ON INTERVAL-FUNCTIONS AND ASSOCIATED SET-FUNCTIONS

by

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An *interval* of an n-dimensional Euclidean space R_n is an n-dimensional rectangular parallelopiped parallel to the coordinate system; thus, in particular, for $n=1$ we have a segment, and for $n=2$ a rectangle parallel to the coordinate-axes. If such an interval is considered without its boundary, the interval is called *open*; if the boundary belongs to the interval, it is called *closed*.

Any rule by which to every closed interval I a number is attached defines an *interval-function* $\chi(I)$.

A finite set S of closed intervals, $I_1, I_2, \ldots, I_k$, no pair of which has any *inner* points in common, may be called a *simple* system of intervals. Then we define:

$$\chi(S)=\chi(I_1)+\chi(I_2)+\ldots+\chi(I_k).$$

The most important example of an interval-function is the volume of the intervals and of the simple systems of intervals. Starting with it, one comes to the Lebesgue measure μ_n in the well known manner. Similary one can ask, when and how it is possible to go from more general interval-functions to closely related set-functions, and, in particular, to totally additive set-functions.

This question has been treated repeatedly by several mathematicians[1], after Lebesgue had developed his classical theory of measure.

(1) J. RADON, Sitzungsberichte Akad. Wiss. Wien 122 (1913), p. 1305; C. DE LA VALLÉE POUSSIN, Trans. Amer. Math. Soc. 16 (1915), p: 458, 478, 493; *In-*

Here now we want to proceed in quite a different way from the given interval-function χ to a set-function which is very closely connected with by certain strong conditions of convergence, and which we shall then call an «*associated*» set-function. Then because of the strong relations between both functions, the conditions for the existence of such an associated set-function will also be rather strong.

This new theory which I want to report upon here(²) has been developed in a section of the unpublished second volume of *H. Hahn's* «Reelle Funktionen», which I prepared and elaborated on the basis of Hahn's left manuscripts. In fact, the fundamentals of the theory were found in Hahn's manuscripts. But for the general case his conditions were incomplete, and thus his theory had to be modified and supplemented in a rather essential manner.

In order to introduce the basic concepts, we could use the n-dimensional Lebesgue measure μ_n. But a generalization is possible by taking a totally additive set-function ψ instead of μ_n. For this purpose let m be a σ-field of sets in R_n, containing the closed intervals; and let ψ be a finite, totally additive(³) set-function in m, satisfying the following additional condition: for every set $M \varepsilon m$, a G_δ-set $B \geqq M$ shall exist with $\overline{\psi}(B) = \overline{\psi}(M)$, where $\overline{\psi}$ designates the absolute-function (or total variation) of ψ. Besides, let m be complete for ψ(⁴).

Let us now consider an infinite system V of closed intervals for which *Vitali*'s covering theorem (with respect to ψ) is valid. And all the intervals with which we are concerned shall belong to this system V.

We shall call *boundary-segment* T any intersection of a

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tégrale de Lebesgue, Fonctions d' ensemble, Classes de Baire, Paris 1916, p. 76, 98; C. Carathéodory, *Vorlesungen über reelle Funktionen,* Leipzig-Berlin 1918, p. 502; J. C. Burkill, Proc. London Math. Soc. (2) 22 (1924), p. 275; S. Saks, *Theory of the integral,* Warszawa-Lwów 1937, p. 59, 93, 105; P. Reichelderfer and L. Ringenberg. Duke Math. Journ. 8 (1941), p. 231.

(²) A short talk about it has already been given at the meeting of the Southwestern Division of the Math. Assoc. of America at the New Mexico State College, (Las Cruces, N. M.), on April 28, 1942.

(³) Not necessarily monotone increasing.

(⁴) I. e.: if $N \varepsilon m$ with $\overline{\psi}(N) = 0$, then every subset of N shall belong also to m.

finite number of closed intervals of V which belong to a simple system of intervals.

Furthermore let S_ν (for every $\nu = 1, 2, \ldots$) be a simple system of intervals of V, and let d_ν be the maximum of the diameters of the intervals of S_ν. If $d_\nu \to 0$, then the sequence $((S_\nu))$ shall be called a *distinguished sequence* of systems of intervals.

Let $((S_\nu))$ be such a distinguished sequence, and let $M \varepsilon m$. Then we shall say: $((S_\nu))$ *converges to* M *for* ψ, written symbolically: $S_\nu \to {}_\psi M$, *if* $\overline{\psi}(S_\nu - M) \to 0$ *and* $\overline{\psi}(M - S_\nu) \to 0$. It can be proved that for every set $M \varepsilon m$ there exist distinguished sequences $((S_\nu))$, such that $S_\nu \to {}_\psi M$.

Now to the given interval-function χ, we can define an associated set-function φ. Let φ be a finite set-function in m. If for every set $M \varepsilon m$ and every distinguished sequence $((S_\nu))$ with $S_\nu \to {}_\psi M$ we have $\chi(S_\nu) \to \varphi(M)$, then we call φ *associated* with χ, (with respect to ψ and V).

If φ is associated with χ, it can be proved that φ must be *totally additive* and ψ-*continuous*(5). This means: $\overline{\psi}(M) = 0$ implies $\varphi(M) = 0$; or (as an equivalent expression): for every sequence $((M_\nu))$ with $\overline{\psi}(M_\nu) \to 0$ we have $\varphi(M_\nu) \to 0$.

Similarly we also say that the interval-function χ es ψ-*continuous* on V, if for every distinguished sequence $((S_\nu))$ of interval-systems of V with $\overline{\psi}(S_\nu) \to 0$, we have $\chi(S_\nu) \to 0$.

Of course, the most essential problem now is the following: *Under what conditions does a set-function* φ *exist which is associated with the given interval-function* χ, (with respect to ψ and V)?

To answer this question, we assume: To every interval $I \varepsilon V$ there shall be a distinguished sequence $((S_\nu))$ of interval-systems of V, such that $S_\nu \leqq I$ and $S_\nu \to {}_\psi I$.

Under this assumption(6), it can be proved that the following four conditions together are *necessary and sufficient* for the existence of φ:

1.) χ *shall be* ψ-*continuous on* V.

2.) *The derivative of* χ *with respect to* ψ *and* V *shall*

(5) In the case $\Psi = \mu^n$, it is also called "totally continuous".

(6) Which certainly is satisfied, if Ψ is μ_n — continuous.

exist «almost everywhere» (i, e. everywhere except perhaps for a set N with $\overline{\psi}(N)=0$).

This derivative, of course, is defined at a point p in the following way: Let $((I_\nu))$ be any sequence of intervals of V, containing p and converging to p; then for every such sequence,

$$\lim_{\nu} \frac{\chi(I_\nu)}{\psi(I_\nu)} = D$$

shall exist, and this value D (which then is independent on the particular sequence $((I_\nu))$ converging to p) is called the *derivative* $D(\chi, \psi)$ at the point p; or written more explicitely: $D(p, \chi, \psi, V)$.

3.) *This derivative* $D(\chi, \psi)$ *shall be* ψ*-summable* (i. e. integrable with respect to ψ in the sense of Radon).

4.) *For every boundary-segment* T, *we shall have*:

$$\int_T D(\chi, \psi)\, d\psi = 0.$$

If these conditions are satisfied, then φ can be represented for every $M \varepsilon m$ in the following way:

$$\varphi(M) = \int_M D(\chi, \psi)\, d\psi,$$

where here and in 4.) the integral is to be taken in the sense of Radon.

In the particular case that ψ is μ_n-continuous, the condition 4.) is satisfied automatically.