

# Generation of dual and self-dual quantum mechanical potential systems



L. Buragohain<sup>1</sup>, S. A. S. Ahmed<sup>2</sup>

<sup>1</sup>Department of Physics, Chaiduar College, Gohpur-784168, India.

<sup>2</sup>Department of Physics, Gauhati University, Guwahati-781014, India.

**E-mail:** lburagohain@yahoo.com and sasa\_gup@hotmail.com

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## Abstract

Extended transformation method is applied to find dual and self-dual potentials for a general quantum mechanical multiterm potential. Exact bound state solutions of the Schrödinger equation for a specific multiterm potentials are obtained in any chosen dimensional space, using extended transformation (ET) method which may find applications in atomic, molecular, nuclear and particle Physics. We have found for multiterm power law potentials, under the framework of ET that a family relationship emerges among the parent and the newly generated exactly solvable potentials (ESPs). The normalizability of bound state solutions of the generated quantum systems is also discussed.

**Keywords:** Exactly solvable potential, Schrödinger equation, extended transformation method.

## Resumen

El Método de transformación extendida se aplica a los potenciales dual y auto-dual para un potencial mecánico cuántico multitérmino. Soluciones exactas de estado acotado a la ecuación de Schrödinger para potenciales multitérminos específicos son obtenidas en cualquier espacio multidimensional escogido, usando un método de transformación extendida (ET) el cual puede encontrar aplicaciones en Física atómica, molecular, nuclear y de partículas. Hemos encontrado que para una ley de potencia para potenciales multitérmino, bajo el esquema ET una relación de familia emerge entre los potenciales padres y los potenciales exactos solubles nuevos generados (ESPs). La normalización de las soluciones de estado acotadas de los sistemas cuánticos generados es también discutida.

**Palabras clave:** Potencial resoluble exactamente, ecuación de Schrödinger, método de transformación extendida.

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## I. INTRODUCTION

Considerable efforts have been made in recent years towards obtaining exact analytic solution (EAS) of the Schrödinger equation for potentials of physical interest [1, 2, 3, 4, 5, 6]. The EAS of the Schrödinger equation of a given quantum system (QS) represents an important problem in many fields of physics with much attention and with numerous applications as it provides us insight into the physical problem in question. However only a very few potentials governing physical systems yield analytical solutions. The study of integral power potential model has relevance in connection with the imaginary time formation of quantum mechanics and its relation with diffusion theory. It has been found widely used in the atomic and in molecular Physics. We have applied the extended transformation (ET) method [7, 8] to generate class of new exactly solvable potentials (ESPs) in any preassigned dimensional spaces in the non-relativistic quantum mechanics. The method of generation of ESP is based on transformation which includes a coordinate transformation

(CT) and is followed by a functional transformation (FT) and a set of plausible standard ansatz to restore the transformed equation to standard Schrödinger equation form. The CT alone can generate new ESPs from an old one. But it leads the problem regarding dimensionality of the Euclidean space of the generated quantum system. The FT allows us to have dimensional extension and/or dimensional reduction of the generated quantum systems. Starting from exactly solved  $k$ -term potential, ET can in principle generate  $2^{k-1}$  different new exactly solved potentials. A very important property of the ET formalism is that the wavefunctions of the generated quantum systems are almost always normalizable provided the behaviour of transformation function  $g_B(r)$  is smooth. The paper is organized as follows: In section II formalism of ET is given. In the section III normalizability condition of the generated quantum system is discussed. In section IV dual and self-dual ESP are generated from the already known singular integral power-law potential. Section V comprises of findings of our investigations.

## II. FORMALISM

We have considered a general quantum mechanical multiterm potential  $V_A(r)$  in  $D_A$ -dimensional space and is given by:

$$V_A(r) = \sum_{i=1}^3 a_i r^{\alpha_i} \tag{1}$$

The parameters of the potential are defined by  $a_i$ 's and is termed as A-quantum system (A-QS).

The radial part of the Schrödinger equation in  $D_A$ -dimensional space for A-QS ( $\hbar = 2m = 1$ ) is

$$\Psi_A''(r) + \frac{D_A-1}{r} \Psi_A'(r) + \left[ E_A - V_A(r) - \frac{l_A(l_A+D_A-2)}{r^2} \right] \Psi_A(r) = 0, \tag{2}$$

where the normalized eigenfunctions  $\Psi_A(r)$  and energy eigenvalue  $E_A$  are known for the given  $V_A(r)$ . Prime denotes the differentiation of the function with respect to its argument.

Under extended transformation (ET), which consists of a coordinate transformation  $r \rightarrow g_B(r)$  and is followed by a functional transformation of the wave function:

$$\Psi_B(r) = f_B^{-1}(r) \Psi_A(g_B(r)), \tag{3}$$

the transformation function  $g_B(r)$  and the modulated amplitude function  $f_B(r)$  have to be specified within the framework of ET. As  $\Psi_A(r)$  is the eigenfunction of an exactly solved potential, hence  $\Psi_B(r)$  also gets specified exactly, henceforth called B-QS.

The transformed B-QS after implementing ET on A-QS becomes:

$$\Psi_B''(r) + \left( \frac{d}{dr} \ln \frac{f_B^2 g_B^{D_A-1}}{g_B'} \right) \Psi_B'(r) + \left[ \left( \frac{d}{dr} \ln f_B \right) \left( \frac{d}{dr} \ln \frac{f_B^2 g_B^{D_A-1}}{g_B'} \right) + g_B'^2 \left( E_A - V_A(g_B(r)) - \frac{l(l_A+D_A-2)}{g_B^2} \right) \right] \Psi_B(r) = 0. \tag{4}$$

To mould the above equation to the standard Schrödinger equation form in a chosen  $D_B$ -dimensional Euclidean space, the coefficient of  $\Psi_B(r)$  in the equation (4) is put as:

$$\frac{d}{dr} \ln \frac{f_B^2 g_B^{D_A-1}}{g_B'} = \frac{D_B-1}{r} = \frac{d}{dr} \ln r^{D_B-1}. \tag{5}$$

This yield:

$$f_B(r) = N g_B^{1/2} g_B^{-D_A-1/2} r^{D_B-1/2}. \tag{6}$$

The transformation function  $g_B(r)$  is at least three times differentiable and  $f_B(r)$  is non-singular function of  $r$ .

The corresponding  $D_B$ -dimensional Schrödinger equation for the B-QS is found to be:

$$\Psi_B''(r) + \frac{D_B-1}{r} \Psi_B'(r) + \left[ \frac{1}{2} \{g_B, r\} - \frac{D_A-1}{2} \frac{D_A-3}{2} \left( \frac{g_B'}{g_B} \right)^2 + \frac{D_B-1}{2} \frac{D_B-3}{2} \frac{1}{r^2} \right. \\ \left. g_B'^2 \left( E_A - V_B(g_B(r)) - \frac{(l_A + \frac{D_A}{2} - 1)^2}{g_B^2} + \frac{(D_A-2)^2}{4g_B^2} \right) \right] \Psi_B(r) = 0, \tag{7}$$

where

$$\{g_B, r\} = \frac{g_B'''(r)}{g_B'(r)} - \frac{3}{2} \frac{g_B''(r)^2}{g_B'(r)^2} \tag{8}$$

is the Schwartzian derivative symbol .

To implement ET on A-QS, we have to select a term of  $V_A(g_B(r))$  as working potential denoted by  $V_A^W(g_B(r))$  and make the following ansatz:

$$g_B'^2(r) V_A^W(g_B(r)) = -E_B, \tag{9}$$

$$-g_B'^2 \left[ E_A - (V_A(g_B) - V_A^W(g_B)) \right] = -[V_B^{(1)}(r) + V_B^{(2)}(r)], \tag{10}$$

$$\frac{g_B'^2 \left( l_A + \frac{D_A}{2} - 1 \right)^2}{g_B^2} = \frac{\left( l_B + \frac{D_B}{2} - 1 \right)^2}{r^2}. \tag{11}$$

Let the working potential be selected as

$$V_A^W(g_B(r)) = a_1 g_B^{\alpha_1}(r), \tag{12}$$

which will specify, by equation (9), the functional form of the transformation function  $g_B(r)$  and is:

$$g_B(r) = \left[ \frac{\alpha_1 + 2}{2} \left( -\frac{E_B}{a_1} \right) r \right]^{\frac{2}{\alpha_1 + 2}}, \tag{13}$$

obtained by a simple integration. The transformation function  $g_B(r)$  has the desirable local property  $g_B(0)=0$  and asymptotic property  $g_B(\infty)=\infty$  by putting the integration constant equal to zero.

Equations (10) and (13) lead to:

$$V_B(r) = A_1 r^{\beta_1} + A_2 r^{\beta_2} + A_3 r^{\beta_3}. \tag{14}$$

We define the parameters of the potential as

$$C_B^2 = -E_A \left( \frac{\alpha_1 + 2}{2} \left( -\frac{E_B}{a_1} \right)^{\frac{1}{2}} \right)^{\frac{4}{\alpha_1 + 2}} \left( \frac{2}{\alpha_1 + 2} \right)^2 = A_1, \tag{15}$$

### III. NORMALIZABILITY OF THE GENERATED QUANTUM SYSTEM

A very important property of the extended transformation is that it seems to preserve the normalizability property of the wavefunctions of generated quantum systems. The normalizability condition for  $D_B$ - dimensional B-QS is can be proved under a fairly general conditions and is given by

$$\int_0^{\infty} |\Psi_B(r)|^2 r^{D_B-1} dr = \frac{1}{|N_B|^2} = \text{finite}. \quad (22)$$

This leads to normalization constant  $N_B$  as;

$$N_B = \left[ \frac{-E_B}{\langle V_A^W(g_B(r)) \rangle} \right]^{\frac{1}{2}}. \quad (23)$$

The expectation value for a potential of exactly solved quantum system is always finite and so a part of it also finite. The A-QS eigenfunction  $\Psi_A(r)$  is the normalized wavefunction of a genuine quantum mechanical system. Its existence also implies that  $\Psi_B(r)$  are also normalizable for  $E_B \neq 0$ , since the behaviour of  $g_B(r)$  is smooth so far local and asymptotic behaviour are concerned. Transformation function carries over the normalizability property of the parent quantum system (QS) to the daughter QSs.

### IV. APPLICATION IN INTEGRAL POWER SINGULAR POTENTIAL

#### A. Dual quantum systems

We have considered the following exactly solved quantum system [9] to investigate the condition of dual and self-duality of a potential:

$$V_A(r) = ar^2 + br^{-4} + cr^{-6}, \quad a > 0; \quad c > 0. \quad (24)$$

The exact analytic solution in 3-dimensional spaces is provided by [9] as

$$\Psi_A(r) = N_A r^{\frac{3+b}{\sqrt{c}}} \exp\left[-\frac{1}{2}(\sqrt{ar^2 + \sqrt{cr^{-2}}})\right]. \quad (25)$$

With a constraint

$$(2\sqrt{c} + b)^2 = c \left[ (2l_A + 1)^2 + 8\sqrt{ca} \right]. \quad (26)$$

The energy eigenvalue for the potential system is provided by [9] and is:

where  $C_B^2$  is the Characteristic Constant of B-QS and

$$A_2 = \left[ \frac{\alpha_1 + 2}{2} \left( -\frac{E_B}{a_1} \right)^{1/2} \right]^{\frac{4+2\alpha_2}{\alpha_1+2}} \left( \frac{2}{\alpha_1 + 2} \right)^2 a_3, \quad (16)$$

$$A_3 = \left[ \frac{\alpha_1 + 2}{2} \left( -\frac{E_B}{a_1} \right)^{1/2} \right]^{\frac{4+2\alpha_3}{\alpha_1+2}} \left( \frac{2}{\alpha_1 + 2} \right)^2 a_2. \quad (17)$$

The exponents of the B-QS potential are defined as:

$$\beta_1 = -\frac{2\alpha_1}{\alpha_1 + 2}, \quad \beta_2 = \frac{-2\alpha_1 + 2\alpha_3}{\alpha_1 + 2} \quad \text{and} \quad \beta_3 = \frac{-2\alpha_1 + 2\alpha_2}{\alpha_1 + 2}. \quad (18)$$

The energy eigenvalue of B-QS is obtained from Eq. (15) and is:

$$E_B = -a_1 \left[ \frac{C_B^2}{(-E_A) \left( \frac{\alpha_1 + 2}{2} \right)^{\frac{2\alpha_1}{\alpha_1 + 2}}} \right]^{\frac{\alpha_1 + 2}{2}}. \quad (19)$$

The corresponding  $D_B$ -dimensional Schrödinger equation for B-QS reduces to

$$\Psi_B''(r) + \frac{D_B - 1}{r} \Psi_B'(r) + \left[ E_B - V_B(r) - \frac{l_B(l_B + D_B - 2)}{r^2} \right] \Psi_B(r) = 0 \quad (20)$$

We have obtained following relations from Eq. (18):

$$(\alpha_1 + 2)(\beta_1 + 2) = 4, \quad (21a)$$

$$(\alpha_1 + 2)(\beta_2 + 2) = 2(\alpha_3 + 2), \quad (21b)$$

$$(\alpha_1 + 2)(\beta_3 + 2) = 2(\alpha_2 + 2). \quad (21c)$$

These equations specify the  $\beta_j$ s of the dual B-QS from the known  $\alpha_i$ s of A-QS and the generalization of the duality relation of the one term potential.

Referring to equations (21a-21c) one finds that if the transformed B-QS is identical to the untransformed A-QS for a particular working potential say,  $a_1 r^{\alpha_1}$ , implies that  $\alpha_1 = \beta_1$  which fixes either  $\alpha_1 = 0$  or  $-4$ . For non-trivial transformation,  $\alpha_1 = -4$ .

A further consequence is  $\alpha_2 + \alpha_3 = -4$ . Therefore we may conclude that a three term potential will be self-dual under ET when the working potential chosen has the exponent (-4), and also the sum of the other two exponents is again (-4). This immediately rules out of any two term self-dual potential.

$$E_A = \sqrt{a} \left( 4 + \frac{b}{\sqrt{c}} \right). \quad (27)$$

Selecting  $V_A^W(r) = ar^2$  as the working potential and utilizing equation (9), yields

$$g_B(r) = \pm 2^{1/2} \left( -\frac{E_B}{a} \right)^{1/4} r^{1/2}. \quad (28)$$

Applying equation (10), we have found the following potential of B-QS as:

$$V_B(r) = -A_1 r^{-1} + B_1 r^{-4} + C_1 r^{-3} \quad (29)$$

Where the parameters of the potential are:

$$A_1 = C_B^2 = \frac{1}{2} \left( -\frac{E_B}{a} \right)^{1/2} E_A, \quad (30)$$

$$B_1 = \frac{c}{16} \left( -\frac{E_B}{a} \right)^{-1}, \quad C_1 = \frac{b}{8} \left( -\frac{E_B}{a} \right)^{-1/2}. \quad (31)$$

The relation between the angular momentum quantum numbers  $l_A$  and  $l_B$  of A-QS and B-QS are obtained from equation (11) as

$$2l_A + 1 = 2(2l_B + D_B - 2). \quad (32)$$

The energy eigenvalue of B-QS comes out from the equation (30) as

$$E_B = -A_1^2 \left( 2 + \frac{C_1}{\sqrt{B_1}} \right)^{-2}. \quad (33)$$

With the constraint equation

$$\left( 2 + \frac{C_1}{\sqrt{B_1}} \right)^2 = (2l_B + D - 2)^2 + 8\sqrt{-E_B} \sqrt{B_1}. \quad (34)$$

The exact energy eigenfunction of B-QS obtained from equation (3) in D-dimensional Euclidean spaces and is:

$$\Psi_B(r) = N_B r^{\frac{3-D+\frac{C_1}{\sqrt{B_1}}}{2}} \exp \left[ -\frac{A_1}{2 + \frac{C_1}{\sqrt{B_1}}} r - \frac{\sqrt{B_1}}{r} \right]. \quad (35)$$

The normalization constant  $N_B$  of B-QS is defined as

$$N_B = \left[ \frac{-E_B}{\langle ar^2 \rangle_A} \right]^{1/2}.$$

The shape of the potential as well as eigenfunctions and corresponding energy eigenvalue for B-QS are shown in Figure 1.

Regarding the values of the parameters of the potential we fix them as follows: for a fixed values of  $B_1$  and  $C_1$  the value of  $A_1$  is determined from the constraint equation as given by equation (34).

### B. Self-dual quantum system

Selecting the second term of equation (24) as working potential

$$V_A^W(r) = br^{-4} \quad (36)$$

and utilizing equation (9), we have found the transformation function as

$$g_C(r) = -\left( -\frac{E_C}{b} \right)^{1/2} r^{-1}. \quad (37)$$

Applying equation (10), we have the following self-dual potential of C-QS:

$$V_C(r) = A_2 r^2 + B_2 r^{-4} + C_2 r^{-6} \quad (38)$$

Where the parameters of the potential are:

$$B_2 = C_C^2 = \frac{bE_A}{E_C}, \quad (39)$$

$$A_2 = \frac{cE_C^2}{b^2}, \quad C_2 = \frac{ab^2}{E_C^2}. \quad (40)$$

Energy eigenvalue of C-QS from equation (34) comes out

$$E_C = \sqrt{A_2} \left( 4 + \frac{B_2}{\sqrt{C_2}} \right), \quad (41)$$

With the following constraint equation between the parameters of the potential

$$\left( 2 + \frac{B_2}{\sqrt{C_2}} \right)^2 = (2l_C + D - 2)^2 + 8\sqrt{A_2} \sqrt{C_2}. \quad (42)$$

V. CONCLUSIONS

In quantum multiterm potentials it is possible to generate a finite set of different exactly solved quantum systems by selecting the working potential. In our present work we are taking one term working potential. But we are not considering two or multiterm working potentials as they offer the following practical difficulties: the integral specifying the transformation function  $g_B(r)$  cannot be extracted analytically in most of the cases and even if such integrals are found they are of the form  $F(r)=r+c$  and the analytical inverse function  $F^{-1}(g)$  cannot be found. The daughter along with the parent potential formed family relationship (multiplet structure) as one can go from one ESP to the other ESPs with the help of ET. The multiplet structure however is no related to representation of any group, as ET doesn't form a group in the conventional sense. In view Self-duality constraints that only monoterm and three terms self dual potential are possible. It is also noteworthy that under ET the constraint equation gets converted in to the energy eigenvalue expression and the energy eigenvalue in to a constraint equation of the parent and daughter quantum systems.

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The relationship between the angular momentum quantum numbers  $l_A$  and  $l_C$  of A and C-QSs are obtained from equation (11) as

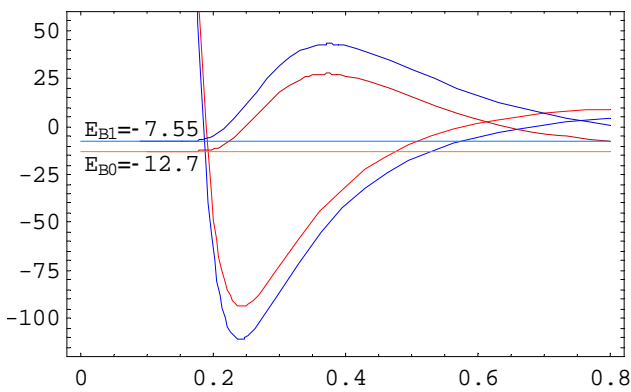
$$2l_A + 1 = -(2l_C + D - 2). \tag{43}$$

The exact energy eigenfunctions of C-QS is obtained in D-dimensional Euclidean spaces from equation (3) and is

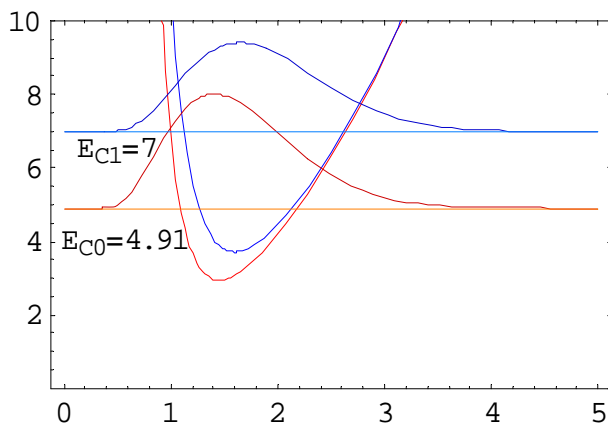
$$\Psi_C(r) = N_C r^{\frac{4-D+\frac{B_2}{\sqrt{C_2}}}{2}} \exp\left[-\frac{1}{2}\left(\sqrt{A_2}r^2 + \sqrt{C_2}r^{-2}\right)\right]. \tag{44}$$

The  $N_C$  is the normalization constant of C-QS.

The shape of the potential as well as eigenfunctions and the corresponding energy eigenvalue for C-QS are shown in Figure 2.



**FIGURE 1.** The continuous red curves are for the B-EQS with the  $\{E_{B0}, V_{B0}(r), \Psi_{B0}(r)\}$ , where the parameter set is  $(E_{B0}=-12.7, A_1=-17.8, B_1=1.5, C_1=-8.57, l_B=0, D=3)$  and the blue curves are for the B-EQS  $\{E_{B1}, V_{B1}(r), \Psi_{B1}(r)\}$ , where the parameter set is  $(E_{B1}=-7.5, A_1=-13.728, B_1=1.5, C_1=-8.57, l_B=1, D=3)$ . The graphs are drawn in arbitrary scale.



**FIGURE 2.** The continuous red curves are for the C-EQS with the  $\{E_{C0}, V_{C0}(r), \Psi_{C0}(r)\}$ , where the parameter set is  $(E_{C0}=4.91, A_2=1, B_2=1.83, C_2=4, l_C=0, D=3)$  and the blue curves are for the C-EQS  $\{E_{C1}, V_{C1}(r), \Psi_{C1}(r)\}$ , where the parameter set is  $(E_{C1}=7, A_2=1, B_2=6, C_2=4, l_C=1, D=3)$ . The graphs are drawn in arbitrary scale.