VOTING FOR THE BEST AND AGAINST THE WORST

José Luis García–Lapresta[(a)], A.A.J. Marley[(b)], Miguel Martínez–Panero[(a)]

[(a)] Dep. de Economía Aplicada, Grupo de Investigación PRESAD, Universidad de Valladolid. Avda. Valle de Esgueva 6, 47011 Valladolid, Spain. E-mail: {lapresta,panero}@eco.uva.es

[(b)] Dep. of Psychology, University of Victoria, PO Box 3050 STN CSC, Victoria BC V8W 3P5, Canada. E-mail: ajmarley@uvic.ca

Abstract

An increasing body of theoretical and empirical work on discrete choice considers a choice design in which a person is asked to select both the best and the worst alternative in an available set of alternatives, in contrast to more traditional tasks such as where the person is asked to: select the best alternative; select the worst alternative; rank the alternatives. Here we consider voting systems motivated by such "best-worst" choice; relate them to approval and disapproval voting systems; and characterize a "best-worst" voting system in terms of a set of axioms in the context of scoring rules.

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1 Introduction

Usually, voters find it difficult to linearly order a reasonable number n of feasible alternatives – say, $n \geq 10$. According to Dummett [16, p. 243],

> "If there are, say, twenty possible outcomes, the task of deciding the precise order of preference in which he ranks them may induce a kind of psychological paralysis in the voter; and, for the tellers, the labour of reckoning the preference scores becomes very tedious. We have, therefore, to devise new or modified procedures for use in this case"[1].

However, voters can easily (provided they are not indifferent between all the alternatives) select reliably the best and the worst alternative, and, perhaps, the p–best and q–worst alternatives for "small" p and q.

Finn and Louviere [19] proposed, and studied, a discrete choice task in which a person selects both the best and the worst option in an available (sub)set of alternatives. Since the publication

[1] Of course, the concern regarding "the labor of reckoning the preference scores" no longer carries much weight.

of that paper, interest in, and use of, such best-worst choice tasks has been increasing, with two recent empirical applications receiving "best paper" awards (Cohen [14]; Cohen and Neira [15]). According to Marley and Louviere [28], best-worst tasks have a number of advantages over traditional discrete choice tasks: 1) a single pair of best-worst choices contains a great deal of information about the person's ranking of options (e.g., if there are 3 items in a set, one obtains the entire ranking of that set; if there are 4 items in a set, one obtains information on the implied best option in 9 of the 11 possible non-empty, non-singleton subsets[2]; and if there are 5 items in a set, one obtains information on the implied best option in 18 of the 26 possible non-empty, non-singleton subsets); 2) best-worst tasks take advantage of a person's propensity to identify and respond more consistently to extreme options; and 3) best-worst tasks seem to be easy for people.

No extensive comparison exists of best-worst methods versus more traditional methods, such as selecting the best option or rating or ranking all of the options. However, the opinion of Flynn, Louviere, Li, Coast and Peters [22] is that "Best-worst tasks provide a balance between relatively inefficient traditional 'pick one' choice tasks and rating/ranking tasks which, although apparently providing far more information on preferences [24], frequently induce behaviour which violates the statistical assumptions inherent in these models [5, 8]."

Despite increasing use of the approach, Marley and Louviere [28] is the first presentation of the theoretical properties of probabilistic models of best-worst choice. Here we develop theoretical properties of deterministic best-worst choice, and its generalizations, as a voting system. We do this in the context of *scoring rules*, which are of major importance in the voting literature (see Chebotarev and Shamis [12] for a referenced survey). In fact, Marley and Louviere [28] present some basic results on the "optimality" of scoring rules in the estimation of parameters in probabilistic models of best-worst choice. That work can be interpreted, in the voting context, as assuming a restricted domain for the possible voting profiles. Here we approach the problem deterministically and without domain restrictions.

As the title of the paper suggests, we focus our attention in characterizing a class of voting systems within a tradition which came from the very beginning of modern Social Choice Theory.

[2] Let $\{a, b, c, d\}$ be the set of options and suppose that we know that a is selected as best and d as worst. If we now check each (sub)set of size $2, 3, 4$ of $\{a, b, c, d\}$ in turn, then we see that the subsets $\{b, c\}$ and $\{b, c, d\}$ are the only ones where the best element is not determined either by the information that a is best or by the information that d is worst.

Indeed, as pointed out by Merlin [30], Arrow's theorem can be understood as an axiomatization of dictatorship. Given that Arrow's theorem [1] is usually considered a negative result, May's theorem [29] (which characterizes simple majority) is commonly considered as the first characterization theorem of voting theory. After May, a relevant issue in Social Choice Theory is to find axiomatic characterizations of voting systems. For references, see Merlin [30] and Marchant [27], among others. In this paper we present two axiomatic characterizations of 1–approval 1–disapproval voting (1–best 1–worst voting, in our scoring context).

The remainder of the paper is organized as follows. Section 2 defines various voting systems, the most general of which we call ranked p–approval q–disapproval voting. As the name suggests, when there are n available alternatives, in this voting system the voter is required to rank order p approved alternatives from most approved ("best") to least approved ("worst") and q disapproved alternatives from most disapproved ("worst") to least disapproved ("best"), with the constraints that the set of approved alternatives is disjoint from the set of disapproved alternatives and $1 \leq p + q \leq n$. Then Marley and Louviere's best-worst choice corresponds to 1–approval 1–disapproval voting. Section 3 introduces the standard scoring rules, i.e., those for which the scores are all nonnegative, and Subsection 3.1 extends that concept to ranked approval-disapproval scoring rules, which can be viewed as extended scoring rules in which negative scores are allowed. Section 4 contains the characterization results. Finally, the paper concludes with a Discussion in Section 5.

2 General Voting Systems

In this section we define various voting systems, the most general of which we call ranked p–approval q–disapproval voting.

Suppose that each of m voters has to indicate an opinion concerning (each of) the n alternatives in a set $X = \{x_1, \ldots, x_n\}$.

1. *Approval voting* allows each voter to approve of any number of alternatives k ($1 \leq k \leq n-1$), where k is up to the voter. The natural social welfare rule assumes that the alternative(s) with the largest number of approval votes is (are) the winner(s)[3]. Note that

[3] For all the voting procedures and scoring rules that we consider, tie scores can occur. If this causes difficulty in

we could assume also that the alternative(s) with the smallest number of approval votes is (are) the loser(s).

2. *Disapproval voting* allows each voter to disapprove of any number of alternatives k ($1 \leq k \leq n-1$), where k is up to the voter. The natural social welfare rule assumes that the alternative(s) with the smallest number of disapproval votes is (are) the winner(s). Note that we could also assume that the alternative(s) with the largest number of disapproval votes is (are) the loser(s).

   Obviously, disapproval voting is equivalent formally – though not necessarily behaviorally – to approval voting.

3. *Approval-disapproval voting* [4] allows each voter to approve of any number of alternatives k ($0 \leq k \leq n$) and to disapprove of any number of alternatives l ($0 \leq l \leq n$) , where k and l are selected by the voter, with the constraints that the set of approved alternatives is disjoint from the set of disapproved alternatives, and $1 \leq k + l \leq n$. The natural social welfare rule assumes that the alternative(s) with the highest difference score – its total approval votes minus its total disapproval votes – is (are) the winner(s). This voting system is related to Yılmaz's [34] *trichotomous procedure* in which each voter classifies each alternative as "favorite" or "acceptable" or "disapproved".

4. *p–approval voting* requires each voter to approve of the same fixed number, p, of alternatives ($1 \leq p \leq n-1$). The natural social welfare rule assumes that the alternative(s) with the highest score is (are) the winner(s).

5. *q–disapproval voting* requires each voter to disapprove of the same fixed number, q, of alternatives ($1 \leq q \leq n-1$). The natural social welfare rule assumes that the alternative(s) with the lowest score is (are) the winner(s).

   If we assume no indifference – i.e., no alternatives that a voter neither approves or disapproves – then we can interpret p–approval voting as equivalent to $(n-q)$–disapproval

selecting the appropriate number of alternatives (winning, losing, etc.), then some additional (normally random) procedure is required to reach the final social decision.

[4]Approval-disapproval voting is called *negative voting* by Brams and Fishburn [11] and is related to *yes-no voting* proposed by the same authors in [10]. These authors point out that it was first proposed by Boehm [7] and then analyzed by Brams (see, for example, [9]) and Felsenthal [18]. It has also recently advocated by Hillinger [25].A natural probabilistic model for approval-disapproval voting is a variant of the *size-independent model* of approval voting (Falmagne and Regenwetter [17]). Basically, one extends that model by allowing approval of some options and disapproval of others.

voting; and q–disapproval voting as equivalent to $(n-q)$–approval voting.

6. *p–approval q–disapproval voting*, which we later call p–best q–worst voting, requires each voter to approve of the same fixed number, p, of alternatives and to disapprove of the same fixed number, q, of alternatives, with the constraints that the set of approved alternatives is disjoint from the set of disapproved alternatives and $1 \leq p+q \leq n$. The natural social welfare rule assumes that the alternative(s) with the highest difference score – i.e., its total approval votes minus its total disapproval votes – is (are) the winner(s).

7. *Ranked p–approval q–disapproval voting*[5] is the extension of p–approval q–disapproval voting where the voter is required to rank order the p approved alternatives from most approved ("best") to least approved ("worst") and the q disapproved alternatives from most disapproved ("worst") to least disapproved ("best"), with the constraints that the set of approved alternatives is disjoint from the set of disapproved alternatives and $1 \leq p+q \leq n$.

8. *Plurality*[6] requires each voter to indicate the most preferred ("best") alternative. The alternative(s) with the most votes is (are) the winner(s). Obviously, assuming no indifference, plurality is equivalent formally – though not necessarily behaviorally – to each of 1–approval voting and $(n-1)$–disapproval voting.

9. *Antiplurality*[7] requires each voter to indicate the worst alternative. The alternative(s) with the least number of negative votes is (are) the winner(s). Obviously, assuming no indifference, antiplurality is equivalent formally – though not necessarily behaviorally – to each of $(n-1)$–approval voting and 1–disapproval voting.

10. *Classic Borda rule*[8] requires each voter to rank order the alternatives from best to worst. Then, for each voter, each alternative is assigned a score equal to the number of alternatives worse than it, and the alternative(s) with the largest total score is (are) the winner(s).

[5] As for approval-disapproval voting, a natural probabilistic model for ranked p–approval q–disapproval voting is a variant of the *size-independent model* of approval voting (Falmagne and Regenwetter [17]). Basically, one extends that model by retaining the ranked information that underpins that model, plus allowing for the (ranked) approval of some options and (ranked) disapproval of others.

[6] Sometimes called *first past the post*, *the best of the best* and *most votes count*.

[7] Sometimes called *negative voting*, *inverse plurality rule*, *avoid the worst of the worst*, *blackball* and *blacklist*.

[8] There are several ways to adjust the Borda rule when each voter is allowed to state a weak order (or complete preorder) over the alternatives, i.e., indifferences are allowed but transitivity is retained (see Black [6]). We call *classic* the case when such indifferences are not allowed.

3 Scoring Rules

Assuming that each voter rank orders the alternatives, a *scoring rule* is a vector of scores $(s_1, \ldots, s_n) \in \mathbb{R}^n$ with $s_1 \geq \cdots \geq s_n$ and $s_1 > s_n$ where, for each voter, s_1 points are assigned to the top-ranked alternative, s_2 points to the second-ranked alternative, and so on. The alternative(s) with the largest total score is (are) the winner(s).

Given a scoring rule with vector of scores $(s_1, \ldots, s_n)$, $a, b \in \mathbb{R}$ such that $a > 0$, the new scoring rule with vector of scores $(s'_1, \ldots, s'_n)$, where $s'_i = as_i + b$ for all $i = 1, \ldots, n$, is equivalent to the previous one, in the sense they provide the same social outcomes.

A scoring rule $(s_1, \ldots, s_n)$ is *standard*[9] if $s_n \geq 0$, otherwise, it is *extended.* Clearly, every extended scoring rule $(s_1, \ldots, s_n)$ is equivalent (in the sense that they always provide the same outcome) to the standard one $(s'_1, \ldots, s'_n)$, where $s'_i = s_i - s_n$. Even more, every scoring rule $(s_1, \ldots, s_n)$ (standard or extended) is equivalent to a standard one $(s'_1, \ldots, s'_n)$ where $s'_1 = 1$ and $s'_n = 0$ – simply take $s'_i = (s_i - s_n)/(s_1 - s_n)$.

The following examples give the scoring rules associated with various voting systems:

- p–approval voting: $(1, \ldots, 1, 0, \ldots, 0)$, with p 1's.
- q–disapproval voting[10]: $(1, \ldots, 1, 0, \ldots, 0)$, with $(n - q)$ 1's.
- p–approval q–disapproval voting[11]: $(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$, with p 2's, $(n - p - q)$ 1's and q 0's.
- Plurality: $(1, 0, \ldots, 0)$.
- Antiplurality: $(1, \ldots, 1, 0)$.
- Classic Borda rule: $(n - 1, n - 2, \ldots, 1, 0)$.

For $n = 2$, 1–approval 1–disapproval voting coincides with both plurality and antiplurality.

[9] Notice that this definition differs from that of Woeginger [33].

[10] We introduced this voting system in a negative way, corresponding to the scoring rule $(0, \ldots, 0, -1, \ldots, -1)$ with q scores of (-1). But as we note above, q–disapproval voting is formally equivalent to $(n - q)$–approval voting.

[11] Initially this voting system is defined as an extended scoring rule (allowing negative scores) with associated vector of scores $(1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)$. Adding 1 to each score gives the equivalent standard scoring rule that we present here.

Notice that approval voting, disapproval voting and approval-disapproval voting are not scoring rules. However, they can be understood as *flexible* scoring rules following Baharad and Nitzan [2].

3.1 Ranked Approval-Disapproval Voting and Scoring Rules

As discussed in Section 1, usually voters find it difficult to linearly order a reasonable number n of feasible alternatives. However, they can usually (provided they are not indifferent between all the alternatives) select reliably the best and the worst alternative, and, perhaps, the (ranked) p–best q–worst alternatives for "small" p and q. We therefore now discuss scoring rules for such voting systems.

Let $(\alpha_1, \ldots, \alpha_p)$, $(\delta_1, \ldots, \delta_q)$ be a pair of nonnegative score vectors[12] such that $\alpha_1 \geq \cdots \geq \alpha_p \geq 0$ and $\delta_1 \geq \cdots \geq \delta_q \geq 0$, where $p \geq 1$, $q \geq 1$ and $p + q \leq n$. The *(ranked) p–best q–worst voting procedure associated with* $[\alpha_1, \ldots, \alpha_p\, ; \delta_1, \ldots, \delta_q]$ is the voting system where, for each voter, α_1 points is assigned to that voter's first-ranked alternative, α_2 points to the second ranked alternative, and so on; δ_1 points is assigned to that voter's last-ranked alternative, δ_2 points to the next-to-last-ranked alternative, and so on. The alternative(s) with the highest total score difference – i.e., the difference between the total α score and the total δ score for the alternative – is (are) the winner(s).

In principle, assuming voters rank order all the alternatives, there exists a clear difference between the aforementioned system and standard scoring rules. However, $[\alpha_1, \ldots, \alpha_p\, ; \delta_1, \ldots, \delta_q]$ can be viewed as an *extended*[13] scoring rule

$$(\alpha_1, \ldots, \alpha_p, 0, \ldots, 0, -\delta_q, \ldots, -\delta_1).$$

If we add δ_1 to each term, we obtain the standard scoring rule

$$(\alpha_1 + \delta_1, \ldots, \alpha_p + \delta_1, \delta_1, \ldots, \delta_1, \delta_1 - \delta_q, \ldots, \delta_1 - \delta_2, 0).$$

In order to shed light on this dual aspect, we show some particular ranked best-worst voting procedures and their translation into the scoring context.

- Plurality, defined by $[1; 0]$, corresponds to the scoring rule $(1, 0, \ldots, 0)$.

[12] We use α (resp., δ) as a reminder that the scores are for ranked **a**pproval (resp., **d**isapproval) votes.
[13] Note that no negative score is allowed in standard scoring rules.

- Antiplurality, defined by $[0;1]$, corresponds to the scoring rule $(0,\ldots,0,-1)$.
- Classic Borda rule, defined by $[n-1,n-2,\ldots,1;0]$, corresponds to the scoring rule $(n-1,n-2,\ldots,1,0)$.
- 1–best 1–worst, defined by $[\alpha;\delta]$, corresponds to the scoring rule $(\alpha,0,\ldots,0,-\delta)$. Notice that the case $[\alpha;\alpha]$, $\alpha\neq 1$, coincides with $[1;1]$ because their respective scoring versions $(\alpha,0,\ldots,0,-\alpha)$ and $(1,\ldots,-1)$ are both equivalent to the same standard scoring rule $(1,1/2,\ldots,1/2,0)$. We will call the latter the *basic* 1–*best* 1–*worst voting system.* Taking into account again their scoring versions, it is easy to see that two 1–best 1–worst voting systems $[\alpha;\delta]$ and $[\alpha';\delta']$ are equivalent if and only if $\alpha'\delta=\alpha\delta'$. In this way, if $\alpha\neq\delta$, then $[\alpha;\delta]$ is essentially different to $[1;1]$.

Thus, mathematically, every (ranked) approval-disapproval voting system is equivalent to an extended (alternatively, a standard) scoring rule. This fact will be crucial in the analysis of properties and axiomatic characterization of some approval-disapproval scoring rules.

It is worth emphasizing that the above identification is within the mathematical framework but not from a behavioral perspective. In particular, the use of ranked approval-disapproval scoring rules in the mathematics does not entail that voters rank the alternatives. The same happens, for instance, with plurality, where voters only indicate their best alternative (they do not need to provide a ranking), but scoring rules are used in its axiomatic characterization (see Richelson [32]).

4 Characterization of 1–Best 1–Worst Voting

In this section we analyze some properties of the 1–best 1–worst voting and give two characterizations: one for the general case $[\alpha;\delta]$, and another one for the basic case $[1\,;1]$. Our approach follows that used in the fundamental paper of Young [36], where he proved that a scoring rule is a social choice function characterized by the following axioms:

A *Anonymity*: There is an egalitarian consideration for the agents.

N *Neutrality*: There is a symmetric status for each alternative.

R *Reinforcement*: If two disjoint subsets of voters have at least one common alternative among their winners, then all such common alternatives keep on being winners for the joined set of voters[14].

C *Continuity*: If two disjoint sets of voters U and V select x and y as winners, respectively, then x is a winner for the superset $(nU) \cup V$ for n sufficiently large[15].

The Borda count was characterized by Young [35], who then characterized it also in the scoring context (Young [36]) – namely, the Borda count is the only scoring rule with the *cancellation* property: in any situation[16] where every preference for alternative x over y for one voter is balanced by another preference for alternative y over x for another voter, all the alternatives win. Also see Merlin [30, Theorem 7] for a characterizations of the Borda rule by adding the *Condorcet loser* property to those of Young [36].

Other researchers have used Young's approach to characterize other scoring rules (or it is possible to translate their results to this framework.) For example, Richelson [32] characterized plurality rule as the only scoring rule (i.e., verifying **A**, **N**, **R** and **C**) that also satisfies *independence of Pareto-dominated alternatives* (or *reduction*, as appearing in Fishburn [20, p. 148]). Even more, Ching [13] demonstrated that continuity is redundant in the previous characterization. On the other hand, Lepelley [26] provided an alternative axiomatization of plurality by adding the *strong Condorcet winner* property to those of Young, and Merlin and Naeve [31] achieve another characterization of plurality by introducing *bottom-invariance* as additional property: "a voter cannot exclude a winner from the choice set by reshuffling her preferences above this winner". Taking into account the symmetry between plurality and antiplurality, Merlin [30] points out that antiplurality can be characterized by introducing *top-invariance* as additional property (see also Barberà and Dutta [4]). More recently, in a similar way, Baharad and Nitzan [3] have developed necessary and sufficient conditions for antiplurality, and in this case the additional property has been formulated as a minimal veto condition. In addition, these authors have also proposed an alternative characterization of plurality rule with another

[14] This property is also considered by Fishburn [21] in his characterization of approval voting, and by Young and Levenglick [37] in their characterization of Kemeny's rule. For other names for this condition (*consistency*, *separability*, etc.), see Merlin [30, page 95].

[15] In this axiom, for U a set of voters and their votes, (nU) means n copies of those voters and votes. In Young [36] this axiom is introduced as a extension of a domain condition for social choice functions.

[16] A *situation* is a set consisting of one preference order for each voter (Gärdenfors [23]).

veto-type condition.

Given the earlier successes of this approach, our goal is to develop properties of best-worst voting that allow us to say that its associated scoring rule is determined by the fulfillment of these conditions in addition of those proposed by Young. As pointed out before, we succeed in this goal for general 1–best 1–worst voting $[\alpha; \delta]$ and for the basic case $[1; 1]$; the first requires three conditions, the second a single condition, in addition to Young's.

Now we develop three properties that, together with Young's, are necessary and sufficient to characterize 1–best 1–worst voting.

TSM *Top Strict Monotonicity*: If x is a non-unique winner in a situation where at least one voter considers x to be the best alternative, then x would not be a winner in the situation obtained where just this voter changes his opinion only about x (preserving his pairwise preferences about the other alternatives).

BSM *Bottom Strict Monotonicity*: If x is a non-unique winner in a situation where at least one voter does not consider x to be the worst alternative, then x would not be a winner in the situation obtained where just this voter changes his opinion about x, and decides that x is the worst alternative in the second situation (preserving his pairwise preferences about the other alternatives).

IMA *Independence of Middle Alternatives*: The winner(s) in a situation are preserved if one or more voters change their opinions about alternatives other than those they have selected as their personal best and worst (i.e., if pairwise preferences containing the best or the worst alternative do not change in a new situation, then the winner(s) are the same).

Theorem 1. An extended scoring rule satisfies **TSM**, **BSM** and **IMA** if and only if it is a 1–best 1–worst voting system.

Proof. Obviously, every 1–best 1–worst voting system satisfies the required properties.

For sufficiency, we consider the cases $n = 2, 3$ in detail so that the reader will more easily understand the general case $n \geq 4$.

Notice that if $n = 2$, just the definition of scoring rule entails $s_1 > s_2$. Thus, in this particular case plurality and antiplurality rules, as well as 1–best 1–worst voting system coincide.

If $n = 3$, **IMA** trivially holds, and it will be proven that **TSM** and **BSM** entail $s_1 > s_2 > s_3$.

Consider the situation:

Voter 1	Voter 2	Voter 3
x_1	x_2	x_3
x_2	x_3	x_1
x_3	x_1	x_2

It is clear that all the alternatives obtain the same total score: $s_1 + s_2 + s_3$, so all of them are winners.

Now we modify the previous situation in two cases:

1. The first voter interchanges the first and the second alternatives, and the opinions of the other voters do not change:

Voter 1	Voter 2	Voter 3
x_2	x_2	x_3
x_1	x_3	x_1
x_3	x_1	x_2

   According to **TSM**, now x_1 is not a winner. Thus, the total score of x_1, $2s_2 + s_3$, is smaller than the total score of at least other alternative. Taking into account that the total score of x_2 is $2s_1 + s_3$, and the total score of x_3 is $s_1 + s_2 + s_3$, then we have that either $2s_2 + s_3 < 2s_1 + s_3$ or $2s_2 + s_3 < s_1 + s_2 + s_3$. In either case, $s_1 > s_2$.

2. The third voter interchanges the second and the third alternatives, and the opinions of the other voters do not change:

Voter 1	Voter 2	Voter 3
x_1	x_2	x_3
x_2	x_3	x_2
x_3	x_1	x_1

   According to **BSM**, now x_1 is not a winner. Thus, the total score of x_1, $s_1 + 2s_3$, is smaller than the total score of at least other alternative. Taking into account that the total score of x_2 is $s_1 + 2s_2$, and the total score of x_3 is $s_1 + s_2 + s_3$, then we have that either $s_1 + 2s_3 < s_1 + 2s_2$ or $s_1 + 2s_3 < s_1 + s_2 + s_3$. In either case $s_2 > s_3$.

Note that the obtained extended scoring rule, (s_1, s_2, s_3), where $s_1 > s_2 > s_3$, is equivalent to $(s_1 - s_2, 0, s_3 - s_2)$ which defines the 1–best 1–worst voting system $[\alpha\,;\delta]$, where $\alpha = s_1 - s_2$ and $\delta = s_2 - s_3$.

Finally, suppose that $n \geq 4$ and consider an extended scoring rule with associated vector of scores $(s_1, \ldots, s_n)$ such that $s_1 \geq \cdots \geq s_n$ and $s_1 > s_n$.

In order to prove $s_1 > s_2 = \cdots = s_{n-1} > s_n$, consider the following situation where, for $k = 1, .., n-1$, the elements in row $k+1$ are obtained from row k by moving the element in column 1 in row k to column n in row $k+1$, and moving the element in column j, $j \neq 1$, in row k to column $j-1$ in row $k+1$.

Voter 1	Voter 2	$\cdots$	Voter $n-1$	Voter n
x_1	x_2	$\cdots$	x_{n-1}	x_n
x_2	x_3	$\cdots$	x_n	x_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_k	x_{k+1}	$\cdots$	x_{k-2}	x_{k-1}
x_{k+1}	x_{k+2}	$\cdots$	x_{k-1}	x_k
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{n-1}	x_n	$\cdots$	x_{n-3}	x_{n-2}
x_n	x_1	$\cdots$	x_{n-2}	x_{n-1}

It is clear that all the alternatives obtain the same total score: $s_1 + \cdots + s_n$, so all of them are winners[17].

Now we modify the previous situation in the following cases:

1. The first voter interchanges the first and the second alternatives, and the opinions of the other voters do not change:

Voter 1	Voter 2	$\cdots$	Voter $n-1$	Voter n
x_2	x_2	$\cdots$	x_{n-1}	x_n
x_1	x_3	$\cdots$	x_n	x_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{n-1}	x_n	$\cdots$	x_{n-3}	x_{n-2}
x_n	x_1	$\cdots$	x_{n-2}	x_{n-1}

According to **TSM**, now x_1 is not a winner. Thus, the total score of x_1, $2s_2 + s_3 + \cdots + s_n$, is smaller than the total score of at least one other alternative. Taking into account that the

[17]Note that it is also true for any social choice function satisfying anonymity and neutrality.

total score of x_2 is $2s_1 + s_3 + \cdots + s_n$, and that of each of $x_3, \ldots, x_n$ is $s_1 + \cdots + s_n$, then we have that either $2s_2 + s_3 + \cdots + s_n < 2s_1 + s_3 + \cdots + s_n$ or $2s_2 + s_3 + \cdots + s_n < s_1 + \cdots + s_n$. In either case, $s_1 > s_2$.

2. The second voter interchanges x_1 and x_n, and the opinions of the other voters do not change:

Voter 1	Voter 2	$\cdots$	Voter $n-1$	Voter n
x_1	x_2	$\cdots$	x_{n-1}	x_n
x_2	x_3	$\cdots$	x_n	x_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{n-1}	x_1	$\cdots$	x_{n-3}	x_{n-2}
x_n	x_n	$\cdots$	x_{n-2}	x_{n-1}

According to **BSM**, now x_n is not a winner. Thus, the total score of x_n, which is $s_1 + \cdots + s_{n-2} + 2s_n$, is smaller than the total score of at least one other alternative. Taking into account that the total score of x_1 is $s_1 + \cdots + s_{n-2} + 2s_{n-1}$, and that of each of $x_2, \ldots, x_{n-1}$ is $s_1 + \cdots + s_n$, then either $s_1 + \cdots + s_{n-2} + 2s_n < s_1 + \cdots + s_{n-2} + 2s_{n-1}$ or $s_1 + \cdots + s_{n-2} + 2s_n < s_1 + \cdots + s_n$. In either case $s_{n-1} > s_n$.

3. The first voter interchanges x_k and x_{k+1}, successively for $k = 2, \ldots, n-2$, and the opinions of the other voters do not change:

Voter 1	Voter 2	$\cdots$	Voter $n-1$	Voter n
x_1	x_2	$\cdots$	x_{n-1}	x_n
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{k+1}	x_{k+1}	$\cdots$	x_{k-2}	x_{k-1}
x_k	x_{k+2}	$\cdots$	x_{k-1}	x_k
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	x_1	$\cdots$	x_{n-2}	x_{n-1}

According to **IMA**, all the alternatives remain winners. In particular, x_k and x_{k+1} should have the same score. Then,

$$(s_1 + \cdots + s_n) - s_k + s_{k+1} = (s_1 + \cdots + s_n) + s_k - s_{k+1}.$$

Consequently, $s_k = s_{k+1}$ for $k = 2, \ldots, n-2$, i.e., $s_2 = s_3 = \cdots = s_{n-1}$.

Thus, we have $s_1 > s_2 = s_3 = \cdots = s_{n-1} > s_n$. This extended[18] scoring rule, $(s_1, s_2, \ldots, s_n)$, is equivalent to the standard $(s_1 - s_2, 0, \ldots, 0, s_n - s_2)$ which defines the 1–best 1–worst voting system $[\alpha\,;\delta]$, where $\alpha = s_1 - s_2$ and $\delta = s_2 - s_n$. ∎

Proposition 1. **TSM**, **BSM** and **IMA** are independent.

PROOF.

1. Plurality satisfies **TSM** and **IMA**, but not **BSM**.
2. Antiplurality satisfies **BSM** and **IMA**, but not **TSM**.
3. The classic Borda rule satisfies **TSM** and **BSM**, but not **IMA**. ∎

Now we present a characterization of the basic 1–best 1–worst voting system $[1\,;1]$ by means of one additional property, in addition to Young's. This additional condition is related to Young's [35] cancellation condition.

TBC *Top Bottom Cancellation*: In any situation where each alternative considered the best by one voter is cancelled by the same alternative considered the worst by another voter, all the alternatives win.

Theorem 2. An extended scoring rule satisfies **TBC** if and only if it is the basic 1–best 1–worst voting system $[1\,;1]$.

PROOF. Obviously, $[1\,;1]$ satisfies **TBC**.

For sufficiency, notice that if $n = 2$, just the definition of scoring rule entails $s_1 > s_2$, and again in this case, plurality, antiplurality and basic 1–best 1–worst voting system coincide.

Now, if $n \geq 3$, consider the following situation:

Voter 1	Voter 2
x_1	x_n
x_2	x_2
$\cdots$	$\cdots$
x_{n-1}	x_{n-1}
x_n	x_1

[18] Notice that no sign conditions on the scores are obtained from the imposed properties.

The total score of each of x_1 and x_n is $s_1 + s_n$, and that of x_i, $i = 2, \ldots, n-1$, is $2s_i$. By **TBC** all the alternatives win, and so all of them have the same total score: $s_1 + s_n = 2s_2 = \cdots = 2s_{n-1}$.

This extended scoring rule, $(s_1, \frac{s_1+s_n}{2}, \ldots, \frac{s_1+s_n}{2}, s_n)$, is equivalent to $(s_1 - s_n, \frac{s_1-s_n}{2}, \ldots, \frac{s_1-s_n}{2}, 0)$, and to $(2, 1, \ldots, 1, 0)$, and to $(1, 0, \ldots, 0, -1)$, which is just $[1\,;1]$. ■

5 Discussion

The general characterization that we have given for the 1–best 1–worst voting system is related to that of plurality (respectively, antiplurality) by bottom invariance (respectively, top invariance) plus the standard conditions (see Barbera and Dutta [4] and Merlin [30]). However, in Merlin [30] "top" means above the winner, whereas our "top" (i.e., "best") always means a winner in the first position of the voters' preference (the same applies, in a symmetric manner, for "bottom"). In fact, our top and bottom conditions are somehow connected to May's [29] *positive responsiveness*, and our **IMA** should be more properly understood as an invariance condition in the sense of Merlin [30].

We have assumed that voters can easily select both their "best" and their "worst" option in a consistent fashion. However, one might ask whether this is the case. This is a legitimate question, which is partially answered, in a positive manner, by the success of the best-worst method in discrete choice experiments (see Section 1). Also, to the extent that voters can respond reliably in elections that decide the winner by the number of best votes (plurality) and those that decide by the number of worst votes (antiplurality), one might expect that they can vote reliably in elections that use 1–best and 1–worst voting. Nonetheless, if each voter may have a partial order, rather than a rank order, over the options, then it may be preferable not to ask a voter to select the best, worst, or best and worst candidate(s), as several candidates may be tied – either as best or as worst – in the voter's partial order. In such a case, approval-disapproval voting seems appropriate, and an interesting open problem (as far as we know) is to characterize that voting method.

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