# An alternative proof of the $g$-lemma of finite-time thermodynamics

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**Abstract**

In this paper we present an alternative proof of the so-named $g$-lemma of finite-time thermodynamics. It is shown that $g$-lemma does not impose any restriction on the heat transfer laws or reflect any kind of criterion on whether a heat transfer law proposal could be valid in nature or not, however, the possibilities to work with new heat exchange models can be reached since combinations of heat transfer laws may be treated by using the invariant form of the $g$ function.

**Keywords:** $g$-lemma, Curzon-Ahlborn engine, heat transfer law.

**Resumen**

En este artículo se presenta una demostración alternativa del llamado lema $g$ de la termodinámica de tiempos finitos. Se muestra que el lema $g$ no impone restricción alguna sobre las funciones de transferencia de calor o que represente algún criterio para validar o no funciones de transferencia de calor para ser consideradas como válidas en la naturaleza, sin embargo, utilizando la forma invariante de la función $g(\eta)$ (que no depende las funciones de transferencia de calor entre los acoplamientos) es posible trabajar con nuevos modelos de intercambio de calor.

**Palabras clave:** Lema $g$**,** máquina de Curzon-Ahlborn, leyes de transferencia de calor.

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## I. INTRODUCTION

Classical equilibrium thermodynamics (CET) has played a very important role in the analysis and design of thermal engines [1]. The main role of CET within the study of thermal engines has consisted in providing superior or inferior bounds for process variables such as efficiency, work, heat and others. However, the CET bounds are only achievable in the reversible limit and usually they are far away from typical real values of the corresponding heat engine characteristics. Since around three decades finite-time thermodynamics (FTT) has been developed [2, 3, 4, 5, 6, 7, 8, 9]. One of the main purposes of FTT has been to formulate heat engine models under more realistic conditions than those of CET. By means of FTT heat engine models a reasonable good agreement between theoretical values of process variables and experimental data has been obtained [10, 11, 12, 13, 14, 15]. Practically all FTT models are elaborated within the context of optimization criteria, such as minimization of entropy generation [16], maximization of power output [10], optimization of profits [17] and maximization of a kind of ecological function [18]. In 1975, Curzon and Ahlborn [10] introduced a Carnot-like thermal engine in which there is no thermal equilibrium between the working fluid and the thermal reservoirs at the isothermal branches of the cycle. These authors demonstrated that such an engine produces nonzero power (contrary to the Carnot reversible engine), and that the power output can be maximized by varying the temperatures of the cycle's isothermal branches. The efficiency under maximum power conditions is [10],

$$\eta_{CA} = 1 - \sqrt{\frac{T_2}{T_1}}, \qquad (1)$$

where $T_1$ and $T_2$ are the temperatures of the hot and cold thermal reservoirs, respectively. Eq. (1) was obtained assuming that heat flows between thermal reservoirs and working fluid obey a Newton's cooling law, so this result depends on the type of heat transfer law. If a different one is used, Eq. (1) is not obtained [19, 20]. In 1991, an ecological optimization criterion for FTT-thermal cycles was proposed [18]. This criterion consists of the maximization of a function $E$ that represents a good trade off between high power output and low entropy production. This function is given by

$$E = P - T_2\sigma, \quad (2)$$

where $P$ is the power output of the cycle, $\sigma$ the total entropy production (system plus surroundings) per cycle, and $T_2$ the cold reservoir temperature. The FTT-regime under maximum-$E$ function leads to the following properties [18, 21]: the semisum property and the 75-25 corollary; that is, the maximum ecological efficiency is,

$$\eta_E \approx \frac{1}{2}(\eta_C + \eta_{CA}), \quad (3)$$

where $\eta_C$ is the Carnot efficiency and $\eta_{CA}$ is the maximum power efficiency. On the other hand, when $E$ function is maximized, the CA-cycle has a configuration that produces around 75% of the maximum power and only about 25% of the entropy produced under the maximum power regime [18, 21]. In ref. [21] it was showed that the semisum formula (Eq. (3)) and the 75-25 corollary are general properties of CA-endoreversible cycles, independent of the heat transfer laws used in a CA-engine model. These demonstrations were made by means of the so-called $g(\eta)$ function ($g$-lemma) [21]. In the present article we show an alternative demonstration of the so called $g$-lemma and we discuss the possibility that through $g$-lemma new heat transfer laws can be proposed. The article is organized as follows: In Sect. II, we present a resume of the $g$-lemma; in Sect. III, we show an alternative proof of this lemma; and we present some examples of unusual heat transfer "laws" that satisfy the $g$-lemma; in Sect. IV we briefly comment the lack of requirements on heat transfer laws necessary to comply $g$-lemma and finally some conclusions are given.

## II. THE $g$ FUNCTION AND $g$-LEMMA

For CA cycles, it was shown in Ref. [11] that the power output P and the universe's entropy production $\sigma$ are linked by $[P_N(x,y)/\sigma_N(x,y)] = g(x,y) = [P_{DP}(x,y)/$ $\sigma DPx,y$, where $gx,y=T1T2T1-T2-x-y$ with $x = T_1 - T_{1w}$ and $y = T_{2w} - T_2$ (see Figure 1), and the subscripts $N$ and $DP$ refer to Newton's and Dulong-Petit's laws of cooling respectively. These expressions were derived in the context of a CA treatment using the variables $x$ and $y$. However, when one adopts a treatment based on only one variable [3], namely the efficiency $\eta$, the expression for $g(x,y)$ becomes [18]

$$g(\eta) = \frac{T_1T_2\eta}{T_1 - T_2 - \eta T_1} = \frac{\eta}{\eta_c - \eta}T_2. \quad (4)$$

Thus, for the two heat transfer laws mentioned, we have

$$P(\eta) = g(\eta)\sigma(\eta), \quad (5)$$

besides, for instance, for heat transfer laws of the form,

$$Q_1 = \alpha(T_1^k - T_{1w}^k)$$

and

$$Q_2 = \beta(T_{2w}^k - T_2^k), \quad (6)$$

$\alpha$, $\beta$ being thermal conductances and $k = 1, 2, \ldots, n$ (for example, $k = 1$ for Newton heat transfer law), the functions $P(\eta)$ and $\sigma(\eta)$ are given by [21, 22]

$$P(\eta) = \gamma\eta \frac{(1-\eta)^k T_1^k - T_2^k}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}, \quad (7)$$

and

$$\sigma(\eta) = \frac{\gamma}{T_1T_2}\frac{[(1-\eta)T_1 - T_2][(1-\eta)^k T_1^k - T_2^k]}{\frac{\alpha}{\alpha+\beta}(1-\eta) + \frac{\beta}{\alpha+\beta}(1-\eta)^k}, \quad (8)$$

with $\gamma = \alpha\beta/(\alpha + \beta)$. On the other hand, if the heat transfer laws at the couplings between the working fluid and heat reservoirs have the form

$$Q_1 = \alpha(T_1 - T_{1w})^k,$$

and

$$Q_2 = \beta(T_{2w} - T_2)^k, \quad (9)$$

with $k > 0$ (for example, $k = 5/4$ for the Dulong-Petit heat transfer law), functions $P(\eta)$ and $\sigma(\eta)$ are given by [21, 22],

$$P(\eta) = \alpha\beta \frac{1}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{(k-1)/k}}\right]^k, \quad (10)$$

and

$$\sigma(\eta) = \frac{\alpha\beta}{T_1T_2}\frac{(1-\eta)T_1 - T_2}{1-\eta}\left[\frac{(1-\eta)T_1 - T_2}{\alpha^{1/k} + \beta^{1/k}(1-\eta)^{(k-1)/k}}\right]^k. \quad (11)$$

It was demonstrated in Ref. [21] that the function $g(\eta)$ given by Eq. (4), which links power output with entropy production is independent of any heat transfer law used in an endoreversible CA cycle. Starting from Figure 1, we have that the universe entropy production per cycle is given by

$$\sigma = \frac{Q_2}{T_2} + \frac{Q_2}{T_{2w}} - \frac{Q_1}{T_{1w}} - \frac{Q_1}{T_1} > 0. \quad (12)$$

By using the endoreversibility hypothesis [10], which is

$$\frac{Q_2}{T_{2w}} = \frac{Q_1}{T_{1w}}, \quad (13)$$

Eq. (12) reduces to

$$\sigma = \frac{Q_2}{T_2} - \frac{Q_1}{T_1} > 0. \quad (14)$$

If we use the first law of thermodynamics, we get

$$\sigma = \frac{Q_1 - P}{T_2} - \frac{Q_1}{T_1} = Q_1\left(\frac{1}{T_2} - \frac{1}{T_1}\right) - \frac{P}{T_2}, \qquad (15)$$

where $P$ is the work per cycle period (power output of one cycle). Multiplying Eq. (15) by $\eta$ we obtain

$$\eta\sigma = \eta Q_1\left(\frac{1}{T_2} - \frac{1}{T_1}\right) - \frac{\eta P}{T_2}, \qquad (16)$$

which becomes

$$P(\eta) = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2}\sigma(\eta), \qquad (17)$$

or

$$P(\eta) = \frac{\eta}{\eta_c - \eta} T_2 \sigma(\eta). \qquad (18)$$

Thus it was determined that the $g(\eta)$ function is independent of the heat transfer law corresponding to $Q_1$ and $Q_2$, which along Eqs. (12-18) are not specified.

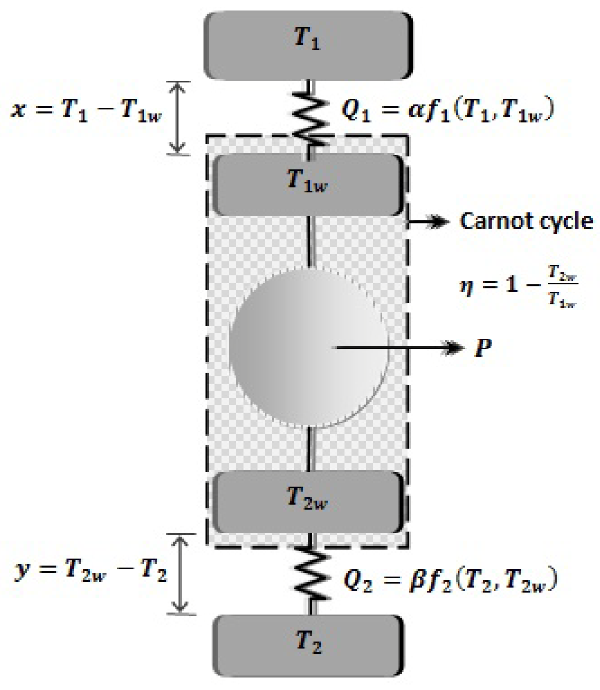


**FIGURE 1.** CA cycle.

## III ALTERNATIVE PROOF OF THE $g$-LEMMA

Let suppose two arbitrary "laws" (laws-like) of heat

$$Q_1 = \alpha f_1(T_1, T_{1w}), \quad Q_2 = \beta f_2(T_2, T_{2w}), \qquad (19)$$

where $T_1 \geq T_{1w} \geq T_{2w} \geq T_2$.

Endoreversible hypothesis (ERH) establishes that [10]:

$$\frac{Q_1}{T_{1w}} = \frac{Q_2}{T_{2w}}. \qquad (20)$$

Because the internal part is a Carnot engine, then

$$\eta = 1 - \frac{T_{2w}}{T_{1w}}. \qquad (21)$$

Substituting Eqs. (19) in Eq. (20) we obtain:

$$\gamma \frac{T_{1w}}{T_{2w}} f_2(T_2, T_{2w}) = f_1(T_1, T_{1w}), \qquad (22)$$

where $\gamma = \frac{\beta}{\alpha}$. Solving for $\frac{T_{1w}}{T_{2w}}$, Eqs. (21) and (22) lead to the following relation,

$$\frac{\gamma}{1-\eta} f_2(T_2, T_{2w}) = f_1(T_1, T_{1w}). \qquad (23)$$

The power output $P$ and the entropy production per cycle $\sigma$ are related to $Q_1$ and $Q_2$ as follows [10],

$$P = |Q_1| - |Q_2|, \qquad \sigma = -\frac{Q_1}{T_1} + \frac{Q_2}{T_2}, \qquad (24)$$

thus, from Eq. (23), both of them are functions of $\eta$,

$$P(\eta) = \left[\frac{\alpha\gamma}{1-\eta} - \beta\right] f_2(T_2, T_{2w}) = \beta \frac{\eta}{1-\eta} f_2(T_2, T_{2w}), \qquad (25)$$

$$\sigma(\eta) = \left[-\frac{\alpha\gamma}{(1-\eta)T_1} + \frac{\beta}{T_2}\right] f_2(T_2, T_{2w}), \qquad (26)$$

and therefore, by rewriting Eq. (26)

$$\sigma(\eta) = \beta\left[\frac{(1-\eta)T_1 - T_2}{(1-\eta)T_1 T_2}\right] f_2(T_2, T_{2w}), \qquad (27)$$

finally, we arrive to the desired result obtained in [21] and [18],

$$g(\eta) = \frac{P(\eta)}{\sigma(\eta)} = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2} = \frac{\eta}{\eta_c - \eta} T_2. \qquad (28)$$

Thus, the function $g(\eta)$ does not depend on the particular form of the functions $f_1(T_1, T_{1w})$ and $f_2(T_2, T_{2w})$.

### A. Examples

Below there are some examples of heat transfer laws-like different from the usual cases. Those heat transfer laws-like in each case permit that heat flows from the hot to the cold bodies.

### A.1 Example 1

Given the heat transport laws,

$$Q_1 = \alpha({T_1}^k - {T_{1w}}^k), \quad Q_2 = \beta(T_{2w} - T_2)^m. \qquad (29)$$

That is, different heat laws at upper and lower thermal couplings. Applying directly the Eqs. (25) and (27) from the last proof,

$$P(\eta) = \beta \frac{\eta}{1-\eta}(T_{2w} - T_2)^m , \qquad (30)$$

$$\sigma(\eta) = \beta \left[\frac{(1-\eta)T_1 - T_2}{(1-\eta)T_1 T_2}\right](T_{2w} - T_2)^m , \qquad (31)$$

and then,

$$g(\eta) = \frac{P(\eta)}{\sigma(\eta)} = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2} = \frac{\eta}{\eta_c - \eta} T_2.$$

That is the $g(\eta)$ function, so the $g$-lemma is fulfilled.
One might consider from Eqs. (30) and (31) that the value of $T_{1w}$ is free to be chosen in the $(T_2, T_1)$ interval (see Figure 1).

From the ERH we have,

$$\gamma \frac{T_{1w}}{T_{2w}}(T_{2w} - T_2)^m = ({T_1}^k - {T_{1w}}^k) , \qquad (32)$$

the solutions to this equation are plotted in Figure 2.

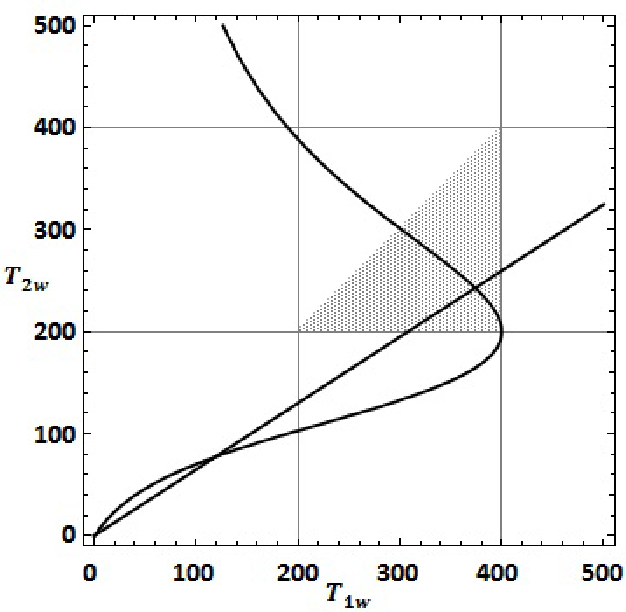


**FIGURE 2.** The solution to the ERH and $\eta = 1 - T_{2w}/T_{1w}$ are plotted for the values of $\eta = 0.35$, $T_1 = 400$, $T_2 = 200$, $k = 1.7$, $m = 2$ and $\gamma = 1$.

Note that some values of $T_{1w}$ and $T_{2w}$ that solve ERH are such that $T_{1w} < T_{2w}$ or $T_1 < T_{1w}$ or $T_{2w} < T_2$ which are not desired, only by restricting $T_{1w}$ and $T_{2w}$ to comply that the internal efficiency is a Carnot efficiency, one physical result is obtained. The shaded triangle in Figure 2 is actually the physical region where $T_{1w}$ and $T_{2w}$ can take values, the point of the curve that cross the right-bottom corner of the triangle correspond to the case where the CA engine is a Carnot engine ($\eta = 1 - T_2/T_1$) and points that cross the hypotenuse of the triangle correspond to zero efficiency. Any laws-like proposed in the examples of this work have a similar behavior.

The reason to have only one valid solution for $T_{1w}$ and $T_{2w}$ in the physical region might not be answered by just considering first and second laws of thermodynamics ($g$-lemma is a compactification of both laws), but another criteria must be applied to answer this question, this is not discussed in the present work.

**A.2 Example 2.**

Let

$$Q_1 = \alpha ln \frac{T_1}{T_{1w}} \quad \text{and} \quad Q_2 = \beta ln \frac{T_{2w}}{T_2} , \qquad (33)$$

be the heat transfer laws-like.

By using the ERH:

$$\frac{Q_1}{T_{1w}} = \frac{Q_2}{T_{2w}} \Rightarrow \left(\gamma \frac{T_{1w}}{T_{2w}}\right) ln \frac{T_{2w}}{T_2} = ln \frac{T_1}{T_{1w}}, \qquad (34)$$

and

$$\eta = 1 - \frac{T_{2w}}{T_{1w}}, \qquad (21)$$

and then replacing Eq. (21) into Eq. (34) yields to,

$$\frac{\gamma}{1-\eta} ln \frac{T_{2w}}{T_2} = ln \frac{T_1}{T_{1w}} . \qquad (35)$$

Thus

$$P(\eta) = \alpha ln \frac{T_1}{T_{1w}} - \beta ln \frac{T_{2w}}{T_2}$$

$$= \alpha \frac{\gamma}{1-\eta} ln \frac{T_{2w}}{T_2} - \beta ln \frac{T_{2w}}{T_2} = \left(\alpha \frac{\gamma}{1-\eta} - \beta\right) ln \frac{T_{2w}}{T_2}$$

$$= \beta \left(\frac{1}{1-\eta} - 1\right) ln \frac{T_{2w}}{T_2} = \beta \left(\frac{\eta}{1-\eta}\right) ln \frac{T_{2w}}{T_2} , \qquad (36)$$

and

$$\sigma(\eta) = -\frac{\alpha ln \frac{T_1}{T_{1w}}}{T_1} + \frac{\beta ln \frac{T_{2w}}{T_2}}{T_2} = \left(-\frac{\frac{\alpha\gamma}{1-\eta}}{T_1} + \frac{\beta}{T_2}\right) ln \frac{T_{2w}}{T_2}$$

$$= \frac{\beta}{T_2(1-\eta)}\left(-\frac{T_2}{T_1} + 1 - \eta\right) ln \frac{T_{2w}}{T_2}, \qquad (37)$$

and therefore,

$$g(\eta) = \frac{\beta\left(\frac{\eta}{1-\eta}\right)}{\frac{\beta}{T_2(1-\eta)}\left(-\frac{T_2}{T_1} + 1 - \eta\right)} = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2} = \frac{\eta}{\eta_c - \eta} T_2.$$

That is, the $g(\eta)$ function again for logarithmic heat transfer laws-like.

**A.3 Example 3**.

Let be,

$$Q_1 = \alpha senh\left(\frac{T_1}{T_{1w}} - 1\right), \text{ and}$$

$$Q_2 = \beta senh\left(\frac{T_{2w}}{T_2} - 1\right). \quad (38)$$

By using the ERH, we get,

$$\left(\frac{\gamma}{1-\eta}\right) senh\left(\frac{T_{2w}}{T_2} - 1\right) = senh\left(\frac{T_1}{T_{1w}} - 1\right). \quad (39)$$

Therefore

$$P(\eta) = (\frac{\beta}{1-\eta} - \beta) senh\left(\frac{T_{2w}}{T_2} - 1\right),$$

$$\sigma(\eta) = \left(-\frac{\beta}{T_1(1-\eta)} + \frac{\beta}{T_2}\right) senh\left(\frac{T_{2w}}{T_2} - 1\right), \quad (40)$$

$$g(\eta) = \frac{\frac{\eta}{1-\eta}}{\frac{1}{T_2(1-\eta)}\left(-\frac{T_2}{T_1} + 1 - \eta\right)} = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2}.$$

The same result that the previous one.

From Ex. 2 and 3, one can see that if $Q_1 = \alpha f_1(T_1/T_{1w})$ and $Q_2 = \beta f_2(T_{2w}/T_2)$, we end up with the following expression:

$$\left(\frac{\gamma}{1-\eta}\right) f_2\left(\frac{T_{2w}}{T_2}\right) = f_1\left(\frac{T_1}{T_{1w}}\right). \quad (41)$$

And following from this point as it was showed in the demonstration (Eqs. 23-28), the same result will be obtained despite the form of the heat transfer functions used at the couplings between working fluid and heat reservoirs.

## IV. REQUIREMENTS ON HEAT TRANSFER LAWS TO COMPLY $g$ LEMMA

The following section is intended to see if it is possible to find heat transfer laws which do not satisfy $g$-lemma.

By definition,

$$g(\eta) = \frac{P(\eta)}{\sigma(\eta)} = \frac{T_1 T_2 \eta}{(1-\eta)T_1 - T_2},$$

where $T_1, T_2$ are fixed parameters.

Let $Q_1$ and $Q_2$ be the heat transfer laws in a CA cycle. Note that the input conditions given are the temperatures $T_1$ and $T_2$. With these transfer laws $P$ and $\sigma$ are calculated. Again, ERH establishes that $\frac{Q_1}{T_{1w}} = \frac{Q_2}{T_{2w}}$, it is known that [3],

$$\sigma = \frac{Q_2}{T_2} - \frac{Q_1}{T_1} > 0, \quad (42)$$

and $P$ being a fraction of the input heat per cycle, it has the form

$$P = \eta Q_1, \quad (43)$$

then:

$$\frac{P}{\sigma} = \frac{\eta Q_1}{\frac{Q_2}{T_2} - \frac{Q_1}{T_1}} = \frac{\eta T_1 T_2}{T_1 \frac{Q_2}{Q_1} - T_2}. \quad (44)$$

Let's inquire which conditions must be satisfied, if any, by $Q_1$ and $Q_2$ to fulfill $g$-lemma. If it is taken for granted that,

$$\frac{\eta T_1 T_2}{T_1 \frac{Q_2}{Q_1} - T_2} = g = \frac{T_1 T_2 \eta}{T_1 - T_2 - \eta T_1}, \quad (45)$$

$$\Longrightarrow T_1 \frac{Q_2}{Q_1} - T_2 = T_1 - T_2 - \eta T_1, \quad (46)$$

we obtain the well known relation

$$\Longrightarrow \frac{Q_2}{Q_1} = 1 - \eta, \quad (47)$$

as it should happen. Moreover, since the internal machine is a Carnot engine, it holds that,

$$\eta = 1 - \frac{T_{2w}}{T_{1w}}. \quad (21)$$

In the CA cycle $T_{1w}$ and $T_{2w}$ are the variables to find. But as $Q_1 = Q_1(T_{1w})$ and $Q_2 = Q_2(T_{2w})$, we expect that

$$\frac{Q_2}{Q_1} = f(T_{1w}, T_{2w}), \quad (48)$$

is a function of $T_{1w}$ and $T_{2w}$ (the real variables to find since $T_1$ and $T_2$ are taken as fixed parameters), which implies that:

$$f(T_{1w}, T_{2w}) = 1 - \eta. \quad (49)$$

Eqs. (21) and (48) form a system of two equations with two unknowns ($T_{1w}$ and $T_{2w}$), since $\eta$ is given and because the system is soluble, $g$-lemma is satisfied. Therefore, it suffices to find the value of $T_{1w}$ (or $T_{2w}$) to find $T_{2w}$(or $T_{1w}$) and with (21) and (48) to find the other one. This is shown in the following example.

### A. Example 4.

Using the heat transfer laws-like from Example 2:

$$Q_1 = \alpha(lnT_1 - lnT_{1w}),$$

$$Q_2 = \beta(lnT_{2w} - lnT_2), \quad (50)$$

$$1 - \eta = \frac{Q_2}{Q_1} = \frac{\beta(lnT_{2w} - lnT_2)}{\alpha(lnT_1 - lnT_{1w})}. \quad (51)$$

From Eq. (21) $T_{2w} = T_{1w}(1-\eta)$, replacing $T_{2w}$ in Eq. (51)

$$1-\eta=\frac{\beta}{\alpha}\frac{[ln(T_{1w}(1-\eta))-lnT_2]}{(lnT_1-lnT_{1w})}, \quad (52)$$

solving for $T_{1w}$ we get:

$$T_{1w}=\left[\frac{T_1^{\frac{\alpha}{\beta}(1-\eta)}T_2}{1-\eta}\right]^{\frac{1}{1+(1-\eta)^{\alpha}/\beta}}, \quad (53)$$

implying that $Q_1$ and $Q_2$ satisfy $g$-lemma (45).

## V. CONCLUSIONS

It has been presented an alternative proof of $g$-lemma of finite time thermodynamics. It is also notable that the heat transfer functions need to be not physically acceptable, nor even real. It was known that $g(\eta)$ does not depend on the heat laws, but it was doubtful whether any heat laws agree whit the $g$-lemma or just a few would satisfy it. The arbitrariness in the choice of the heat transfer laws in Eqs. (19) is an indicative that apparently shows that the nature of the heat exchange is not subject to any restriction imposed by $g$-lemma. However, it is quite interesting that the nature of the heat exchange between the stores $T_1 \to T_{1w}$ and $T_{2w} \to T_2$ may be different (for example, radiative the first one and convective the second one) and the $g$-lemma is still valid.

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