GEOMETRIC PROPERTIES OF ISOTHERMAL FAMILIES

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Introduction. In this paper, we wish to discuss the various geometrical properties of an isothermal family of curves. Although isothermal families are important in the theory of functions of a complex variable $z = x + iy$, especially in connection with the Laplace equation, it seems that only a few elementary characterizations of such families are known. We shall present here some of the old and many new characterizations of isothermal families. The outstanding theorems of the ten which characterize isothermal families are those numbered 1, 2, 4, 5, 8, 9.

1. The Laplace equation of isothermal families. An isothermal family of curves may be defined geometrically as the conformal image of a parallel pencil of straight lines. Therefore, if $H(x, y)$ is a harmonic function, that is, H satisfies the Laplace equation

$$H_{xx} + H_{yy} = 0, \tag{1}$$

then the family of curves $H(x, y) = \text{const.}$ is isothermal.

Let $F(x, y) = \text{const.}$ be an isothermal family. There must exist a function $G(F)$, which is harmonic in (x, y). Therefore the family of curves $F(x, y) = \text{const.}$ is isothermal if and only if the function F satisfies the partial differential equation of third order

$$(2) \qquad \left(F_y\frac{\partial}{\partial x} - F_x\frac{\partial}{\partial y}\right)\left(\frac{F_{xx}+F_{yy}}{F_x^2+F_y^2}\right)=0.$$

A simple family of curves (called a sheaf) is the integral curves of a field of ∞^2 lineal elements, and is defined by a general differential equation of first order

$$(3) \qquad y' = \tan\Theta(x,y),$$

where Θ is the inclination of the lineal element of the field through the point (x,y).

Theorem 1. The family of curves defined by the differential equation (3) is isothermal if and only if the inclination Θ is a harmonic function of (x,y).

By this theorem, which is due to Lie, it may be shown that if the differential equation (3) is known to represent an isothermal family, then it can be integrated by quadratures only.

There is a conformal group of *three* parameters carrying a given isothermal family into itself. Of course, there is an infinite group of point transformacions preserving a given isothermal family, but the conformal transformations form only a three-parameter subgroup.

2. *Isothermal nets.* If a simple family is isothermal, than the related orthogonal family is also isothermal. The orthogonal net thus formed is called an isothermal net.

Any isothermal net is the conformal image of two orthogonal parallel pencils of lines. We may regard an isothermal net as dividing a region of the plane into infinitesimal squares.

For any given simple family F_0, construct the $(n-1)$ simple families F_j $(j=1, 2, \ldots, n-1)$ which cut F_0 at an angle of $2j\pi/n$. These n families $F_0, F_1, \ldots, F_{(n-1)}$, are said to form a *symmetric n-web*. An orthogonal net is a symmetric four-web.

Through any point p of the plane, there are m curves $C_0, C_1, \ldots, C_{(m-1)}$ of the symmetric n-web where $m=n$ if n is odd and $m=n/2$ if n is even. Let $\Gamma_j, \Gamma'_j, \Gamma''_j, \Gamma'''_j, \ldots,$ denote the curvature and the successive rates of variation of the curvature with respect to the arc length of the curve

C_j $(j=0,\ 1,\ 2,\ \ldots,\ m-1)$ of the web at the point p. This notation is used in Theorems 2, 8 and 9.

Theorem 2. An orthogonal net is isothermal if and only if the sum of the first rates of variations of the curvature with respect to the arc length is zero. That is,

$$\Gamma_0'+\Gamma_1'=0. \tag{4}$$

If the curvature of the evolute of C_i is denoted by K_i, then Lamé's Theorem 2 may be put into the following form. An orthogonal net is isothermal if and only if

$$\frac{\Gamma_0^3}{K_0}+\frac{\Gamma_1^3}{K_1}=0. \tag{5}$$

3. Velocity system. Before continuing further with our discussion of isothermal families, it is found convenient to discuss some other material which simplifies the development of other properties of isothermal families. First we wish to introduce the term *wex*. This denotes the total set of ∞^2 integral curves of a general differential equation of second order $y''=f(x, y, y')$.

In our study of dynamical trajectories(1), we have encountered an important class of wexes, which we have termed *velocity system*. Any such system is given by a differential equation of the form

$$y''=(1+y'^2)\,(\psi-y'\phi), \tag{6}$$

where ϕ and ψ are general functions of (x, y).

A set of ∞^2 curves is a velocity system if and only if the osculating circles at a fixed point p of the ∞^1 curves through p form a pencil. Therefore these circles pass through another point P given by

$$X=x+\frac{2\phi}{\phi^2+\psi^2}, \qquad Y=y+\frac{2\psi}{\phi^2+\psi^2}. \tag{7}$$

4. The conformal rectilinear wexes. Two important types of velocity systems are natural families and isogonal wexes.

The velocity system (6) is natural or isogonal according as $\phi_y - \psi_x = 0$ or $\phi_x + \psi_y = 0$ (See reference 1).

Theorem 3. A system of ∞^2 curves is both natural and isogonal if and only if it is the complete set of isogonal trajectories of an isothermal family.

Any system of this theorem is called a *conformal rectilinear wex,* and is denoted by R_0. The reason for the name is that any wex of this special kind is conformally equivalent to the ∞^2 straight lines of the plane.

Theorem 4. The complete system of isogonal trajectories of a given simple family F is linear if and only if F is isothermal(2).

The total group preserving a given conformal rectilinear wex R_0 consists of *eight* parameters. The conformal subgroup contains *four* parameters.

Any wex for which the sum of the angles of a curvilinear triangle formed by any three curves of the system is a straight angle, must be a conformal rectilinear wex.

5. *The associated point transformation T of a conformal rectilinear wex.* The velocity system (6) induces a transformation from the point p to the point P. This correspondence is called the *associated point transformation T* of the velocity system (6) and is defined by (7).

Theorem 5. A system of ∞^2 curves is the complete set of isogonal trajectories of an isothermal family if and only if it is a velocity system whose associated point transformation T is direct conformal.

If the associated point correspondence T of an isogonal wex is reverse conformal, then the wex is a parabolic pencil of circles. In that event, T is degenerate, carrying the ∞^2 points into a single point.

6. *Reciprocal velocity systems.* By (7), it is seen that any correspondence (except the identity) may be regarded as the associated point transformation T of a unique velocity system S. Two velocity systems S and S^{-1} are said to be *reciprocal* or *conjugate* if their associated transformations are inverses of one another.

Theorem 6. The reciprocal of a velocity system S is a conformal rectilinear wex if and only if S is conformal rectilinear.

The reciprocal of a natural family (or isogonal wex) is a natural family (or isogonal wex)(1).

7. *The associated point transformation T_1 of a velocity system.* For a given velocity system S, there is an associated point transformation T. Now we shall associate *another* point transformation T_1 with S as follows. Under T, let the point p correspond to the point P. The perpendicular bisector of the line determined by p and P is the line of centers of the osculating circles of S through the point p. Construct the pole P_1 with respect to the unit circle with p as center. The transformation T_1 carrying p into P_1 is also said to be *associated* with our velocity system S. Our new transformation T_1 is defined by

$$X_1 = x + \phi, \qquad Y_1 = y + \psi. \tag{8}$$

Under any transformation T_1, there is a pair of lineal elements e_1 and e_2 at the point p, each of which is carried into a parallel element at the corresponding point P_1.

Theorem 7. For a correspondence to be the associated point transformation T_1 of a conformal rectilinear wex Γ_0, it is necessary and sufficient that the pair of lineal elements e_1 and e_2 described above make respectively, angles of $45^\circ = \pi/4$ radians with the lineal element e_x parallel to the x-axis(3).

The inverse of a transformation T_1 described in the above theorem is not of the same type. Therefore the reciprocal S_1^{-1} of a conformal rectilinear wex with respect to the transformation T_1 is not conformal rectilinear.

8. *Symmetric three-webs*(4). For a symmetric three-web, the sum of the curvatures of the curves through any point p is zero. That is,

$$\Gamma_0 + \Gamma_1 + \Gamma_2 = 0. \tag{9}$$

Theorem 8. A symmetric three-web is isothermal if and *only if the sum of the rates of variation of the curvature is zero. That is,*

(10) $$\Gamma_0' + \Gamma_1' + \Gamma_2' = 0.$$

In terms of the curvatures of the evolutes of the three curves, this test is

(11) $$\frac{\Gamma_0^3}{K_0} + \frac{\Gamma_1^3}{K_1} + \frac{\Gamma_2^3}{K_2} = 0.$$

Theorem 9. A symmetric three-web is isothermal if and only if their curvatures and the first two rates of variation of the curvature satisfy the equations

$$2\Gamma'_1(\Gamma_2 - \Gamma_0) + 2\Gamma'_2(\Gamma_1 - \Gamma_0) - \sqrt{3}(\Gamma''_2 - \Gamma''_1) = 0,$$

(12) $$2\Gamma'_2(\Gamma_0 - \Gamma_1) + 2\Gamma'_0(\Gamma_2 - \Gamma_1) - \sqrt{3}(\Gamma''_0 - \Gamma''_2) = 0,$$

$$2\Gamma'_0(\Gamma_1 - \Gamma_2) + 2\Gamma'_1(\Gamma_0 - \Gamma_2) - \sqrt{3}(\Gamma''_1 - \Gamma''_0) = 0.$$

Only two of these three equations are independent. The sum of the left hand members of these three equations is zero.

9. *Addenda.* We give here some additional elementary properties of isothermal families.

Theorem 10. The angle between two independent isothermal families is a harmonic function of (x, y). Conversely if an unknown family F intersects a given isothermal family G in an angle which is a harmonic function, then F is an isothermal family.

As a special case of this result, we note that any simple system of isogonal, multiplicative, or additive-multiplicative trajectories of a given isothermal family is isothermal(5).

The isocline field of a given isothermal field is isothermal(6). If the isocline field of a field F is isothermal, then the inclination Θ of F is a function of a harmonic function (in general, F is not isothermal).

Let us consider two independent simple families of curves F and G.

The locus of all points p where the angle between the curves of F and G is a fixed number α is a curve C_α. By varying α, there will result a simple family A of all such curves C_α. There is also a simple family R of curves which divides the angle between F and G in a constant ratio.

If F and G are isothermal then the families A and K described above are also isothermal.

The results outlined above are special cases of the general transformation theory of isothermal families developed elsewhere (see references 5, 6, 7).

In later papers, we shall study the Laplace equation in three dimensions and the physically important theory of isothermal families of surfaces and their orthogonal stream lines. In space of four dimensions J. De Cicco is investigating bi-isothermal systems connected with functions of two complex variables.

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2) KASNER, *A characteristic property of isothermal systems*, Mathematische Annalen, Vol. 59, pp. 352-4 (1904). DE CICCO, *New proofs of the theorems of Beltrami and* KASNER *on linear families of curves*, Bulletin of the American Mathematical Society, 1943.

3) In an entirely different connection, Darboux investigated transformations of space which carry three mutually perpendicular directions e_1, e_2, e_3 at any point p into three directions E_1, E_2, E_3 at the corresponding point P so that e_j is parallel to E_j. See DARBOUX, Proceedings of the London Mathematical Society, 1900.

4) The results on isothermal symmetric three-webs were developed by DE CICCO. See *The two conformal covariants of a field*, Revista de matemáticas, Vol. 2, pp. 59-66, 1941.

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6) KASNER and DE CICCO, *Generalized transformation theory of isothermal and dual families*, Proceedings of the National Academy of Sciences, Vol. 28, pp. 52-55, February, 1942. *An extensive class of transformations of isothermal families*, Revista de Matemáticas, Vol. 3, 1942.

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