# ON THE ROLE OF PUBLIC DEBT IN AN OLG MODEL WITH ENDOGENOUS LABOR SUPPLY

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revised version: April 2007

*Abstract:* This paper argues that some propositions reported in a recent paper by Fanti and Spataro (2006) are not warranted. They claim that including an endogenous labor supply in an overlapping generations model may change the conclusions concerning the capital accumulation and welfare effects of (internal) public debt issue. We show that their results are not the consequence of the Cobb-Douglas preferences they posit, but of a rather incomplete development of their model. When this incompleteness is corrected, and under general assumptions on preferences and technology, the propositions arrived at originally by Diamond (1965) in a model that does not take the labor-leisure decision into account continue to hold. In particular, no matter whether the starting point is a dynamically efficient or inefficient steady state, an increase in the stock of public debt per taxpayer unambiguously depresses the capital-labor ratio and raises the interest rate. Moreover, the welfare level will increase (decrease) when the starting point is a dynamically inefficient (efficient) steady state.

*JEL Classification:* D91, E62, H63, J22

*Keywords:* overlapping generations, endogenous labor supply, dynamic (in)efficiency, public debt

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(*) The author is indebted to two anonymous referees and to Ted Palivos for their very helpful comments on an earlier version. Needless to say, the usual caveat applies. Research support from the Instituto de Estudios Fiscales (Ministry of Finances, Spain), as well as institutional support from the Spanish Science and Technology System (Project No. SEJ2006-4444) and the Catalan Government Science Network (Project No. SGR2005-177 and Xarxa de Referencia d'R+D+I en Economia i Politica Publiques, XREPP) is gratefully acknowledged.

## 1. Introduction

Ever since the celebrated paper by Diamond (1965) the overlapping generations (OLG) model has become a standard tool to deal with both macroeconomic and public finance issues. In a recent contribution to this Journal, Fanti and Spataro (2006) (F-S henceforth) introduce an endogenous labor supply in Diamond's (1965) model and reconsider the role of public debt. This is an important extension, since the standard approach focuses on savings decisions and neglects labor-leisure decisions. In particular, F-S posit Cobb-Douglas preferences and discuss the consequences of the government issuing internal public debt on steady state capital accumulation and welfare when taxes (subsidies) fall on (are received by) the younger generation. They claim to have shown that when the starting point is a dynamically inefficient steady state (i.e., a situation where the interest rate falls short of the population growth rate), the relationship between public debt issue and the steady-state levels of the interest rate and welfare is ambiguous: it may even be the case that an increase in debt is welfare worsening. These results are thus different from those arising from Diamond's (1965) setting.

The purpose of this paper is to show that F-S's claims are not warranted since they are the consequence of a rather incomplete development of their model. When this incompleteness is corrected, and under general conditions (i.e., without restricting the analysis to Cobb-Douglas preferences), it is shown that the introduction of an elastic labor supply does not change the propositions arrived at by Diamond (1965) in his OLG model without a labor-leisure decision. In particular, no matter whether the starting point is a dynamically efficient or inefficient steady state, an increase in the stock of public debt per taxpayer unambiguously depresses the capital-labor ratio and raises the interest rate. The welfare level will increase when the starting point is a dynamically inefficient steady state and will decrease when the economy's capital-labor ratio is below its golden rule level.

## 2. The model

The framework is the OLG model developed by Diamond (1965), extended to include a labor-leisure decision. All the individuals are identical and live for two periods. In the first one they consume $c_1$ and work an amount of time defined as their time endowment (normalized to unity) minus their leisure, $l$. In the second one they fully retire and consume $c_2$. Denoting $N$ ($R$) the number of workers (retirees), the demographic structure is $N = (1+n)R$ , where $n$ is the rate of population growth. The technology is represented by a constant-returns-to-scale production function $Y = F(K,L)$ which relates output $Y$ to capital $K$ and labor $L$, where $L = N(1\text{-}l)$. This can be rewritten as $y = f(k)$, with $f'(.) > 0$ and $f''(.) < 0$, where $k$ and $y$ are capital and output per unit of labor. Under competition, the rate of return to capital $r$ and the wage rate $w$ are given, respectively, by $r = f'(k)$ and $w = f(k) - kf'(k)$.

As in Diamond (1965), the government issues internal public debt and levies (grants) lump-sum taxes (subsidies) upon the young. F-S assume that the stock of debt per taxpayer, $B/N$, is kept constant at a certain ratio $b$. If $\tau$ is the lump-sum tax/subsidy on each young individual, the government budget constraint in each period is $b(1+n) + \tau = b(1+r)$, so that $\tau = (r - n)b$, which will be positive or negative according to whether $r$ is greater or less than $n$.

The individuals save for life-cycle reasons, and maximize a utility function $U = U(c_1,l,c_2)$ subject to the lifetime budget constraint:

$$c_1 + \frac{c_2}{(1+r)} = w(1\text{-}l) - \tau \tag{1}$$

The demands for consumption and leisure are thus $c_i = c_i(\tau,w,r)$, $i = 1,2$, and $l = l(\tau,w,r)$, where $\partial c_i/\partial\tau < 0$ if both $c_1$ and $c_2$ are normal goods, and $\partial(1\text{-}l)/\partial\tau > 0$ assuming that leisure is also a normal good. For later use, it is convenient to notice the following relationship:

$$\frac{\partial c_1}{\partial \tau} + \frac{1}{(1+r)}\frac{\partial c_2}{\partial \tau} - w\frac{\partial(1\text{-}l)}{\partial \tau} = -1 \tag{2}$$

The capital stock plus the amount of public debt will correspond to the savings made by the older generation. In a steady state this implies that $K + B = \left[w(1\text{-}l) - \tau - c_1\right] R$ , so that the capital market equilibrium condition becomes:

$$(1+n)\left[k(1\text{-}l) + b\right] = [f(k) - kf'(k)](1\text{-}l) - [f'(k) - n]b - c_1 \tag{3}$$

This expression implicitly characterizes the steady-state capital-labor ratio $k$ as a function of the stock of debt per young taxpayer $b$. In what follows it will be assumed that the steady-state solutions of (3) are unique and stable, so that $k = k(b)$. Using the relationship between $r$ and $k$, the steady-state interest rate can thus be written as $r = f'[k(b)] = r(b)$. It is well known that in these models a steady state can be either dynamically efficient ($r > n$) or inefficient ($r < n$).

Concerning welfare, since both consumptions and leisure depend on $\tau$, $w$, and $r$, one can write $U = U\left[c_1(\tau,w,r),l(\tau,w,r),c_2(\tau,w,r)\right] = G(\tau,w,r)$, which is an indirect utility function providing the maximum steady-state welfare level that can be attained for given values of $\tau$, $w$ and $r$. Taking into account that $\tau = (r - n)b$, that both $r$ and $w$ depend on $k$, and using the locus $k = k(b)$, we end up with the function $U = V(b)$. This can be used to discuss the steady-state welfare effects of changes of $b$.

### 3. Public debt, capital accumulation and welfare

The explicit purpose of F-S's paper is to "reconsider the analysis of Diamond (1965) on the possibility of correcting the dynamic inefficiency (DI) of an OLG economy via debt issuing, in the presence of endogenous labor supply and well-behaved preferences" (p. 437). They claim that they provide a general discussion of the relationship between the equilibrium interest rate and the level of public debt, and that with Cobb-Douglas preferences (i) "its sign can be negative in the DI case so as to potentially invalidate Diamond's rule" (p. 429, italics added). On the basis of their results, they contend that (ii) "Diamond's rule is to be considered still valid, in the presence of elastic labor supply, when the economy is dynamically efficient" […but…] (iii) "in the case of DI, the ambiguity brought about by the change of the interest rate mentioned above leads to the possibility that debt issuing be welfare worsening in the presence of endogenous labor supply" (p. 429, italics added).

Using the $r = r(b)$ function, F-S work out the derivative d$r(b)$/d$b$ and obtain their expression (19), which in their own words is "a general result" (p. 433). The authors then go on and put Cobb-Douglas demand responses into (19) to find (20) and state their Proposition 1: when $r > n$, d$r$/d$b$ is always positive, but when $r < n$, d$r$/d$b$ may become negative if a certain inequality holds. F-S provide some numerical examples that assume Cobb-Douglas preferences and a CES production function that are presented as support for their claims (Tables 1 and 2). As will be shown below, on the one hand, the result that d$r$/d$b > 0$ when $r > n$ is not restricted to the Cobb-Douglas case. On the other hand, the sufficient condition for d$r$/d$b < 0$ when $r < n$ would be correct for those particular preferences… if this condition could be fulfilled. Actually, as discussed immediately, this condition cannot be fulfilled when d$r$/d$b$ is evaluated at a locally stable steady state.

The reason for our claim that F-S's model development is incomplete can be seen from the following discussion. From the capital market equilibrium condition (3):

$$\frac{\mathrm{d}k(b)}{\mathrm{d}b} = \frac{-1}{D(.)}\left\{(1+n) + [r(b) - n](1+n)k\frac{\partial(1\text{-}l)}{\partial\tau} - [r(b) - n]\left(-1 - \frac{\partial c_1}{\partial\tau} + w\frac{\partial(1\text{-}l)}{\partial\tau}\right)\right\} \tag{4}$$

where $D(.)$ is positive for a steady state to be locally stable. Denoting, respectively, $\varepsilon_{1\text{-}l} = (\partial(1\text{-}l)/\partial\tau)(\tau/(1\text{-}l))$ and $\varepsilon_{c_1} = (\partial c_1/\partial\tau)(\tau/c_1)$ the elasticities of the supply of labor and the demand for first-period consumption with respect to $\tau$, and with the elasticities $\theta$ (with respect to $w$) and $\eta$ (with respect to $r$) defined in a similar way, $D(.)$ can be written as:

$$D(.) = (1+n)(1\text{-}l) + f''[k(1\text{-}l) + b] + (1+r)bf''\left\{-\frac{b}{\tau}\varepsilon_{1\text{-}l} + \frac{k}{w}\theta_{1\text{-}l} - \frac{1}{r}\eta_{1\text{-}l}\right\} + \tag{5}$$

$$+ c_1 f''\left\{-\frac{b}{\tau}[\varepsilon_{1\text{-}l} - \varepsilon_{c_1}] + \frac{k}{w}[\theta_{1\text{-}l} - \theta_{c_1}] - \frac{1}{r}[\eta_{1\text{-}l} - \eta_{c_1}]\right\}$$

When there is no labor-leisure decision, i.e., when $l = 0$, the numerator in (4) reduces to $\{1 + r(b)(1 + \partial c_1/\partial\tau) - n\,(\partial c_1 \partial\tau)\}$, so that (4) becomes expression (27) in Diamond (1965), p. 1142: assuming normality of both consumptions (so that $1 + \partial c_1/\partial\tau > 0$), the derivative $\mathrm{d}k(b)/\mathrm{d}b$ is negative regardless of the relationship between $r$ and $n$. Turning now to the case where labor supply is endogenous, and using (2), it is clear from mere inspection of (4) that if the steady state taken as a starting point is dynamically efficient, i.e., when $r(b) > n$, an increase in $b$ entails a reduction in the capital-labor ratio and an increase in the interest rate. This provides a generalization of F-S's Proposition 1 that $\mathrm{d}k(b)/\mathrm{d}b < 0$ [or, similarly, that $\mathrm{d}r(b)/\mathrm{d}b > 0$] when $r$ is greater than $n$.

However, F-S also contend in their Proposition 1 that $\mathrm{d}k(b)/\mathrm{d}b$ [$\mathrm{d}r(b)/\mathrm{d}b$] may be positive [negative] when $r(b) < n$, and advance a sufficient condition for this to be the case. Although we have stated above that this claim does not hold, (4) does not seem to provide a clear answer for the effects of $b$ on $k$ when the starting point is a dynamically inefficient steady state. After all, when $r < n$, the first term in brackets has a positive sign while the sum of the second and the third is negative. A definite answer can be found rewriting the capital market equilibrium condition (3) as $(1+n)k = w - (1+r)b/(1\text{-}l) - c_1/(1\text{-}l)$ and replacing it in (4) to find:

$$\frac{\mathrm{d}k(b)}{\mathrm{d}b} = \frac{-1}{D(.)}\left\{[1+r(b)] - \frac{[r(b) - n][1+r(b)]\,b}{(1\text{-}l)}\frac{\partial(1\text{-}l)}{\partial\tau} - \frac{c_1}{b}[\varepsilon_{1\text{-}l} - \varepsilon_{c_1}]\right\} \tag{6}$$

where $\varepsilon_{1\text{-}l}$ and $\varepsilon_{c_1}$ are $\tau$-elasticities defined above. When the starting point is a dynamically inefficient steady state, i.e., when $r(b) < n$, the sum of the first two terms in square brackets in (6) is positive. It can easily be seen that the third one will also be positive. In fact, at a dynamically inefficient steady state it will be the case that $\tau = (r - n)b$ is negative, which (assuming normality) implies that $\varepsilon_{1\text{-}l} < 0$ and $\varepsilon_{c_1} > 0$, thus rendering positive the whole bracket in (6) and implying that $\mathrm{d}k(b)/\mathrm{d}b < 0$.

Summarizing, in contrast to the discussion in F-S, with an elastic labor supply in the first period of life, and regardless of whether the preferences are of the Cobb-Douglas type or not, an increase in $b$ translates into a reduction in $k$ and a rise in $r$ when the starting point is a dynamically inefficient steady state. This result, when $r(b) < n$, in conjunction with the previous one when $r(b) > n$, gives rise to the following proposition.

**Proposition 1**: *When a labor-leisure decision is included in an OLG model à la Diamond (1965), an increase of the stock of debt per taxpayer depresses the capital-labor ratio and increases the interest rate regardless of whether the starting point is a dynamically efficient* ($r > n$) *or inefficient* ($r < n$) *steady state.*

Before turning to the analysis of the welfare effects of public debt issue in the presence of an elastic labor supply, a word must be said concerning the numerical examples provided by F-S. As a matter of fact, Proposition 1 above implies that there must be something in the numerical simulations that explains F-S' results. They do not provide details on their procedure farther than pointing out that their model does not admit closed-form solutions and that the simulations have been obtained using the software Maple 5.1 (p. 435). However, a reasonable conjecture is that the values for $\mathrm{d}r/\mathrm{d}b$ reported in Table 2 are the consequence of evaluating this derivative at an unstable steady state. In effect, the capital market condition (3) can be written as $k = \left[w(1\text{-}l) - (r - n)b - c_1 - (1+n)b\right] / (1+n)(1\text{-}l)$. With an obvious notation, this amounts to $k = g(k,b)$, so that $\mathrm{d}k/\mathrm{d}b = (\partial g/\partial b) / (1 - \partial g/\partial k)$. Local stability requires that $\partial g/\partial k < 1$, i.e., that $D(.)$ in (5) be positive. Thus, the evaluation of $\mathrm{d}k/\mathrm{d}b$ at an unstable steady state where $(1 - \partial g/\partial k) < 0$ will reverse its sign!.

The above discussion has also implications for Propositions 2 and 3 in F-S, which focus on the welfare effects of debt issue. On the one hand, on account of our previous argument, their Proposition 2 is based on a result (that $\mathrm{d}r/\mathrm{d}b > 0$ when $r > n$) that is not restricted to Cobb-Douglas preferences. On the other hand, because Proposition 3 hinges upon a claim that stems from an incomplete development of their model (that it may be the case that $\mathrm{d}r/\mathrm{d}b < 0$ when $r < n$). Deriving with respect to $b$ in the $U = V(b)$ function discussed above, F-S arrive at an expression [their (24)] which is the same as in Diamond (1965, p. 1143). Since sign $\{\mathrm{d}V/\mathrm{d}b\}$ = sign$\{n - r\}$, we are back to Diamond's result, i.e., "utility is decreased in the efficient case and increased in the inefficient case" (ibid). This can be summarized in the following proposition that entails that Diamond's result holds when his model is extended to deal with an elastic labor supply:

**Proposition 2**: *When a labor-leisure decision is included in an OLG model à la Diamond (1965), an increase of the stock of debt per taxpayer decreases (increases) the welfare level whenever the starting point is a dynamically efficient (inefficient) steady state.*

## 4. Concluding remarks

This paper has argued that some propositions reported in a recent paper by Fanti and Spataro (2006) published in this Journal are not warranted. They claim that including an endogenous labor supply in an OLG model à la Diamond (1965) may change the conclusions concerning the capital accumulation and welfare effects of internal public debt issue. We have shown that their results are not the consequence of the rather stringent Cobb-Douglas preferences they posit, but of an incomplete development of their model. When this incompleteness is corrected, and under general assumptions on preferences and technology, the results arrived at originally by Diamond (1965) in a model that does not take the labor-leisure decision into account continue to hold. In particular, no matter whether the starting point is a dynamically efficient or inefficient steady state, an increase in the stock of public debt per taxpayer unambiguously depresses the capital-labor ratio and raises the interest rate. Moreover, the welfare level will increase (decrease) when the starting point is a dynamically inefficient (efficient) steady state.

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