Congestive Public Capital in an OLG Model*

Emma Moreno-García
Universidad de Salamanca.
e-mail: emmam@usal.es

Ramón J. Torregrosa-Montaner
Universidad de Salamanca.
e-mail: rtorregr@usal.es

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Abstract. We consider an overlapping generation model where a constant returns to scale technology uses both private and public inputs. We analyze the effect that changes in the public capital have in the private capital when we start from a stable steady state. The main results support the intuition that such effect depends crucially on the elasticities of the marginal product of both factors. We also present some particular examples which exhibit special economic interest.

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1 Introduction

The idea that public capital stock, as for example roads, airports, bridges, hospitals and streets, plays an important role in production of private commodities seems to be agreed by consensus among economists since Aschauer (1989) showed the positive effect that public capital stock exerts in productivity of private capital.

One of the key points of both the empirical and theoretical treatment of the issue of public capital as an input is the way in which this enters in production function. On the one hand, when constant returns to scale affects only private inputs, zero-profits are reached under competitive conditions. In such a case the public input works as pure or factor-augmenting public input, the so-called "atmospheric" public input in words of Meade (1952). On the other hand, when production function shows constant returns to scale in all inputs, including the public one, economic profits arise under competitive equilibrium as long as production function shows decreasing returns to scale in private inputs. This feature causes a rent-dissipation phenomenon which drives the economy to a congestion in the use of that inputs up to when economic profits disappear.

It goes without saying that, as was pointed out by Stiglitz (1988), a large part of public capital stock is subject to congestion, thus the last case constitutes the most interesting one. In this trend there have been different ways in which the problem of public capital stock affected by congestion has been modeled in macroeconomics setups. For example, Uzawa (1988) models the congestion of infrastructure as an externality which is internalized by the policy maker by means of a service charged on the users of this infrastructure; and Glom and Ravikumar (1994), using a Ramsey-Cass-Koopmans growth model, characterize the congestion by adjusting the stock of public capital to the aggregated use of private factors in production function. In this trend, the purpose of this paper is to study the effects of public capital stock in a overlapping generation economy when congestion takes place. For this purpose we follow the approach made by Feehan and Batina (2007) to characterize the congestion due to public capital stock. This approach characterizes congestion by means the rent dissipation caused by the returns of public capital stock. These returns lead up the firms to hire up private factors until their average products equal their prices with the consequence of an excess amount of private factors hired. Our modelization allows us to share the rents of public capital stock between the remain private factors (labor and private capital stock) and, in addition, to study the effects of changes in the amount of public capital stock in the amount of private capital stock in the steady state, and how these changes depends on how the dissipated rent is shared between the private factors.

The paper is structured as follows: section 2 presents the model and the concept of equilibrium. Section 3 develops our main result. Section 4 analyzes two examples. Finally, section 5 is devoted to final remarks.

2 The Model

Let us consider an overlapping-generations economy where agents live for two periods. During the first period, when individuals are young, they work and in the second period, when they are old, they are retired from the labor force. An agent born at period t is endowed with one unit of labor that is supplied inelastically. Consumers can save during youth in order to consume when they are old.

In the economy there is one private good that agents value. Actually, the preferences of an agent born at time t are represented by an utility function $U(C_t^y, C_{t+1}^o)$, where C_t^y is the consumption of a young agent at time t and C_{t+1}^o is the consumption of an old agent at time $t+1$. We assume that the utility function is increasing in both C_t^y and C_{t+1}^o and concave. Then, given a wage w_t and a rental rate r_{t+1}, the individual problem of a consumer born at t is given by:

$$\begin{aligned} Max \quad & U(C_t^y, C_{t+1}^o) \\ s.t. \quad & C_t^y + S_t = w_t - T_t \\ & C_{t+1}^o = (1 + r_{t+1})S_t, \end{aligned}$$

where S_t denotes savings and T_t is a lump-sum tax. Let $C_t^y(w_t, r_{t+1}, T_t)$ and $C_{t+1}^o(w_t, r_{t+1}, T_t)$ denote the consumption demands when young and old, respectively. We assume that preferences ensure that the corresponding income and substitution effects lead to the following properties of demands: $0 < \frac{\partial C_t^y}{\partial w_t} < 1$ and $\frac{\partial C_{t+1}^o}{\partial r_{t+1}} = 0$. For instance, the utility function $U(C_t^y, C_{t+1}^o) = \log C_t^y + \beta \log C_{t+1}^o$, with $0 < \beta < 1$, verifies our requirements.

The private good is produced by a technology that uses two private factors, capital and labor, denoted by K and L respectively; and a public input, namely, public capital, denoted by G. Private capital is supplied by the old and labor supplied by the young. The technology displays constant returns to scale and is represented by a production function F which is homogeneous of degree one in all inputs.

We assume that the population grows at a constant rate n, that is, there are $L_t = (1+n)L_{t-1}$ consumers at every date t, with L_0 given. Thus, $k_t = K_t/L_t$ units of capital per capita joint with the unit of labor per capita and with an amount $g_t = \frac{G_t}{L_t}$ of public input per capita produce $f(k_t, g_t) = F(K_t, L_t, G_t)/L_t = F(k_t, 1, g_t)$ units of the private good per capita.

Regarding this technology with constant returns to scale we state the following assumptions:

(A.1) $f(0, g_t) = f(k_t, 0) = 0$,

(A.2) $f_k = \frac{\partial f}{\partial k_t} > 0$ and $f_g = \frac{\partial f}{\partial g_t} > 0$,

(A.3) $f_{kk} = \frac{\partial f^2}{\partial k_t^2} < 0$ and $f_{gg} = \frac{\partial f^2}{\partial g_t^2} < 0$; and

(A.4) $f_{kg} = \frac{\partial f^2}{\partial k_t \partial g_t} > 0$ and $f_k - k f_{kg} > 0$

Assumption (A.1) states that both private and public capital are essential in production, i.e., output is zero if either input is zero. Assumption (A.2) means that the marginal product of both private and public capital are positive whereas (A.3) ensures that the marginal productivity is decreasing for private capital and public capital. Finally, assumption (A.4) implies that $0 < \eta_k^{f_g} < 1$, where $\eta_k^{f_g}$ denote the elasticity of f_g with respect to k.

In order to provide the public capital, the agents in the economy devise a "government" that finances the public input by means of lump-sum taxes at every date t. A government policy is thus an infinite sequence of taxes T_t and levels of public investment G_t such that $T_t L_t = G_t$.

Let us consider an exogenous parameter $\gamma \in (0,1)$ that specifies the share of the contribution to output from the public capital that goes to labor income. Then $1-\gamma$ is the share of the contribution to output from the public capital that goes to private capital income. Thus, following Feehan and Batina (2007), at the null-profit equilibrium, the next conditions hold:

$$r_t = f_k(k_t, g_t) + \gamma f_g(k_t, g_t)\frac{g_t}{k_t} \quad \text{and}$$

$$w_t = f(k_t, g_t) - f_k(k_t, g_t)k_t - \gamma f_g(k_t, g_t)g_t.$$

Note that the equalities above imply that $w_t + r_t k_t = f(k_t, g_t)$. This characterization of rent dissipation differs from those of Uzawa (1988) and Glom and Ravikumar (1994) and implies that the quantity of public capital stock is available to private industries without charge or rationing, that is, as a common property resource.

Definition 2.1 *Given $\gamma \in (0,1)$, an initial private capital level K_0 and an initial public capital level G_0 a competitive equilibrium is defined as a sequence of allocation $\{C_t^y, C_{t+1}^o, K_t, G_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$ and lump-sum taxes $\{T_t\}_{t=0}^{\infty}$ such that:*

(i) Given the factor prices and the taxes, the allocation solves the maximization problem of each consumer;

(ii) given the allocation and taking into account γ, the factor prices are consistent with the firms' profit maximization,

(iii) the market for the consumption commodity clears at every date t; and

(iv) $S_t L_t = K_{t+1}$ and $T_t L_t = G_{t+1}$.

We now have a complete description of the economy. Observe that the accumulation expression for capital provided by the notion of equilibrium we address can be written in per-capita terms as follows:

$$w_t - c_t^y(w_t, r_{t+1}, T_t) - (1+n)g_{t+1} = (1+n)k_{t+1}.$$

3 Main Result

Our aim now is to analyze how the public capital investment affects the private capital regarding the steady state.

For this, given a function $\mathcal{F}$ depending on x, let $\eta_x^{\mathcal{F}}$ denote the elasticity of $\mathcal{F}$ with respect to x, i.e., $\eta_x^{\mathcal{F}}(\cdot) = \frac{\partial \mathcal{F}(\cdot)}{\partial x} \frac{x}{\mathcal{F}(\cdot)}$.

Theorem 3.1 *If $\gamma \left(1 + \eta_g^{f_g}\right) > 1 - \eta_k^{f_g}$ and $\eta_k^{f_k} + \gamma \eta_g^{f_k} < 0$, then an increase of the public capital investment depresses the private capital stock whenever the starting point is a dynamically efficient steady state.*

The previous result provides conditions which suffice to ensure that if our starting point is a dynamically efficient steady state, then an increase in public capital investment results in a decrease of private capital stock. Note that, since $\eta_g^{f_g} < 0$, $\eta_k^{f_g} > 0$ and $\gamma \in (0,1)$, we have that $\frac{1-\eta_k^{f_g}}{1+\eta_g^{f_g}} = \frac{1-\eta_k^{f_g}}{1-|\eta_g^{f_g}|} < 1$ implies $|\ \eta_g^{f_g}\ | < \eta_k^{f_g}$. In other words, these sufficient conditions support the intuition that the following property of the marginal product of public capital must be implicitly required: f_g is more sensible to changes in private capital than in public investment, that is, in percentage terms, the decrease of the public capital marginal product induced by an increase of such public input is less than the increase of this marginal product caused by an increase of the private capital. Moreover, the second requirement is also not surprising, provided that it guarantees that the sensibility of the marginal product of private capital regarding itself is greater than the elasticity of such a marginal product with respect the public capital taking into account the parameter γ.

4 Some Examples

Now we consider two particular scenarios which have special economic interest. First, we analyze the model for a technology with constant elasticity of of substitution (CES technology) and then we address the particular case of a Cobb-Douglas production function. In both situations, we consider preferences relation represented by the utility function:

$$U(C_t^y, C_{t+1}^o) = \log C_t^y + \beta \log C_{t+1}^o, \text{ with } 0 < \beta < 1.$$

As we have already remarked this utility satisfies the requirements on demands stated in Section 2. Indeed, this preference relation leads to the following consumption demand when young

$$c_t^y(w_t, T_t) = (1+\beta)^{-1}(w_t - T_t).$$

4.1 CES Technology

Let us consider the production function:

$$F(K_t, L_t, G_t) = (aK_t^\rho + bL_t^\rho + cG_t^\rho)^{1/\rho}$$

with $0 \neq \rho < 1$ and $a + b + c = 1$.

In per-capita terms: $f(k_t, g_t) = F(k_t, 1, g_t) = (ak_t^\rho + b + cg_t^\rho)^{1/\rho}$.

Proposition 4.1 *If $\rho < 0$, then an increase of the public capital investment depresses the private capital stock whenever the starting point is a dynamically efficient steady state.*

4.2 Cobb-Douglas Technology

Let us consider the next technology:

$$F(K_t, L_t, G_t) = K_t^a L_t^b G_t^c$$

with , a, b, c strictly positive and $a + b + c = 1$.

In per-capita terms: $f(k_t, g_t) = F(k_t, 1, g_t) = k_t^a g_t^c$.

Note that $\dfrac{ck}{ag} = \dfrac{f_g(k,g)}{f_k(k,g)}$.

Proposition 4.2 *Let us consider a stable steady state as a starting point. Then, the following statements hold:*

(i) If $g = k$ then $\dfrac{dk}{dg} < 0$ if and only if $\beta a > (1+\beta)c$.

(ii) If $\beta a g > (1+\beta)ck$ then $\dfrac{dk}{dg} < 0$.

5 Final Remarks

As we have seen this paper concerns with the issue of how the public capital stock financed by means of taxes affects in the long run the amount of private capital stock. For that purpose we have considered a overlapping generations model where public capital stock enters in production function in such way that causes congestion in demand of private factors. This setting allow us to point out the conditions for which an increase in the amount of public capital stock declines the amount of private capital stock in the stable steady state. These conditions are, on the one hand, that an increase in the amount of private capital stock prompts higher increase in the marginal product of public capital that the decrease that would cause an increase of public capital stock on this magnitude. On the other hand, the share of private capital stock in the rents dissipated by public capital stock has to be high enough.

Appendix

In this Appendix, we present the proof or our main results. For it, recall that the first order conditions that characterizes the equilibrium in our model are given by:

$$
\begin{aligned}
r_t = &\quad f_k(k_t, g_t) + \gamma f_g(k_t, g_t)\tfrac{g_t}{k_t}, \\
w_t = &\quad f(k_t, g_t) - f_k(k_t, g_t)k_t - \gamma(k_t, g_t)g_t
\end{aligned}
$$

Therefore, the partial derivatives of r_t and w_t with respect to k_t and g_t, respectively, are given by:

$$
\begin{aligned}
\tfrac{\partial r_t(\cdot)}{\partial k_t} = &\quad f_{kk}(\cdot) + \gamma f_{gk}(\cdot)\tfrac{g_t}{k_t} - \gamma f_g(\cdot)\tfrac{g_t}{k_t^2} < 0, \\
\tfrac{\partial r_t(\cdot)}{\partial g_t} = &\quad f_{kg}(\cdot) + \gamma f_{gg}(\cdot)\tfrac{g_t}{k_t} + \tfrac{\gamma}{k_t} f_g(\cdot) + \left(f_{kk}(\cdot) + \gamma f_{gk}(\cdot)\tfrac{g_t}{k_t} - \gamma f_g(\cdot)\tfrac{g_t}{k_t^2}\right)\tfrac{dk_t}{dg_t}, \\
\tfrac{\partial w_t(\cdot)}{\partial k_t} = &\quad -f_{kk}(\cdot)k_t - \gamma f_{gk}(\cdot)g_t, \text{ and finally} \\
\tfrac{\partial w_t(\cdot)}{\partial g_t} = &\quad (1-\gamma)f_g(\cdot) - f_{gk}(\cdot)k_t - \gamma f_{gg}g_t - \left(f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t\right)\tfrac{dk_t}{dg_t}.
\end{aligned}
$$

Furthermore, at equilibrium, we have $T_t = (1+n)g_t$ and $S_t = (1+n)k_{t+1}$. Then, we obtain the next equality

$$
w_t - c_t^y(w_t, r_{t+1}, T_t) - (1+n)g_{t+1} = (1+n)k_{t+1}.
$$

By calculating the derivative of the above expression with respect to k_t we can deduce:

$$
\frac{dk_{t+1}}{dk_t} = -\frac{\left(1 - \frac{\partial c_t^y(\cdot)}{\partial w_t}\right)\left(f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t\right)}{1+n}
$$

Let $\mathcal{N}$ denote the numerator so that we can write $\frac{dk_{t+1}}{dk_t} = -\frac{\mathcal{N}}{1+n}$. The stability conditions require $\left|\frac{dk_{t+1}}{dk_t}\right| < 1$.

Proof of Theorem 3.1. Let us consider a stable steady state equilibrium $\{w, r, k, g, T\}$. Then, we have $w - c^y(w, r, T) - (1+n)g = (1+n)k$. Taking the derivative with respect the public capital g we obtain:

$$
\frac{dk}{dg} = (\mathcal{N} + 1 + n)^{-1}\left[\left(1 - \frac{\partial c^y(\cdot)}{\partial w}\right)\left(f_g(\cdot) - f_{kg}(\cdot)k - \gamma\left(f_g(\cdot) + f_{gg}(\cdot)g\right)\right)\right]
$$

Let us state the following notation:

$$\mathcal{A} = \quad (f_g(\cdot) - f_{kg}(\cdot)k - \gamma\,(f_g(\cdot) + f_{gg}(\cdot)g))$$

$$\mathcal{B} = \quad f_{kk}(\cdot)k_t + \gamma f_{gk}(\cdot)g_t$$

Note that $\frac{dk_{t+1}}{dk_t} = -\frac{\mathcal{N}}{1+n}$ and $\mathcal{N} = \left(1 - \frac{\partial c_t^y(\cdot)}{\partial w_t}\right)\mathcal{B}$. Note also that if $\mathcal{N} < 0$ then $\frac{dk_{t+1}}{dk_t} > 0$ and, in this case, the stability condition allows us to conclude that $\mathcal{N} + 1 + n > 0$. Therefore, it remains to show that both $\mathcal{A}$ and $\mathcal{B}$ are negative. For it, note that we can write $\mathcal{A}$ and $\mathcal{B}$ in terms of elasticities of f_g and f_k, respectively, as follows:

$$\mathcal{A} = f_g\left(1 - \eta_k^{f_g} - \gamma\left(1 + \eta_g^{f_g}\right)\right) \quad \text{and}$$
$$\mathcal{B} = f_g\left(\eta_k^{f_k} + \gamma\eta_g^{f_k}\right).$$

Now, since $\gamma\left(1 + \eta_g^{f_g}\right) > 1 - \eta_k^{f_g}$ and $\eta_k^{f_k} + \gamma\eta_g^{f_k} < 0$, it is inmediate to conclude that $\mathcal{A} < 0$ and $\mathcal{B} < 0$, which implies $\frac{dk}{dg} < 0$.

Q.E.D.

Proof of Proposition 4.1. With the CES technology, we have

$$r_t = f_k(k_t, g_t) + \gamma f_g(k_t, g_t)\tfrac{g_t}{k_t} = (ak_t^\rho + b + cg_t^\rho)^{(1-\rho)/\rho}\left(ak_t^{\rho-1} + b + \gamma c\tfrac{g_t^\rho}{k_t}\right)$$
$$w_t = f(k_t, g_t) - f_k(k_t, g_t)k_t - \gamma f_g(k_t, g_t)g_t = (ak_t^\rho + b + cg_t^\rho)^{(1-\rho)/\rho}\left(b + c(1-\gamma)g_t^\rho\right).$$

Then, the equilibrium is characterized by the equation:

$$\frac{\beta}{1+\beta}\left(ak_t^\rho + b + cg_t^\rho\right)^{(1-\rho)/\rho}\left(b + c(1-\gamma)g_t^\rho\right) - (1+n)g_{t+1} + \frac{T_t}{1+\beta} = (1+n)k_{t+1}.$$

Then, $\dfrac{dk_{t+1}}{dk_t} = \frac{\beta}{(1+\beta)(1+n)}\left(ak_t^\rho + b + cg_t^\rho\right)^{(1-2\rho)/\rho}\left(b + c(1-\gamma)g_t^\rho\right)(1-\rho)ak_t^{\rho-1}$.

Since $\dfrac{dk_{t+1}}{dk_t} > 0$, the stability condition implies that

$$D(\cdot) = (1+n) - \frac{\beta}{(1+\beta)}\left(ak_t^\rho + b + cg_t^\rho\right)^{(1-2\rho)/\rho}\left(b + c(1-\gamma)g_t^\rho\right)(1-\rho)ak_t^{\rho-1} > 0.$$

Considering a stable steady state, we have:

$$\frac{dk}{dg} = \frac{1}{D}\left(\frac{\beta}{1+\beta}H^{(1-\rho)/\rho}c(1-\gamma)\rho g^{\rho-1} - \frac{\beta(1+n)}{1+\beta}\right)$$

where $D = (1+n) - \dfrac{\beta}{(1+\beta)} H^{(1-2\rho)/\rho} \left(b + c(1-\gamma)g^{\rho}\right)(1-\rho)ak^{\rho-1} > 0$ and $H = ak^{\rho} + b + cg^{\rho}$.

Therefore, we conclude that if $\rho < 0$ then $\dfrac{dk}{dg} < 0$.

Q.E.D.

Proof of Proposition 4.2. For the Cobb-Douglas technology, we obtain $r_t = (a + \gamma c)\, k_t^{a-1} g_t^c$ and $w_t == (1 - a - \gamma c)\, k_t^a g_t^c$.

The equation which characterizes the equilibrium is

$$\frac{\beta}{1+\beta}(1-a-\gamma c)\, k_t^a g_t^c - (1+n)g_{t+1} + \frac{T_t}{1+\beta} = (1+n)k_{t+1}.$$

Then, $\dfrac{dk_{t+1}}{dk_t} = \frac{\beta}{(1+\beta)(1+n)}(1-a-\gamma c)\, ak_t^{a-1}g_t^c$.

Since $\gamma \in (0,1)$ and $a+b+c=1$, we have $\dfrac{dk_{t+1}}{dk_t} > 0$; and then the stability condition implies that

$$D(\cdot) = (1+n) - \frac{\beta}{1+\beta}(1-a-\gamma c)\, ak_t^{a-1}g_t^c > 0.$$

Considering a stable steady state:

$$\frac{dk}{dg} = \frac{1}{D}\left(\frac{\beta\,(1-a-\gamma c)}{1+\beta}k^a c g^{c-1} - \frac{\beta(1+n)}{1+\beta}\right)$$

where $D = (1+n) - \dfrac{\beta\,(1-a-\gamma c)}{1+\beta} ak^{a-1}g^c$

Therefore, it is immediate now to conclude that both statements *(i)* and *(ii)* hold.

Q.E.D.

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