Hilbert action from Einstein´s equations

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Abstract

Starting from Einstein´s equations of General Relativity we obtain the Hilbert Action. We show this, considering the metric tensor and the Christoffel symbols as independent variables.

Keywords: General Relativity, metric tensor, Christoffel symbols.

Resumen

A partir de las ecuaciones de Einstein de la Relatividad General obtenemos la Acción de Hilbert. La demostración considera al tensor métrico y a los símbolos de Christoffel como variables independientes.

Palabras clave: Relatividad General, tensor métrico, símbolos de Christoffel.

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I. INTRODUCTION

The equations of the gravitational field were derived almost simultaneously by Einstein [1] and Hilbert [2] in November 1915. Einstein based his discovery in two principles: (i) the principle of general covariance, and (ii) the principle of equivalence. Essential requirements necessary to obtain his theory is that, in the absence of gravitational fields it should reduce to the special theory of relativity and that, in the limit of weak gravitational fields, Newtonian theory is recovered. In Einstein´s theory both the effects of gravitation and the geometry of space-time are described in terms of the metric tensor . The equations, in the absence of nongravitational fields, are [3]

Where and are the Ricci curvature tensor, the scalar of curvature and the enrgy-momentum tensor of matter fields, respectively. The Ricci tensor is related to the Riemann-Christoffel curvature tensor by , with

We employ the notation of a comma for partial derivative; then, for example

The Christoffel symbol is given in terms of and its derivatives as

Then, an expression like this must be substituted for each Christoffel symbol in (2). In terms of the metric tensor, the Ricci tensor is given by

Where, in the last line, we have to interchange the indexes ν and λ of the first two summands. Contracting the free indexes with the metric tensor we obtain the Ricci scalar .

The tensor is the inverse of in the sense that and they have null covariant derivative

Here we employ the semicolon notation for the covariant derivative; then, for example

Hilbert deduced Eqs.(1) from a variational principle, with an action given by

The choice for is based on the following consideration: since is the invariant volume element the Lagrangian density can be written as times a scalar, where is the determinant of the metric tensor. Furthermore, the metric can be set equal to its canonical form, and its first derivatives set to zero at any point, any nontrivial scalar must involve at least second derivatives of the metric. Thus, the only scalar we could construct from the Riemann tensor, which is in fact second order in the derivatives of the metric, is the Ricci scalar. Then, Hilbert figured that the simplest choice for is

The equations of motion (1) come out from the variation of (6) with respect to the variation of the metric tensor . The algebra to do this is quite complicated [4]. In many texts on General Relativity we find the use of one or both methods, briefly described above, for deducing the Einstein's equations.

On the other hand, it has been shown in [5] that it is possible to obtain the Lagrangian function from the equations of motion. This works well for the case of Maxwell equations. However, when one apply the method to (1) to obtain (7), we find that an easier procedure is to consider variations of the Christoffel symbols as independent variables along with the metric tensor. This remit us to the variational Palatini method to obtain (1), where both are independent variablers. In the next section we will do this.

II. HILBERT LAGRANGIAN FROM EINSTEIN´S EQUATIONS

As a first step we multiply (1) by , where is the variation of the metric tensor, and integrate over

Our goal is to construct the variation of the Hilbert action

We will use a set of known results involved in the study of General Relativity [3]. The identity [3, 7]

and the definition of the scalar allow us to rewrite (8) as

We observe that in the first integral of Eq.(10), the term is lacking to get the variation of the Hilbert action (9)

This term can be constructed as follows. From the property in Eq.(5), and the result [3, 7]

We obtain

Then, we can write the identity

Now, we multiply (13) by the variation of the Christoffel symbol and integrate over the volume of space-time

Where

With the given in terms of the metric tensor and its derivatives, as in (3). Recalling that , we integrate by parts to transform (14) into

Some surface terms have been set to zero by making the variations vanish at the boundary surface. From the definition of the Ricci tensor, we obtain

After a long algebra, we can rewrite (17) as

Substitution of this result in (16) leads to

Finally we add Eqs. (10) and (19) to obtain

which is

The first term is, as expected, the variation of Hilbert´s action and the corresponding integrand is Hilbert's Lagrangian.

III. CONCLUSIONS

We have applied the method of reversing Hamilton's principle of Ref. [5] to obtain Hilbert's Lagrangian from the equation of motion of the Gravitational Field. We assume, for the sake of simplicity, that the metric tensor and the Christoffel symbols are independent variables. We believe that the deduction presented in this paper, can be used in undergraduate and graduate courses on General Relativity to obtain Hilbert Lagrangian. The price to pay in this method is that the beauty analysis of the properties of this Lagrangian are not discussed up to this stage. This can be done once we know the explicit form of .

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REFERENCES

[1] Einstein, A., *Die Feldgleichungten der Gravitation*, Sitzungsberitche der Preussischen Akademie der Wissenschaften zu Berlin, 844 (1915).
[2] Hilbert, D., *Die Grundlagen der Physik (Erste Mitteilung),* Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen, Matematisch-physikalische Klasse, 53 (1915).
[3] We follow signature conventions of Weinberg, S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley and Sons, New York, 1972).
[4] d'Inverno, R., *Introducing Einstein's Relativity,* (Oxford University Press, New York, 1992).
[5] Young-Sea, Huang and Chyi-Lung, Lin, *A systematic method to determine the Lagrangian directly from the equations of motion*, Am. J. Phys. **70**, 741 (2002).
[6] In Palatini's method not only Einstein's equations are obtained but also the formula (3), relating the Christoffel symbols to the metric tensor. See, e. g. Chapter 11 in Ref. [4].
[7] See, for instance, Carmeli, M., *Classical Fields: General Relativity and Gauge Theory*, (John Wiley and Sons, New York, 1982), Chap. 3.