

RESOLUTION OF THE LINEAR FRACTIONAL GOAL PROGRAMMING PROBLEM

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RESUMEN: En este trabajo se resuelve el problema de Programación por Metas cuando dichas metas adoptan una formulación fraccional lineal. La gran dificultad de este tipo de problemas radica en las restricciones no lineales que aparecen al resolver el correspondiente problema de programación asociado al problema de metas. Cuando existen soluciones alcanzando todas las metas, el problema es fácil de resolver a través de la resolución de un problema lineal asociado. En este trabajo nos centramos en el caso en el que no todas las metas se consiguen satisfacer. En dicho caso, encontramos la solución que más se acerca a los niveles de aspiración establecidos por el decisor desde todas las aproximaciones del problema de metas que se contemplan en la literatura: Ponderadas, Minimax y Lexicográficas. Cada sección se termina con un análisis de sensibilidad de dichos niveles de aspiración.

Palabras claves: Programación Matemática No Lineal, Programación Fraccional, Programación Multiobjetivo, Programación por Metas.

ABSTRACT: This work deals with the resolution of the goal programming problem with linear fractional criteria. The main difficulty of these problems is the non-linear constraints of the mathematical programming models that have to be solved. When there exist solutions satisfying all target values, the problem is easy to solve by solving a linear problem. So, in this paper we deal with those instances where there is no guarantee such solutions exist, and therefore we look for those points in the opportunity set closest to the target values. This study has been done taking into account all the different approaches available for solving a goal programming problem, creating solution-search algorithms based on these approaches, and performing a sensitivity analysis of the target values.

Keywords: Non Linear Programming, Fractional Programming, Multi-objective Programming, Goal programming.

1. Introduction

The main interest in fractional programming was generated by the fact that a lot of optimization problems from engineering, natural resources and economics require the optimization of a ratio between physical and/or economic functions. Such problems, where the objective functions appear as a ratio or quotient of other functions, constitute a fractional programming problem. When these quotients have to verify certain target values, we would have a set of fractional goals. In goal programming (GP) problems it is assumed that the decision-maker gives up optimization while he/she establishes target values for each objective. When the levels are verified, the expectations or desires of the decision-maker are satisfied. In addition, the decision-maker usually sets a series of preferences regarding the objectives and orders them according to their relative importance.

In GP problems, when the objectives are linear fractional functions, the formulation of the corresponding goal problem to be solved is rather complex due to its nonlinear constraints. The resolution of this type of problems by direct linearization is not feasible [1]. In [2], Hannan presents a characterization of the conditions for the linear problem such that it is equivalent to the original fractional problem. However, Soyster and Lev [3] argued that Hannan's result was erroneous by using a counterexample. These authors developed a test problem which, once solved, could be used to find out whether we are dealing with equivalent problems or not. See [4] for further information and references about this issue. After these studies, and with the exception of the work of Kornbluth and Steuer [5], there have been very few references of goal programming with fractional goals. Finally, Audet *et al* [6] have used global optimization techniques to solve this problem.

In this paper, we propose solution algorithms for the goal programming problem where the goals take a linear fractional form under all goal programming formulations. Practical examples of fractional goal programming problems occur in many decision problems where the criteria are expressed as a ratio of two given functions such as profit/capital, cost/time, cost/volume, output/input, etc. Thus, this model has important applications in areas such as finance, transportation, information theory, forestry management, agricultural economics, education, resource allocation and others (see [7] and [8]). Recently, the fractional problem have appeared in environmental and natural resources problems ([9], [10], [11], among others). Previous work by Caballero and Hernández [12] shows that by using a simple linear test it is possible to determine beforehand the existence or absence of solutions that satisfy all the goals of the problem under study. Here we assume that not all goals can be satisfied and solve the linear fractional goal programming problem directly. Then we carry out a sensitivity analysis for the target values set by the decision-maker, that have been too constrictive.

As said, in GP problems the decision-maker establishes target values for each objective. In general, the purpose of GP is to reach these target values as closest as

possible, minimizing the deviation between the target value and the level achieved for the corresponding attribute. The decision-maker's preferences regarding the achievement of the targets may be incorporated in this scheme following different ways. Thus, there are several GP variants such as weighted GP, minmax GP and lexicographic GP, among others. Weighted GP implies a mathematical (cardinal) specification of weights for each goal and seeks for a solution that minimizes the weighted sum of all the unwanted deviation variables. The minmax GP approach minimizes the maximum deviation from the targets. Finally, Lexicographic GP involves a preemptive, ordinal weighting of goals that allocates them into priority levels and minimizes the unwanted deviation variables in a lexicographic order. Formally, weighted GP and minmax GP can be viewed as a particular case of lexicographic GP with a single priority level. Therefore, without loss of generality, we assume that we are dealing with the latter approach. A lexicographic GP problem can have a single goal or more within each priority level. This paper studies and solves the model generated in both instances. When there is more than one goal at a single priority level, we will take into account which approach (i.e. weighted or minmax) has been used to aggregate the goals of the level. When appropriate, a worked example is included in the section in order to assist the understanding of the algorithm.

Thus, next section deals with the case of a single goal for a certain priority level. Following sections describe instances of more than one goal within a given priority level. In Section 3, we assume that goals were added using the weighted approach, and in Section 4 it is assumed that the minmax approach was the method chosen for adding goals in the level. Finally, we summarize the most significant conclusions of this work followed by the references.

2. One goal per priority level

We assume that our problem has p linear fractional objectives and a constraint set that is a convex polyhedron. Without loss of generality, we assume that the decision-maker imposes a minimum target value for each objective. Thus, the unwanted deviation variables are the negative ones, that is, n_i for $i = 1, \dots, p$. In the present section, we assume that, using the Lexicographic GP approach, the priority levels are imposed in such a way that in a given level s there is only one goal, corresponding to the i -th objective. In this case, in level s , (where the index set of the goals in s will be denoted by N_s) the problem to be solved is as follows:

$$\begin{aligned}
 & \min \quad n_i \\
 & s.t \quad Ax \leq b \\
 & \quad \frac{f_j(x)}{g_j(x)} \geq u_j \quad j \in N_1, \dots, N_{s-1} \\
 & \quad \frac{f_i(x)}{g_i(x)} + n_i - p_i = u_i \\
 & \quad x, n_i, p_i \geq 0
 \end{aligned} \tag{1}$$

where $A \in M_{m \times n}(\mathbf{R})$ and $b \in \mathbf{R}^m$.

Let $X_s = \{x \in \mathbf{R}^n / Ax \leq b, x \geq 0, f_j(x)/g_j(x) \geq u_j \quad j \in N_1, \dots, N_{s-1}\}$, which includes the constraints (linear) that impose the satisfaction of goals in the previous levels; and $f_j(x) = c_j'x + \alpha_j$, $g_j(x) = d_j'x + \beta_j$ where $c_j, d_j \in \mathbf{R}^n$, $\alpha_j, \beta_j \in \mathbf{R}$, $j = 1, \dots, p$. In addition, we assume that $g_j(x)$, $j = 1, \dots, p$ are strictly positive for every $x \in X$. Let us call $f_j(x)/g_j(x) = \varphi_j(x) \quad j = 1, \dots, p$.

It is clear that problem (1) is not easy to solve due to the non-linear constraint corresponding to the goal we are attempting to satisfy. Taking into account the following associated problem,

$$\begin{aligned} \min \quad & n_i' \\ \text{s.t.} \quad & x \in X_s \\ & f_i(x) - g_i(x) \cdot u_i + n_i' - p_i' = 0 \\ & n_i', p_i' \geq 0 \end{aligned} \quad (2)$$

Awerbuch *et al* in [1] showed that the solution of the linear problem (2) x^* is not necessarily the solution of (1). However, in [12] it is shown that if the solution of (2) in the optimum is zero, then this point will be also the point solution of (1). In the other case, where the optimum of (2) is $n_i'^* > 0$, the following theorem determines the solution point of (1).

Theorem 2.1. Let x^{**} be the solution of the single-objective linear fractional problem

$$\begin{aligned} \max \quad & \varphi_i(x) \\ \text{s.t.} \quad & x \in X_s \end{aligned}$$

If when solving (2) the solution is such that $n_i'^* > 0$, then $(x^{**}, n_i'^*, 0)$ is the solution of (1) where $n_i'^* = u_i - \varphi_i(x^{**})$.

Proof.

On the one hand, the constraints of (1) are verified by point $(x^{**}, u_i - \varphi_i(x^{**}), 0)$ because:

- $x^{**} \in X_s$
- $\varphi_i(x^{**}) + (u_i - \varphi_i(x^{**})) - 0 = u_i$.
- By hypothesis, the goal is not verified in any point of X_s . Therefore it is not verified in x^{**} , which means $\varphi_i(x^{**}) < u_i$. That is, $n_i'^* = u_i - \varphi_i(x^{**}) > 0$.
- Obviously $p_i'^* = 0 \geq 0$.

On the other hand, this point also minimizes the value of n_i among those that verify the constraints. By *reductio ad absurdum*, suppose (x^*, n_i^*, p_i^*) is a solution of (1). According to our hypothesis and Theorem 1 in Caballero and Hernández [12], given that $n_i'^* > 0$, then also $n_i^* > 0$, that is, $u_i > \varphi_i(x^*)$. Given that this point has to verify the

constraints of the problem, then necessarily $\varphi_i(x^*) + n_i^* - p_i^* = u_i$, where $n_i^* > 0$ and $p_i^* \geq 0$, and therefore, $p_i^* = 0$. In other words $\varphi_i(x^*) + n_i^* = u_i$, which means that $n_i^* = u_i - \varphi_i(x^*)$.

However, assuming that $(x^*, n_i^*, 0)$ is a solution of (1) would mean that $n_i^* < n_i^{**}$. That is, $u_i - \varphi_i(x^*) < u_i - \varphi_i(x^{**})$. This would mean that $\varphi_i(x^*) > \varphi_i(x^{**})$, where x^{**} is the maximum of the function φ_i in X_s . This is a contradiction and thus point $(x^{**}, n_i^{**}, 0)$ is the solution of (1). \square

As said in the introduction, Soyster and Lev [3] designed a test problem that can be used to establish the equivalence between problems (1) and (2). If the optimal value of such a problem is zero, the equivalence between solutions for (1) and (2) is guaranteed. Using our notation, this test problem is as follows:

“Let (x^, n'^*, p'^*) be the solution of (2) with $n'^* > 0$. Then, if the optimal solution of the problem*

$$\begin{aligned} \max \quad & f_i(x) - \varphi_i(x^*) g_i(x) \\ \text{s.t.} \quad & x \in X_s \end{aligned}$$

is zero, then x^ also solves (1).”*

So, Theorem 1 is no more than a generalization of Soyster and Lev’ result because all this test problem does is to use the method of Dinkelbach [13] to verify whether x^* is or is not the solution of

$$\begin{aligned} \max \quad & \varphi_i(x) \\ \text{s.t.} \quad & x \in X_s \end{aligned}$$

Once the goal programming problem has been solved, it makes sense to perform a sensitivity analysis of the target values which have been set. In this sense, the solution point of problem (1) itself offers the sensitivity analysis we are looking for and provides us with an interval within which we can be sure that there are points which satisfy the i -th goal. Indeed, if x^{**} is the solution of (1), with $n_i^{**} > 0$, x^{**} is the point of X_s that minimizes the unachievement of the goal and this minimum unachievement is precisely n_i^{**} . Therefore, the sensitivity interval is $(-\infty, u_i - n_i^{**}]$. Thus, given a target value for the goal i which is within the said interval, we can guarantee the existence of at least one point that will satisfy such a goal (satisfying previous levels). On the other hand, if the target value is out of the interval, we guarantee that the goal is not going to be satisfied.

3. More than one goal per priority level. Weighted GP approach.

In this section we will solve the following problem:

$$\begin{aligned}
& \min \sum_{i=1}^k \lambda_i n_i \\
& \text{s.t. } x \in X_s \\
& \frac{f_i(x)}{g_i(x)} + n_i - p_i = u_i \quad i=1, \dots, k \\
& n_i, p_i \geq 0 \quad i=1, \dots, k
\end{aligned} \tag{3}$$

Where parameter λ_i represents the weight of each goal in the weighted GP approach. So these weights must verify that $\sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0 \quad i=1, \dots, k$.

In order to solve problem (3), we consider its associated linear problem

$$\begin{aligned}
& \min \sum_{i=1}^k \lambda_i n'_i \\
& \text{s.t. } x \in X_s \\
& f_i(x) - g_i(x) \cdot u_i + n'_i - p'_i = 0 \quad i=1, \dots, k \\
& n'_i, p'_i \geq 0 \quad i=1, \dots, k
\end{aligned} \tag{4}$$

Similarly to the case of the previous section, if when solving (4) we obtain a solution such that $\sum_{i=1}^k \lambda_i n'_i > 0$, in such a case there is not solution that satisfies all goals in such priority level and the value $\sum_{i=1}^k \lambda_i n'_i$ does not necessarily provide us with the minimum unachievement in X_s of the goals in the current priority level.

Let us solve problem (3) directly, that is, we seek a feasible solution $x \in X_s$ which minimizes the weighted sum of deviations of the fractional criteria from their target values. Given an $x \in X_s$, for every $i \in \{1, \dots, k\}$, when constraint $f_i(x) - g_i(x) u_i + n'_i - p'_i = 0$ is verified, we have:

- if $(-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i) > 0$, then $n'_i = (-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i)$
- if $(-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i) \leq 0$, then $n'_i = 0$

Given that $n_i = n'_i / g_i(x)$, then we can establish the following expressions of the variables n_i in relation to x , for a given i within 1 and k :

- if $(-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i) > 0$, $n_i(x) = ((-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i)) / (d_i x + \beta_i)$ (5)
- if $(-c_i + u_i d_i) x + (-\alpha_i + u_i \beta_i) \leq 0$, $n_i(x) = 0$

In this way, to find the solution point of (3), we just have to minimize in X_s the weighted sum of expressions (5). However, in order to do this, we first have to establish the goals that can give rise to two different expressions of $n_i(x)$ within X_s . These goals are the ones that verify $\{x / \varphi_i(x) = u_i\} \cap X_s \neq \emptyset$ and split the set X_s into two subsets: for one of them, $n_i(x) = 0$; while for the other, $n_i(x)$ is given by expression (5). To locate those goals we propose solving for each i with $1 \leq i \leq k$ a GP problem with a single goal,

where the achievement function is $h(n_i, p_i) = n_i + p_i$ and then check whether the value of function h in the optimal solution is zero. Let us assume that out of the k goals in the current level, the first m goals fulfil such requirement, where $m \leq k$.

We consider a partition of X_s that includes all possible combinations of the constraints $\varphi_i(x) \geq u_i$ on the one hand, and of $\varphi_i(x) \leq u_i$ on the other for $i = 1, \dots, m$. The constraints that have to be added to those existing in X_s to create the subsets of the partition are showed in Table 1:

Table 1. Constraints to be added to X_s

Subset	Constraints to be added to the constraints of X_s
Y_1	$\varphi_1(x) \geq u_1, \varphi_2(x) \geq u_2, \varphi_3(x) \geq u_3, \dots, \varphi_m(x) \geq u_m$
Y_2	$\varphi_1(x) \leq u_1, \varphi_2(x) \geq u_2, \varphi_3(x) \geq u_3, \dots, \varphi_m(x) \geq u_m$
...	...
Y_m	$\varphi_1(x) \geq u_1, \varphi_2(x) \geq u_2, \varphi_3(x) \geq u_3, \dots, \varphi_m(x) \leq u_m$
Y_{m+1}	$\varphi_1(x) \leq u_1, \varphi_2(x) \leq u_2, \varphi_3(x) \geq u_3, \dots, \varphi_m(x) \geq u_m$
...	...
Y_{2^m}	$\varphi_1(x) \leq u_1, \varphi_2(x) \leq u_2, \varphi_3(x) \leq u_3, \dots, \varphi_m(x) \leq u_m$

This partition of X_s is made up of, at most, 2^m polyhedral subsets, where some of these subsets might be the empty set. To find the solution of (3), we propose to minimize the weighted sum of the corresponding expressions of $n_i(x)$ for each of these subsets.

Therefore, the following 2^m problems have to be solved, for each r from 1 to 2^m :

$$\begin{aligned} \min \quad & \sum_{i \in I_r} \lambda_i \frac{(-c_i + u_i d_i)^t x + (-\alpha_i + u_i \beta_i)}{d_i^t x + \beta_i} \\ \text{s.t.} \quad & x \in Y_r \end{aligned}$$

where $I_r = \{i = 1, \dots, k \text{ such that } \varphi_i(x) \leq u_i \text{ in } Y_r\}$ and λ_i is the weight corresponding to the i -th goal which we assume has already been normalized.

Let us consider the optimal solution x_r^* , with a value in the objective function O_r^* , for $r = 1, \dots, 2^m$, and let $\min_{r=1, \dots, 2^m} O_r^* = O_{r_0}^*$. Then $x_{r_0}^*$ is the optimal solution to the

original goal programming problem (3). Thus, for solving (3), the following algorithm is suggested:

Algorithm 3.1.

- *Step 1.* Let $\{i = 1, \dots, k / \{\varphi_i(x) = u_i\} \cap X_s \neq \emptyset\} = \{1, \dots, m\}$.
- *Step 2.* Make a partition of the set X_s into 2^m subsets Y_r as was described earlier. Let $r = 1$.

- *Step 3.* Given r , if $Y_r = \emptyset$, go to step 4.
If $Y_r \neq \emptyset$, using the algorithm in [14], solve the problem

$$\begin{aligned} \min \quad & \sum_{i \in I_r} \lambda_i \frac{(-c_i + u_i d_i)^t x + (-\alpha_i + u_i \beta_i)}{d_i^t x + \beta_i} \\ \text{s.t.} \quad & x \in Y_r \end{aligned}$$

where $I_r = \{i = 1, \dots, k \text{ such that } \varphi_i(x) \leq u_i \text{ in } Y_r\}$.

Let x_r^* be its solution with a value of the objective function O_r^* .

- *Step 4.* If $r = 2^m$, go to step 5.
If $r \neq 2^m$, let $r = r + 1$ and go back to Step 3.
- *Step 5.* Let $J = \{r = 1, \dots, 2^m / Y_r \neq \emptyset\}$. Calculate $\min_{r \in J} O_r^* = O^*$.
If $O_i^* = \min_{r \in J} O_r^*$, **STOP**: x_i^* is the solution of (3).

Before carrying out the sensitivity analysis, note that if there was only one goal at this level, the previous process would give the same results as those described in Section 2. Indeed, in the case of a single goal per level, given that the goal has not being satisfied, the number m of possible goals to be satisfied is $m = 0$. This means that the partition for X_s is made by a single subset which is the actual X_s . Given that in X_s we have $\varphi_i(x) \leq u_i$, the expression of $n_i(x)$ is $n_i(x) = ((-c_i + u_i d_i)^t x + (-\alpha_i + u_i \beta_i)) / (d_i^t x + \beta_i)$, that is, $n_i(x) = -f_i(x) + u_i g_i(x) / g_i(x) = -\varphi_i(x) + u_i$. Given that u_i is a constant, the point in X_s that minimizes this expression of $n_i(x)$ also maximizes the function $\varphi_i(x)$ in X_s .

Regarding the sensitivity analysis, let us assume that the solution of problem (3), found with the suggested algorithm, is $(x^*, n_1^*, \dots, n_k^*, p_1^*, \dots, p_k^*)$. Thus, and using similar reasoning to that used in the previous section, we reach the conclusion that, at least, we have to lower the target values to the corresponding value of $\varphi_i(x^*)$ for each $i = 1, \dots, k$. This assertion is based on the fact that point x^* , obtained as a solution to problem (3), is the point that minimizes the weighted sum of the unachievements of the goals established in this priority level.

Let us conclude this section with a worked example that tries to assist in the understanding of the solution algorithm.

Example 3.1

Assume a problem whose opportunity set is $X = \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \leq 4, x_2 \leq 4, x_1 + x_2 \geq 5\}$ and suppose that in the first priority level we have the following goals, all with the same weight:

$$\begin{aligned} \varphi_1(x) &= (x_2 - 1)/(x_1 + x_2 - 1) \geq 1; \\ \varphi_2(x) &= (-x_1 - x_2 + 5)/(-x_1 + 5) \geq 1; \\ \varphi_3(x) &= (x_1 + x_2 - 8)/(-x_1 + 5) \geq -1; \\ \varphi_4(x) &= (x_1)/(-x_2 + 7) \geq 1. \end{aligned}$$

So that the problem that has to be solved is the following:

$$\begin{aligned} \min \quad & \sum_{i=1}^4 n_i \\ \text{s.t.} \quad & x_1 \leq 4 \\ & x_2 \leq 4 \\ & x_1 + x_2 \geq 5 \\ & (x_2 - 1)/(x_1 + x_2 - 1) + n_1 - p_1 = 1 \\ & (-x_1 - x_2 + 5)/(-x_1 + 5) + n_2 - p_2 = 1 \\ & (x_1 + x_2 - 8)/(-x_1 + 5) + n_3 - p_3 = -1 \\ & (x_1)/(-x_2 + 7) + n_4 - p_4 = 1 \\ & n_i, p_i \geq 0 \quad i = 1, \dots, 4 \end{aligned} \tag{6}$$

After solving the linear problem associated with (6), the solution in the optimum is greater than zero, so we conclude that there are no points that satisfy all the goals.

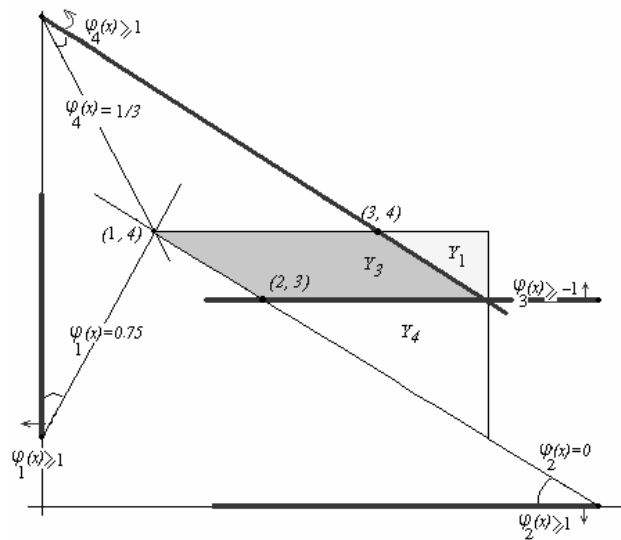


Figure 1. Figure of Example 3.1 problem

So we have to apply the algorithm proposed in order to solve (6) directly. The first step is to identify which of the four goals could be satisfied in this level. Figure 1

clearly shows that only goals $\varphi_3(x) \geq -1$ and $\varphi_4(x) \geq 1$ satisfy this requirement, and therefore, $m = 2$.

Thus, we have to partition the set X into 4 subsets, such as shown in Figure 1:

$$\begin{aligned} Y_1 &= \{x \in X / \varphi_3(x) \geq -1, \varphi_4(x) \geq 1\}, \\ Y_2 &= \{x \in X / \varphi_3(x) \leq -1, \varphi_4(x) \geq 1\} = \emptyset \\ Y_3 &= \{x \in X / \varphi_3(x) \geq -1, \varphi_4(x) \leq 1\}, \\ Y_4 &= \{x \in X / \varphi_3(x) \leq -1, \varphi_4(x) \leq 1\}. \end{aligned}$$

The following is to solve three optimisation problems, one for each non-empty subset of X . Because the two first goals cannot be satisfied in either of these four subsets, in all of them we have to consider that $\varphi_1(x) \leq 1$ and $\varphi_2(x) \leq 1$.

Therefore, the first problem to solve is the following:

$$\begin{aligned} \min \quad & \frac{x_1}{x_1 + x_2 - 1} + \frac{x_2}{-x_1 + 5} \\ \text{s.t.} \quad & x \in Y_1 \end{aligned}$$

After applying the algorithm in [14], we find that its solution is $x_1^* = (3, 4)$ where the value of the objective function is $O_1^* = 2.5$.

The second problem to solve is

$$\begin{aligned} \min \quad & \frac{x_1}{x_1 + x_2 - 1} + \frac{x_2}{-x_1 + 5} + \frac{-x_1 - x_2 + 7}{-x_2 + 7} \\ \text{s.t.} \quad & x \in Y_3 \end{aligned}$$

whose solution is $x_3^* = (1, 4)$ with a value for the objective function of $O_3^* = 1.91667$.

The third problem to solve is

$$\begin{aligned} \min \quad & \frac{x_1}{x_1 + x_2 - 1} + \frac{x_2}{-x_1 + 5} + \frac{-x_1 - x_2 + 7}{-x_2 + 7} + \frac{-x_2 + 3}{-x_1 + 5} \\ \text{s.t.} \quad & x \in Y_4 \end{aligned}$$

whose solution is $x_4^* = (2, 3)$ with a value for the objective function of $O_4^* = 2$.

Therefore, taking $O^* = \min \{O_1^*, O_3^*, O_4^*\} = O_3^* = 1.91667$, the point that solves (6) is point $x_3^* = (1, 4)$, since the weighted sum of deviations of the fractional criteria from their target values to the set X are minimized in this point.

The values of the goals in this solution point are $\varphi_1(x^*) = 0.75$, $\varphi_2(x^*) = 0$, $\varphi_3(x^*) = -0.75$, $\varphi_4(x^*) = 1/3$ and therefore, the sensitivity analysis establishes that the target values of the first two goals have to be dropped at least to values $u_1' = 0.75$ and $u_2' = 0$ and in the fourth goal to value $u_4' = 1/3$ in order to obtain points satisfying the goals in this level. We even can increase the target value for the third goal up to value $u_3' = -0.75$. We want to point out that those are the values that minimized the sum of the unachievments. In this case, this minimum sum of unachievement is $O^* = 1.91667$. Although there are other ways of lowering the target values that lead to solutions satisfying the goals, these other ways imply a sum of deviations from the original target values greater than O^* .

4. More than one goal per priority level. Minimax GP approach.

In this section we suppose that the k goals in the current level s are added under the minimax approach. In this case the problem to solve is as follows:

$$\begin{aligned}
 & \min \quad d \\
 & \text{s.t.} \quad x \in X_s \\
 & \quad \frac{f_i(x)}{g_i(x)} + n_i - p_i = u_i \quad i = 1, \dots, k \\
 & \quad \lambda_i n_i \leq d \quad i = 1, \dots, k \\
 & \quad n_i, p_i \geq 0 \quad i = 1, \dots, k
 \end{aligned} \tag{7}$$

As in previous cases, we associate a linear problem with the original problem which, once solved, will tell us whether there are any solutions that satisfy all the goals in that level or not. The linear problem is expressed as follows:

$$\begin{aligned}
 & \min \quad d' \\
 & \text{s.t.} \quad x \in X_s \\
 & \quad f_i(x) - g_i(x) \cdot u_i + n'_i - p'_i = 0 \quad i = 1, \dots, k \\
 & \quad \lambda_i n'_i \leq d' \quad i = 1, \dots, k \\
 & \quad n'_i, p'_i \geq 0 \quad i = 1, \dots, k
 \end{aligned} \tag{8}$$

However, to solve (7) directly we should bear in mind that this problem is equivalent to

$$\min_{x \in X_s} \left\{ \max_{i=1, \dots, k} (\lambda_i (u_i - \varphi_i(x))) \right\} \tag{9}$$

Given that for $i = 1, \dots, k$ $\varphi_i(x)$ are linear fractional functions and the u_i values are previously given, the functions of this minimax problem are also linear fractional functions and therefore we are dealing with *Generalized Fractional Programming*. This problem has been widely studied in the literature and different algorithms are proposed to solve it. See, for example, the ‘‘Newmodm’’ algorithm [15].

Next, we will carry out the sensitivity analysis. Once (7) has been solved, where x^* is its solution, if x^* is the unique solution, then the target values of the goals which have not been satisfied have to decrease to the value the goals take at this specific point.

However, if x^* is not the unique solution of (7), we have to continue with the analysis because there might be some objective functions for which it might be possible to establish target values higher than those reached in x^* and still have points satisfying all the goals. For this case we propose an iterative algorithm that would lead us to the point where the unachievement of the goals at this level is minimized and so, provide us with the sensitivity analysis sought. In each iteration of this algorithm we take, from the goals not satisfied, those where the deviation variable d has reached its highest value d^* . For these goals, (we assume that are the first m goals) we consider the following constraints:

$$\varphi_i(x) = u_i - d^* = \varphi_i(x^*) \quad i = 1, \dots, m$$

We add these to problem (7) as hard constraints, and eliminate the soft constraints corresponding to those goals. We continue with the process in this way until we either reach a unique solution for some of these problems or have finished improving all possible maximum values for the target values of the goals that have not been satisfied.

The algorithm suggested is as follows:

Algorithm 4.1

- *Step 1.* Let $k = 1$; $X_1 = X_s$ and $I_1 = \{1, \dots, k\}$.
- *Step 2.* Solve the linear fractional minimax problem using the Newmodm algorithm

$$\min_{x \in X_k} \left\{ \max_{i \in I_k} (\lambda_i (u_i - \varphi_i(x))) \right\}$$

Let x_k^* be the solution point and θ_k^* the value of the objective in x_k^* .

- *Step 3.* Establish the following index set J_k , within I_k :

$$J_k = \{i \in I_k / \theta_k^* = u_i - \varphi_i(x_k^*)\}.$$

Let $I_{k+1} = I_k \setminus J_k$.

- *Step 4.* If $I_{k+1} = \emptyset$, **STOP:** $x_k^* = x^*$ is the point that minimizes the unachievement of the goals.
If $I_{k+1} \neq \emptyset$, then let $X_{k+1} = X_k \cap \{x / \varphi_i(x) = u_i - \theta_k^* / i \in J_k\}$.
Let $k = k + 1$, and return to step 2.

This is a fully convergent algorithm because in each iteration the set I_k becomes smaller because, given the actual construction of the value θ_k^* , the index set J_k can never be an empty set.

So, if x^* is the point obtained after applying the algorithm, for $i = 1, \dots, k$, let be $u'_i = \varphi_i(x^*)$ if $u_i > \varphi_i(x^*)$, and $u'_i = u_i$ if $u_i \leq \varphi_i(x^*)$. With these new target values, we make sure that the minimax GP problem corresponding to the current priority level s has a solution that satisfies all the goals, given that at least point x^* is one of them. However, with target values higher than u'_i (for those goals where $u_i > \varphi_i(x^*)$) we guarantee the non-existence of points satisfying all the goals of the problem. This is ensured because x^* is the point that minimizes the maximum unachievement of the goals established. Nevertheless, for target values such that $u_i \leq \varphi_i(x^*)$, we can assert that these target values could still be increased up to the value of $\varphi_i(x^*)$ and still be points that satisfy all the goals.

5. Conclusions

In this work we have examined the goal programming (GP) problem from all possible perspectives when the objectives associated with the goals are linear fractional functions.

We have carried out a comprehensive study and have derived algorithms capable of finding the points that solve the problem in a given priority level. We have explored all the possible cases, one goal per priority level and several goals per priority level. In this later case, the problem was also separately analysed for goals grouped according to the weighted GP approach or the minimax GP approach. In all the cases the non-linear problem derived has been solved.

Once the linear fractional goal programming problem was solved for each case, we also carried out a sensitivity analysis of the target values set by the decision-maker. Thus, each section of the work includes relaxation values for the targets which guarantee the existence of solutions that satisfy all the goals in the current priority level.

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