m^Y\}$ be a finite family of open subsets of $Y$ such that $Q \subseteq \bigcup_{i=1}^m U_i^Y$. For every $i = 1, \ldots, m$, there exists an open subset $U_i$ of $X$ such that $U_i^Y = Y \cap U_i$. We set $c = \{U_1, \ldots, U_m\}$. The family $c$ is a finite family of open subsets of $X$ such that $Q \subseteq \bigcup_{i=1}^m U_i$. Since $r \text{-dim}(Q, X) = n$, there exists a finite family $r$ of open subsets of $X$ refinement of $c$ such that $Q \subseteq \bigcup \{V : V \in r\}$ and $\text{ord}(r) \leq n$. We consider the family

$$r_Y = \{V^Y = Y \cap V : V \in r\}.$$
Since \( Q \subseteq Y \), the family \( r_Y \) is a finite family of open subsets of \( Y \) refinement of \( c_Y \) such that
\[
Q \subseteq \bigcup \{ V^Y : V^Y \in r_Y \}.
\]
Also, since the family \( r_Y \) is refinement of the family \( r \) and \( \text{ord}(r) \leq n \), we have that \( \text{ord}(r_Y) \leq n \). Thus, \( r \text{-dim}(Q, Y) \leq n \).

4. Sum theorems

In this section we give sum theorems for the dimension \( r \text{-dim} \).

**Proposition 4.1.** Let \( Q \) be a subspace of a space \( X \). If \( X = X_1 \cup X_2 \), where \( Q \subseteq X_1 \cap X_2 \), \( r \text{-dim}(Q, X_1) \leq n \), and \( r \text{-dim}(Q, X_2) \leq n \), then \( r \text{-dim}(Q, X) \leq n \).

**Proof.** Let \( c = \{ U_1, \ldots, U_m \} \) be a finite family of open subsets of \( X \) with \( Q \subseteq \bigcup_{i=1}^m U_i \). By Proposition 2.3 (3) it suffices to prove that there exists a finite family \( s \) of open subsets of \( X \) shrinking of \( c \) such that \( Q \subseteq \bigcup \{ V : V \in s \} \) and \( \text{ord}(s) \leq n \). Since the family \( \{ X_1 \cap U_1, \ldots, X_1 \cap U_m \} \) is a finite family of open subsets of \( X_1 \) with \( Q \subseteq \bigcup_{i=1}^m (X_1 \cap U_i) \) and \( r \text{-dim}(Q, X_1) \leq n \), by Proposition 2.3 (3) there exists a family \( \{ V_1^1, \ldots, V_m^1 \} \) of open subsets of \( X_1 \) such that \( V_i^1 \subseteq X_1 \cap U_i \) for \( i = 1, \ldots, m \), \( Q \subseteq \bigcup_{i=1}^m V_i^1 \), and \( \text{ord}(\{ V_1^1, \ldots, V_m^1 \}) \leq n \). For \( i = 1, \ldots, m \) there exists an open subset \( V_i \) of \( X \) such that \( V_i^1 = X_1 \cap V_i \). We set
\[
W_i = U_i \cap V_i, \quad i = 1, \ldots, m.
\]
Obviously, we have \( W_i \subseteq U_i \) for \( i = 1, \ldots, m \) and \( Q \subseteq \bigcup_{i=1}^m W_i \). Moreover, since \( X_1 \cap W_i = U_i \cap V_i^1 \subseteq V_i^1 \) for \( i = 1, \ldots, m \) and \( \text{ord}(\{ V_1^1, \ldots, V_m^1 \}) \leq n \), we have
\[
\text{ord}(\{ X_1 \cap W_1, \ldots, X_1 \cap W_m \}) \leq n. \tag{4}
\]
The family \( \{ X_2 \cap W_1, \ldots, X_2 \cap W_m \} \) is a finite family of open subsets of \( X_2 \) with \( Q \subseteq \bigcup_{i=1}^m (X_2 \cap W_i) \). Also, since \( r \text{-dim}(Q, X_2) \leq n \), by Proposition 2.3 (3) there exists a family \( \{ V_1^2, \ldots, V_m^2 \} \) of open subsets of \( X_2 \) such that \( V_i^2 \subseteq X_2 \cap W_i \) for \( i = 1, \ldots, m \), \( Q \subseteq \bigcup_{i=1}^m V_i^2 \), and \( \text{ord}(\{ V_1^2, \ldots, V_m^2 \}) \leq n \). For \( i = 1, \ldots, m \) there exists an open subset \( V_i^1 \) of \( X \) such that \( V_i^2 = X_2 \cap V_i^1 \). We consider the family \( s = \{ H_1, \ldots, H_m \} \), where \( H_i = W_i \cap V_i^1 \) for \( i = 1, \ldots, m \). Obviously, we have \( H_i \subseteq W_i \subseteq U_i \) for \( i = 1, \ldots, m \) and \( Q \subseteq \bigcup_{i=1}^m H_i \). Moreover, since \( X_2 \cap H_i = W_i \cap V_i^2 \subseteq V_i^2 \) for \( i = 1, \ldots, m \) and \( \text{ord}(\{ V_1^2, \ldots, V_m^2 \}) \leq n \), we have
\[
\text{ord}(\{ X_2 \cap H_1, \ldots, X_2 \cap H_m \}) \leq n. \tag{5}
\]
We prove that $\text{ord}(s) \leq n$. Let $H_1, \ldots, H_{n+2}$ be pairwise distinct elements of $s$, and $x \in H_1 \cap \ldots \cap H_{n+2} \neq \emptyset$. Since $X = X_1 \cup X_2$, $x \in X_1$ or $x \in X_2$. If $x \in X_1$, then

$$x \in (X_1 \cap H_1) \cap \ldots \cap (X_1 \cap H_{n+2}) \subseteq (X_1 \cap W_1) \cap \ldots \cap (X_1 \cap W_{n+2}),$$

which contradicts the relation (4). If $x \in X_2$, then

$$x \in (X_2 \cap H_1) \cap \ldots \cap (X_2 \cap H_{n+2}),$$

which contradicts the relation (5). Thus, $\text{ord}(s) \leq n$ and, therefore, $\text{r-dim}(Q, X) \leq n$.

**Corollary 4.2.** Let $Q$ be a subspace of a space $X$. For every subset $A$ of $X$ such that $Q \subseteq A$ we have

$$\text{r-dim}(Q, X) \leq \max \{ \text{r-dim}(Q, A), \text{r-dim}(Q, (X \setminus A) \cup Q) \}.$$

**Proof.** Follows by Proposition 4.1 for $X_1 = A$ and $X_2 = (X \setminus A) \cup Q$.

**Corollary 4.3.** Let $Q$ be a subspace of a space $X$. If $X = X_1 \cup X_2$, where $Q \subseteq X_1 \cap X_2$, then

$$\text{r-dim}(Q, X) = \max \{ \text{r-dim}(Q, X_1), \text{r-dim}(Q, X_2) \}.$$

**Proof.** Let $\text{r-dim}(Q, X_1) = n_1$ and $\text{r-dim}(Q, X_2) = n_2$, where $n_1, n_2 \in \omega \cup \{ \infty \}$. We set $n = \max\{n_1, n_2\}$. Then, $\text{r-dim}(Q, X_1) \leq n$ and $\text{r-dim}(Q, X_2) \leq n$. By Proposition 4.1 we have $\text{r-dim}(Q, X) \leq n$. Also, by Proposition 3.2, $n_1 \leq \text{r-dim}(Q, X)$ and $n_2 \leq \text{r-dim}(Q, X)$. Thus, $n \leq \text{r-dim}(Q, X)$. By the above, $\text{r-dim}(Q, X) = n$.

**Proposition 4.4.** Let $Q_1$ and $Q_2$ be two subsets of a space $X$. Then,

$$\text{r-dim}(Q_1 \cup Q_2, X) \leq \text{r-dim}(Q_1, X) + \text{r-dim}(Q_2, X) + 1.$$

**Proof.** Let

$$\text{r-dim}(Q_1, X) = n_1 \quad \text{and} \quad \text{r-dim}(Q_2, X) = n_2.$$

We prove that

$$\text{r-dim}(Q_1 \cup Q_2, X) \leq n_1 + n_2 + 1.$$
Let $c$ be a finite family of open subsets of $X$ with $Q_1 \cup Q_2 \subseteq \cup \{U : U \in c\}$. Since $r\text{-}\dim(Q_1, X) = n_1$, there exists a finite family $r_1$ of open subsets of $X$ refinement of $c$ such that $Q_1 \subseteq \cup \{U : U \in r_1\}$ and $\text{ord}(r_1) \leq n_1$. Moreover, since $r\text{-}\dim(Q_2, X) = n_2$, there exists a finite family $r_2$ of open subsets of $X$ refinement of $c$ such that $Q_2 \subseteq \cup \{U : U \in r_2\}$ and $\text{ord}(r_2) \leq n_2$. We set $r = r_1 \cup r_2$. Then, $r$ is a family of open subsets of $X$ refinement of $c$ such that

$$Q_1 \cup Q_2 \subseteq \cup \{U : U \in r\} \quad \text{and} \quad \text{ord}(r) \leq n_1 + n_2 + 1.$$ 

\section{5. Partition and Product Theorems}

In this section we give partition, product, and compactification theorems for the dimension $r\text{-}\dim$.

\textbf{Proposition 5.1.} Let $Q$ be a normal subspace of a space $X$. If for every family $\{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\}$ of $n+1$ pairs of disjoint subsets of $X$, where $A_i$'s are closed in $X$ and $B_i$'s are closed in $Q$, there exist partitions $L_i$ between $A_i$ and $B_i$ such that

$$Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset,$$

then $r\text{-}\dim(Q, X) \leq n$.

\textbf{Proof.} By Proposition 2.3 (4) it suffices to show that for any family $\{U_1, \ldots, U_{n+2}\}$ of open subsets of $X$ with $Q \subseteq \cup_{i=1}^{n+2} U_i$ there exists a family $\{V_1, \ldots, V_{n+2}\}$ of open subsets of $X$ such that

$$V_i \subseteq U_i \quad \text{for} \quad i = 1, \ldots, n + 2, \quad Q \subseteq \cup_{i=1}^{n+2} V_i \quad \text{and} \quad \bigcap_{i=1}^{n+2} V_i = \emptyset.$$

Let $\{U_1, \ldots, U_{n+2}\}$ be a family of open subsets of $X$ with $Q \subseteq \cup_{i=1}^{n+2} U_i$. Since the space $Q$ is normal, there exists a closed cover $\{B_1, \ldots, B_{n+2}\}$ of $Q$ such that $B_i \subseteq U_i \cap Q$ for $i = 1, \ldots, n + 2$. We set

$$A_i = X \setminus U_i \quad \text{for} \quad i = 1, \ldots, n + 1.$$

The family $\{(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})\}$ consists of $n+1$ pairs of disjoint subsets of $X$, where $A_i$'s are closed in $X$ and $B_i$'s are closed in $Q$. Therefore by hypothesis there exist partitions $L_i$ between $A_i$ and $B_i$ such that $Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset$. That is, there exist open subsets $W_i, V_i$ of $X$ such that:

$$A_i \subseteq W_i, \quad B_i \subseteq V_i,$$

$$W_i \cap V_i = \emptyset,$$

$$X \setminus L_i = W_i \cup V_i \quad \text{for} \quad i = 1, \ldots, n + 1.$$
We set \( V_{n+2} = U_{n+2} \cap \bigcup_{i=1}^{n+1} W_i \). By the definition of \( A_i \)'s and (6), (7) we have that \( V_i \subseteq U_i \) for \( i = 1, 2, \ldots, n + 2 \). We prove that \( Q \subseteq \bigcup_{i=1}^{n+2} V_i \). We observe that
\[
\bigcap_{i=1}^{n+2} V_i = \bigcap_{i=1}^{n+2} (W_i \cup V_i) = \bigcap_{i=1}^{n+2} (X \setminus L_i) = X \setminus \bigcap_{i=1}^{n+2} L_i \supseteq Q. \tag{9}
\]
From (6), (9) and the relation \( B_{n+2} \subseteq U_{n+2} \) it follows that
\[
\bigcup_{i=1}^{n+2} V_i = \bigcup_{i=1}^{n+2} V_i \cup \left( U_{n+2} \cap \bigcup_{i=1}^{n+2} W_i \right)
= \left( \bigcup_{i=1}^{n+2} V_i \cup U_{n+2} \right) \cap \left( \bigcup_{i=1}^{n+2} V_i \cup \bigcup_{i=1}^{n+2} W_i \right)
\supseteq \bigcup_{i=1}^{n+2} B_i \cap Q = Q \cap Q = Q.
\]
We prove that \( \bigcap_{i=1}^{n+2} V_i = \emptyset \). From (7) we have
\[
\bigcap_{i=1}^{n+2} V_i = \bigcap_{i=1}^{n+2} V_i \cap \left( U_{n+2} \cap \bigcup_{i=1}^{n+2} W_i \right) \subseteq \bigcap_{i=1}^{n+2} V_i \cap \bigcup_{i=1}^{n+2} W_i = \emptyset.
\]

**Remark.** It was proved (see Proposition 2.2) that if the subset \( Q \) of \( X \) is closed, then
\[
\dim^*(Q, X) = \text{r-dim}(Q, X).
\]
So, by Proposition 2.4, Proposition 3.1, Corollary 3.2, Proposition 3.3, Corollary 3.4, and Proposition 4.2 of [4] we have the following propositions and product theorem for the dimension invariant r-dim.

**Proposition 5.2.** Let \( Q \) be a closed subspace of a normal space \( X \) satisfying \( \text{r-dim}(Q, X) \leq n \). Then, for every family \( \{(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})\} \) of \( n + 1 \) pairs of disjoint closed subsets of \( X \) there exist partitions \( L_i \) between \( A_i \) and \( B_i \) such that \( Q \cap \bigcap_{i=1}^{n+1} L_i = \emptyset \).

**Proposition 5.3.** For every closed subspace \( Q \) of a normal space \( X \) we have
\[
\text{r-dim}(Q, X) = \text{r-dim}(Q, \beta X) = \text{r-dim}(\beta Q, \beta X).
\]
Proposition 5.4. Let $Q^X$ be a closed subspace of a compact Hausdorff space $X$ and $Q^Y$ a closed subspace of a compact Hausdorff space $Y$. Then,

$$r\text{-dim}(Q^X \times Q^Y, X \times Y) \leq r\text{-dim}(Q^X, X) + r\text{-dim}(Q^Y, Y).$$

6. Questions

Question 1. Is it true the property of universality for dimension $r\text{-dim}$? That is, does there exists a universal element in the class $IP$ of all pairs $(Q^X, X)$, where $Q^X$ is a subset of a space $X$ such that $r\text{-dim}(Q^X, X) \leq n$?

Question 2. Is it true the product theorem for $r\text{-dim}$ in the realm of all metrizable spaces?

For some other questions on relative covering dimensions see [3] and [4].

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References