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AN APPROXIMATE ORTHOGONAL DECOMPOSITION METHOD FOR THE SOLUTION OF THE GENERALIZED LIOUVILLE EQUATION

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Abstract. We consider an approximate integration method of the Cauchy problem for the generalized Liouville equation using symbolic and numeric computer computations. This method is based on the probability density function orthonormal series expansion in the small and initial time space domains. We are investigating several expansions and determine their convergence conditions to ensure the convergence of the asymptotic expansion to the solution of the considered problem.

To illustrate the applicability of the introduced asymptotic orthogonal decompositions [18] we took the describing bidimensional integrable dispersive shallow water equation developed by Roberto Camassa and Darryl D. Holm, Los Alamos National Laboratory. Since CH-equation solutions are represented by a superposition of arbitrary number of peakons (peaked solitons) [9],[16], one can compare the coincidence of the "peakon" solutions character provided by numerical modeling along some trajectories for truncated asymptotic series expansions obtained by symbolic computations.

Key Words and Phrases. Liouville equation, orthonormal system, eigenfunction, strong and weak convergence, mean convergence, Camassa-Holm equation, Hermite functions.

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RESUMEN. Consideramos un método de integración aproximada del problema de Cauchy para la ecuación generalizada de Liouville utilizando cálculos simbólicos y numéricos en computador. Este método se basa en la expansión en serie ortonormal de la función de densidad de probablilidad en dominios iniciales pequeños de espacio-tiempo. Se investigan diversas expansiones y se determinan sus condiciones de convergencia con el fin de asegurar la convergencia de las expansiones asintóticas a la solución del problema considerado.

Para ilustrar la aplicabilidad de las descomposiciones asintóticas ortogonales [18] se toma la ecuación integral bidimensional que describe la dispersión en agua panda descrita por Roberto Camassa y Darryl D. Holm, Los Alamos National Laboratory. Puesto que las soluciones de la ecuación CH se representan por la superposición de un número arbitrario de picones (solitones pico) [9],[16], se puede comparar el carácter de la coincidencia de las soluciones "picón", que resultan del modelo numérico a lo largo de algunas trayectorias de las expansiones asintóticas truncadas en serie obtenidas por cálculos simbólicos.

PALABRAS CLAVES. Ecuación de Liouville, sistema ortonormal, función propia, convergencia fuerte y débil, convergencia principal, Ecuación de Camassa-Holm, funciones de Hermite.

1. INTRODUCTION

At the beginning we have to outline some aspects, concerning classic Liouville theorem (Liouville 1838 [35]) and equation for the ODE system

$$\dot{x} = X(x,t),\ x(t_0) = x^0 \in \mathbb{R}^n \tag{1}$$

where $x \in \mathbb{R}^n$; $J = \{t : t_0 \le t < +\infty\}$; $X_i(x,t) \in C_{xt}^{(1,1)}(G)$; $G = \Omega \times J$; $\Omega \subseteq \mathbb{R}^n$ — a bounded domain. If the named conditions are fulfilled, then at arbitrary moment t_0 a unique solution $x(t) = x(x^0, t_0, t)$ of the Cauchy problem (1) passes through each point $x^0 \in \mathbb{R}^n$.

It is well known (see Nemitskij, Stepanov 1949 [41], Kaplan 1953 [31]), that ODE system (1) has a corresponding Liouville equation

$$\frac{\partial}{\partial t} f(x,t) = Lf(x,t),\ f(x,t_0) = f_0(x)\ . \tag{2}$$

Here

$$L\cdot = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[X_i(x,t)\cdot] = -\mathrm{div}[X(x,t)\cdot] \tag{3}$$

is a Liouville operator. Assume $f(x,t) \in L_2(\mathbb{R})$, $t \in J$ and suppose L acting like

$$L : C_0^\infty(\mathbb{R}^n) \to L_2(\mathbb{R}^n) \ . \tag{4}$$

Here function $f_0(x)$ is defined below:

$$f_0(x) \geq 0,\ f_0(x) \in C_0^\infty(\mathbb{R}^n),\ \int_{\mathbb{R}^n} f_0(x)dx = 1 \ . \tag{5}$$

- $\chi(x,t) = \mathrm{div} X(x,t)$ — divergence of vector field for ODE (1);
- $D(x(x^0,t_0,t),t) = \det\left[\frac{\partial x(x^0,t_0,t)}{\partial x^0}\right]$ — Jacobian of inverse transition $x^0 \to x(x(x^0,t_0,t);$
- $S(x,t) = \det\left[\frac{\partial x^0(x,t,t_0)}{\partial x}\right]$ — Jacobian of transition $x(x^0,t_0,t) \to x^0$;
- $\chi(x(x^0,t_0,t),t)$ — divergence of vector field for ODE (1) calculated along the solution trajectory $x(t) = x(x^0,t_0,t)$.

As usual we treat the Gibbs ensemble of represented points (Gibbs 1946 [22]) for the system of equations (1) as a set of the identical systems (1) with the same equation right parts, but having different initial states.

Let $\Omega_{t_0} \subset \Omega$ be a compact Lebesgue mes Ω_{t_0} measure set filled by a Gibbs ensemble of represented points for the ODE system (1) at the moment $t = t_0$. Each of the represented points $x^0 \in \Omega_{t_0}$, moving along with the ODE (1) trajectories, is shifted to a new state $x(x^0,t_0,t) = T(t,t_0)x^0 \in \Omega_t \subset \Omega$ starting from moment t_0 to t. Here $T(t,t_0)$ is a shift operator along the ODE system (1) trajectories (Krasnoselskij 1966 [32]); $\Omega_t = \{x(x^0,t_0,t) = T(t,t_0)x^0 : x^0 \in \Omega_{t_0}\}$ — an image of the set Ω_{t_0} according to ODE system (1). It means that $\Omega_t = T(t,t_0)\Omega_{t_0}$. Let mes Ω_t be a Lebesgue measure of the set $\Omega_t \subset \Omega^n$.

Function $f_0(x)$ which complies the conditions (5) could be considered as a distribution density function for the Gibbs ensemble of the represented points from Ω_{t_0}, system (1). The current value of distribution density function $f(x,t) \in L_2(\mathbb{R}^n)$, $t \in J$, is defined by the initial conditions of the Cauchy problem (2), (3). It describes the state of the Gibbs ensemble of the represented points for the ODE system (1) in the image Ω_t of the set Ω_{t_0}.

To indroduce a uniqueness and existence result we need to introduce the following assumption:

ASSUMPTION (**A**) *holds for the ODE system (1), if the solution $x(t) = x(x^0,t_0,t)$ is nonlocally continuable (Krasnoselskij 1966* [32]*) on J for all represented points $x^0 \subset \Omega_{t_0}$ and is kept in Ω, $\forall t \geq t_0$.*

Definition 1.1. *We call a function $f(x,t) \in L_2(\mathbb{R}^n)$ a* classical solution *of the Cauchy problem (2) with operator (3) acting according to (4), if the substitution of this function $f(x,t)$ into the Liouville equation (2) turns it into identity.*

Then there holds a theorem [50].

Theorem 1.1. First. *Let assumption (A) holds for the ODE system (1).*
Second. *The corresponding ensemble of Gibbs represented points has an initial distribution density function $f_0(x)$ satisfying conditions (5) in the compact set $\Omega_{t_0} \subset \Omega$.*
Third. *Let $\Omega_t = \{x(x^0, t_0, t) = T(t, t_0)x^0 : \ x^0 \in \Omega_{t_0}\}$ be an image of the set Ω_{t_0} defined according to system (1) and $D(x(x^0, t_0, t), t) \neq 0$.*
Hence the shift operator $T(t, t_0)$ along the solution trajectories (1) defines a set homeomorphism $\Omega_{t_0} \subset \Omega$ onto the set $\Omega_t \subset \Omega$; there exists the unique solution of the Cauchy problem (2)–(4) $\forall t \in J$ and it complies with conditions

$$f(x,t) \geq 0,\ f(x,t) \in \mathrm{C}_0^\infty(\mathbb{R}^n),\ \int\limits_{\mathbb{R}^n} f(x,t)\mathrm{d}x = 1 \ , \tag{6}$$

$$f(x(x^0,t_0,t),t) = f_0(x^0) e^{\left[-\int\limits_{t_0}^{t} \chi(x(x^0,t_0,t),t)\mathrm{d}t\right]} = f_0(x^0)/D(x(x^0,t^0,t),t) \ , \tag{7}$$

$$f(x,t) = f_0(p(x,t,t_0)) e^{\left[-\int\limits_{t_0}^{t} \chi(x(p(x,t,t_0),t_0,\tau),\tau)\mathrm{d}\tau\right]} = f_0(p(x,t,t_0))S(x,t) \ . \tag{8}$$

If we denote L — Liouville operator (3); $p(x,t,t_0) = T^{-1}(t,t_0)x = x^0$, then there hold the relations introduced below:

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln D(x(x^0,t_0,t),t) = \chi(x(x^0,t_0,t),t),\ D(x(x^0,t_0,t),t)\big|_{t=t_0} = 1 \ , \tag{9}$$

$$\frac{\partial S(x,t)}{\partial t} = LS(x,t),\ \ S(x,t)|_{t=t_0} = 1 \ , \tag{10}$$

$$\text{mes } \Omega_t = \int\limits_{\Omega_{t_0}} e^{\left[\int\limits_{t_0}^{t} \chi(x(x^0,t_0,t),t)\mathrm{d}t\right]} \mathrm{d}x^0 \ , \tag{11}$$

$$\text{mes } \Omega_t = \int\limits_{t_0}^{t} \int\limits_{\Omega_t} \chi(x,\tau)\, \mathrm{d}x\mathrm{d}\tau + \text{mes } \Omega_{t_0} \ . \tag{12}$$

The above theorem 1.1 has some other interpretations. For example, Nemitskij, Stepanov 1949 [41]; Zubov 1982 [60] treated the function $\rho(x,t)$ satisfying Liouville equation (2),(3) as a kernel or the density of the integral invariant. Resolving n equations $x = x(x^0, t_0, t)$ with respect to n initial conditions x^0, we have

$$x^0 = T^{-1}(t,t_0) \equiv p(x,t,t_0) \ . \tag{13}$$

Here the functions $p(x, t, t_0)$ are the first n independent integrals of the ODE system (1). The transformation mentioned above could be made since a transition involved by the shift operator $T(t, t_0)$ is homeomorphic and conditions on the implicit function are hold.

Theorem 1.2 (Zubov 1982, [60]). *Assume*

1 *Let the solution* $x = x(x^0, t_0, t)$ *of the system (1) exists for* $t \in (-\infty, +\infty)$, $t_0 \in (-\infty, +\infty)$, $x^0 \in \mathbb{R}^n$*;*
2 *Let the vector function (13) exists for* $t \in (-\infty, +\infty)$, $t_0 \in (-\infty, +\infty)$, $x \in \mathbb{R}^n$,

then each nonnegative function $\rho_0(x) \neq 0$ *on* $x \in \mathbb{R}^n$ *is continuously differentiable by all arguments and posses a unique nonnegative solution* $\rho(x, t)$ *of the equation*

$$\frac{\partial \rho(x,t)}{\partial t} + \bigtriangledown \cdot [X(x,t)\rho(x,t)] = 0 \ ,$$

such that $\rho(x, t) = \rho_0(x)$ *at* $t = t_0$*. The function* $\rho(x, t)$ *is a kernel of integral invariant for the system (1) also.*

Liouville theorem and equation play a great role (Hinchin 1943 [27]; Krylov 1950 [33]; Bogolubov [7]; Prigozhin 1964 [47]) in statistical mechanics. Providing a statistical justification of the principles and by discovering the structure of the multibody and/or multiprocess systems tending to the equilibrium state. Invariant measure Liouville theorem (Arnold, Kozlov, Neĭshtadt 1985 [5]) is the basis for qualitative studying methods of the n–body problem (Hilmi 1951 [26]). Liouville equation was taken as an initial point for the ergodicity theory (Cornfeld, Fomin, Sinai 1980 [14]); for the kinetic theory of irreversible processes; in derivation of Vlasov-Maxwell integro-differential equations (Vlasov 1950 [58]). Exactly speaking, Vlasov equation [58] could be derived from Liouville equation for the charged particles distribution function, neglecting particle correlations and supposing many-particle distribution function as a direct product of proper single-particle distribution functions. An application of Liouville theorem for studying VM equation also could be found in (Maslov, Fedoryuk 1985 [38]; Lewis, Barnes, Melendez 1987 [34]; Horst 1990 [29]) and some recent papers. An infinite dimensional formal Hamiltonian approach for the infinite dimensional VM system was developed by (Morrison 1980 [40]; Marsden, Weinstein 1982 [37]). The mentioned papers introduce Poisson bracket evaluation technique for the VM system and prove that it is an infinite dimensional Hamiltonian system, i.e. could be written as Liouville equation.

It seems that Bogolubov was the first mathematician who introduced the "classical" representation of the Liouville equation

$$\frac{\partial}{\partial t}f(q,p,t) = [H(q,p,t), f(q,p,t)],\ f(q,p,t_0) = f_0(q,p) \tag{14}$$

to describe the probabilistic properties of the canonical Hamilton equation system

$$\dot{q}_i = \frac{\partial}{\partial p_i}H(q,p,t),\ \dot{p}_i = -\frac{\partial}{\partial q_i}H(q,p,t),\ q_i(t_0) = q_i^0,\ \ p_i(t_0) = p_i^0 \tag{15}$$

with an arbitrary initial states distributed in $\mathbb{R}^{2n}$ phase space.
Here $q, p \in \mathbb{R}^n$ — are the generalized coordinate and generalized conjugate impulse vectors correspondingly; $t \in \mathbb{R} = (-\infty, +\infty)$; $H(q,p,t) : G \to \mathbb{R}$, $G \subset \mathbb{R}^{2n+1}$ — Hamilton function from C^2 with respect to coordinates q, p;

$$[H, f] = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$$

is the Poisson bracket; $f_0(q,p)$ and $f(q,p,t)$ are the Gibbs ensemble of represented points [22] for the system (15) in $\mathbb{R}^{2n}$. This functions satisfy the probability conditions

$$f_0(q,p) \geq 0,\ \int_{\mathbb{R}^{2n}} f_0(q,p)\mathrm{d}q\mathrm{d}p = 1,\ \int_{\mathbb{R}^{2n}} f(q,p,t)\mathrm{d}q\mathrm{d}p = 1.$$

The "classical" equation (14) is a particular case of the so called "generalized" Liouville equation (16) which deals with compressible or dissipative dynamics. This equation is discussed further. One of the most representative modern digests concerning generalized Liouville equations was made by Ezra [19]. An interested reader also could find there a wide list of basic and recent bibliographic references.
At present Liouville theorem and generalized Liouville equation are widely used for proving existence theorems (Povzner [43]), optimal control synthesis for beam trajectories (Ovsiannikov [42], [17]); stability researches (Fronteau [20]; Rudykh [49]; Zhukov [59]); dynamic properties analysis (Misra [39]; Steeb [52]; Fronteau [21]; Rudykh [50]); qualitative investigation of dynamic systems (Cornfeld, Fomin, Sinai [14]); discovering the stochastic behavior of dynamic systems (Sinai [51]) and molecular dynamics with applications to chemistry (Tuckerman, Martyna [55]. Look for some other developments of Tuckerman's group in section 3).
Due to the problem importance the construction of iterative analytical integration methods becomes our primary goal. In section 2 we provide a complete problem statement and introduce some remarks about analytical iterative solutions. In section 3 we introduce some of the latest results to be discussed and compared later in the text. We need them for numerical modeling of bidimentional Camassa-Holm equation. Section 4 contains new results concerning the

other classical approach for evaluating asymptotical orthonormal decompositions — eigenvalue/eigenvector operator decomposition. Section 5 is a reminder for the results published in [18] necessary for the numerical modeling presented in the next section.
As an application example we chose a lately discovered Camassa–Holm equation of the shallow water [9], [16],[28] to study its dynamics in section 6 and drawing its solution in section 8.

2. Problem Statement

In this paper we consider a method of approximate integration for Cauchy problem of the generalized Liouville equation [45],[8]

$$\frac{\partial}{\partial t} f(q,p,t) = Lf(q,p,t), \;\; f(q,p,t)|_{t=0} = f_0(q,p) \tag{16}$$

corresponding to the autonomous system of quasicanonical Hamilton equations

$$\dot{q}_i = \frac{\partial}{\partial p_i} H(q,p), \;\; \dot{p}_i = -\frac{\partial}{\partial q_i} H(q,p) + Q_i^*(q,p), \tag{17}$$
$$q_i(t)|_{t=0} = q_i^0, \quad p_i(t)|_{t=0} = p_i^0 \; .$$

An additive inclusion of the nonpotential term $Q^*(q,p)$ allows to construct proper measure for existence theorem more easily. Definitely speaking,

$$L\cdot = [H(q,p), \cdot] - \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \{Q_i^*(q,p)\cdot\} \tag{18}$$

is a generalized Liouville operator, $f(q,p,t) \in L_2(\mathbb{R}^{2n})$. Here the $\dfrac{\partial}{\partial p_i}\{Q_i^*(q,p)\cdot\}$ term corresponds to the divergent criterium of the solution stability–instability issue.
Assume the transition rule as given below:

$$L\cdot \; = \; \mathfrak{D}(L) = C_0^{\infty}(\mathbb{R}^{2n}) \to R(L) = L_2(\mathbb{R}^{2n}) \; . \tag{19}$$

$q, p \in \mathbb{R}^n$ are a vector of generalized coordinates and generalized conjugate impulse vector; $H(q,p) \in C_{qp}^{(2,2)}(\mathbb{R}^{2n})$ — Hamiltonian of the system; $Q_i^*(q,p) \in C_{qp}^{(1,1)}(\mathbb{R}^{2n})$ nonpotential generalized forces; $\chi(q,p) \triangleq \sum_{i=1}^{n} \frac{\partial}{\partial p_i} Q_i^*(q,p)$ divergence of vector field for the system (17); $[\cdot,\cdot]$ is Poisson bracket; $f_0(q,p)$, $f(q,p,t)$ are the initial and current values of probability density function for the Gibbs ensemble of represented points in the system of equations (17) in $\mathbb{R}^{2n}$;

$$\int_{\mathbb{R}^{2n}} f_0(q,p)\mathrm{d}q\mathrm{d}p = 1 \;, t \in \mathbb{R}^+ \triangleq \{t : 0 \le t \le \infty\} \; .$$

The equation (16) and the corresponding operator (18), (19) are widely studied. Research papers [45],[46] seem to be the most earlier ones. In the semigroup

theory the Cauchy problem (16) with operator (18) acting as $C_0^\infty(\Omega) \to L_2(\Omega)$, $\Omega \subset \mathbb{R}^{2n}$ was studied in papers [48]. In particular, it was proved an existence and uniqueness theorem for the problem (16). But this theorem can not provide an effective algorithm to the analytical problem solution. Such development is of a great practical importance for the complex multidimensional generalized Liouville equation (16) and its solution $f(q,p,t)$.
To obtain a solution one can integrate (16) numerically [10], but it is hardly acceptable, since $f(q,p,t)$ completely describes the system properties depending on q, p, t. Thus it is more valuable to obtain such a relationship in analytical form. Moreover, a numerical integration takes a lot of computational time even for small dimensions n and advanced integration methods as quasi-Monte Carlo on the low discrepancy lattices. Just note that taking $n = 2,\ 3$ turns into $d = 5,\ 7$ for q-MC over some restricted coordinate and potentials power domain. Any domain change "voids" previously made calculations.

Remark 2.1. *The other great problem for q-MC approach concerns the lattice choice. Being studied with attention, nearly all articles available became low reliable ones, since no one payed the necessary attention to provide their numerical modeling results electing the appropriate low discrepancy lattice.*

On the other hand, since the probability density function $f(q,p,t)$ completely describes a system (17), we need it to describe the time dependence of the mean and dispersion for generalized coordinate and impulse vectors q, p also. Using this prior statistical knowledge we can try to reduce the dimension of the problem, keeping the statistical properties unchanged, applying specific group renormalization method. Or to replace the initial problem with an equivalent one (also of smaller dimension). The term "equal" should be defined very carefully for each problem type.
Nevertheless, this idea could be applied more effectively if we want to study the local means, for example. This method, i.e. modeling the mean properties of the dynamic systems, combined with advanced Monte-Carlo numerical integration techniques - proved to be very useful for the group of Berkley scientists lead by Chorin (see articles [11]–[13] for example). Called "Stochastic optimal prediction" in general, it is compatible with Hamiltonian formalism and becomes very useful for a preliminary research.
Therefore we are interested in evaluating $f(q,p,t)$ as analytical converging series whose coefficients could be simply evaluated and the convergence properties could be analytically studied. If the orthonormal function system $\{h_k(q,p)\}_{k=0}^{\infty} \in$

$L_2(\mathbb{R}^{2n})$ is a priory known or already constructed, the solution of the initial problem (16) is translated into the solution of the infinite system of differential equations to determine the d_k coefficient values in the expansion $f(q,p,t) = \sum_{k=0}^{\infty} d_k(t)h_k(q,p)$.

On the other hand, we can find the solution $f(q,p,t)$ of the problem (16) on the basis of iterative operator method in the small time space over the a priori constructed system of orthonormal functions $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$:

$$f(q,p,\tau) = \sum_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p) \ .$$

A proposed approach makes it possible to bypass the solution of the infinite system of differential equations and settle some convergence propositions. The method introduced below could be treated as a combination of the mentioned approaches for equation (16).

3. The overview of preceding results

Here we state some earlier results for problem (16). Only some of them were published before internationally. Since the approaches used there are rather simple, we do not focus on their details.

3.1. **An approach of M.E. Tuckerman group on the classical mechanics of non-Hamiltonian systems.** Here we will outline some basic results obtained by Professor M.E. Tuckerman and his work group [56, 57]) on the generalized Liouville equation and non-Hamiltonian dynamical systems.

It is known that Hamiltonian flow preserves the measure of phase space treated as a Euclidean manifold. It differs for non-Hamiltonian flow, since measure is not preserved in general case. Thus we could state another question: “there exists an *invariant measure* keeping phase volume unchanged for non-Hamiltonian flows?” A partial answer could be found in the discussed articles. It is shown that such key concepts of Hamiltonian systems such as “invariant measure” and “continuity” can be generalized to the non-Hamiltonian case by a proper treatment of the geometry of the phase space and that an invariant measure on the phase space manifold can be derived. Thus we introduce a general Riemannian manifold to derive the generalized Liouville equation for non-Hamiltonian systems of the type

$$\frac{\partial}{\partial t}(f\sqrt{g}) + \nabla \cdot (f\sqrt{g}\dot{x}) = 0 \ , \tag{20}$$

with

$$\dot{x}^i = \xi^i(x,t) \tag{21}$$

a non-Hamiltonian dynamical system for the evolution of the n coordinates $x = x^1, \ldots, x^n$ with initial values $x_0^1, \ldots, x_0^n$. The n coordinates describe a point P of an n-dimensional Riemannian manifold $\mathbb{G}$ with metric G. The phase space must be treated as a general Riemannian manifold with arbitrary curvature, and the volume n-form, which determines the volume element in an arbitrary coordinate system, should be expressed as $\tilde{w} = \sqrt{g}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$. The general statement of Liouville's theorem for non-Hamiltonian system becomes

$$f(x_t, t)e^{-w(x_t)}\mathrm{d}x_t^1 \ldots \mathrm{d}x_t^n = f(x_0, 0)e^{-w(x_0)}\mathrm{d}x_0^1 \ldots \mathrm{d}x_0^n \tag{22}$$

with $\sqrt{g} = \exp[-w(x)]$.
Right now we are able to establish the correspondence between (20)–(22) and the problem stated in this paper. Liouville equation (3) for the ODE system (2) coincides with (21) for $\sqrt{g} = 1$, and the generalized Liouville equation (16) for non-Hamiltonian system (17) is a partial case.
Hence the Liouville theorem for non-Hamiltonian dynamics allows us to find more than just an invariant measure construction. We are able to derive the initial distribution function (22) f_0 used for Liouville operator Lf_0. All that we need — is a Liouville operator Lf_0.
While discussing this possibilities one should have note invariant measure construction for the generalized Liouville equation written for a real non-Hamiltonian system usually appears to be very difficult. Hence we take an initial distribution function first without paying attention on the existence of such invariant measure. Second — we evaluate the Poisson bracket or Liouville operator to study the dynamics of the initial Hamiltonian or non-Hamiltonian system.
Moreover, the Tukermann's approach is not concerned with the solution of the Liouville equation. One just uses the generalized law on the conservation of the phase volume. It allows to construct the invariant measure without the proper solution of the Liouville equation. Differently speaking, we "spread" some statistical ensembles up to the non-Hamiltonian systems using only the conservation laws.
Such an approach used for derivation of the (21) is based on the differential geometry and has been well known for ages. The new term is its application for the nonequilibrium dissipative thermodynamics.

3.2. **Small–time parameter method.** Considering the formal asymptotic solution of Cauchy problem (16) by the method of the small time space, proposed in [44] and defined by the mapping $\tau = 1 - e^{-st}$, $s > 0$ we transform the initial infinite time interval $\mathbb{R}^+$ in a small, finite one $J \triangleq \{\tau : 0 \leq \tau < 1\}$. This technique is quite general and was applied to Liouville equation by G. Rudykh and A. Sinitsyn at the earlier 80's of the past century. This results were not translated into English earlier but are still valuable.
The small–time transformation is also well known to the applied mathematicians, because it suits for numerical integration over the semi-infinite intervals.

The obvious benefit for a such kind of transform lies in power series expansion over time scale. Using the finite interval $[0,\ 1]$ one can pay the attention to the series coefficients and their bounds.
In our particular case a transformed Cauchy problem becomes

$$(1-\tau)\frac{\partial}{\partial \tau} f(q,p,\tau) = \frac{1}{s} L f(q,p,\tau),\ \ f(q,p,\tau)|_{\tau=0} = f_0(q,p)\ . \tag{23}$$

A solution of the Cauchy problem (23) in the small time space was studied in the form of asymptotic expansion

$$f(q,p,\tau) = \sum_{k=0}^{\infty} f_k(q,p)\cdot \tau^k\ . \tag{24}$$

Substituting (24) into (23) and equating the coefficients of the same τ orders, one can obtain

$$f_k(q,p) = \frac{k-1}{k} f_{k-1}(q,p) + \frac{1}{sk}[H(q,p), f_{k-1}(q,p)] - \frac{1}{sk}\sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^*(q,p)\cdot f_{k-1}(q,p)\}\ .$$

Hence

$$f_k(q,p) = \frac{1}{sk} L \left[\prod_{r=1}^{k-1}\left(1+\frac{1}{sr}L\right)\right] f_0(q,p),\ \ k=2,3,\ldots \tag{25}$$

with

$$\prod_{r=1}^{k-1}(1+\frac{1}{sr}L) = a_{k-1} + \frac{1}{s}a_{k-2}L + \ldots + \frac{1}{s^{k-1}}a_0 L^{k-1}$$

where $a_0 = 1/(k-1)!$, $a_{k-1} = 1$, and

$$a_i = \frac{(-1)^i}{i!}a_0 \cdot \begin{vmatrix} s_1 & 1 & 0 & 0 & \ldots & 0 \\ s_2 & s_1 & 2 & 0 & \ldots & 0 \\ s_3 & s_2 & s_1 & 3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ s_i & s_{i-1} & s_{i-2} & s_{i-3} & \ldots & s_1 \end{vmatrix},\ \ i = 1,2,\ldots,k-1,$$

$$s_i = \sum_{l=1}^{k-1} z_l^i,\ \ z_l = -l.$$

So, the expression (25) can be written in brief form

$$f_k(q,p) = b_{1k} L f_0(q,p) + b_{2k} L^2 f_0(q,p) + \ldots + b_{kk} L^k f_0(q,p)\ . \tag{26}$$

Here $b_{jk} = \dfrac{a_{k-j}}{ks^j} > 0$, $\sum_{j=1}^{k} b_{jk} s^j = 1$ for $k = 1,2,\ldots$.
The recurrent relation (26) expresses $f_k(q,p)$ in terms of initial probability density function $f_0(q,p)$. Consequent evaluation of the $f_k(q,p)$ terms by equation (26) with further backward substitution into (24) gives a formal asymptotic solution of the Cauchy problem (23). Using the definition of a small time

parameter method, one can derive the solution $f(q,p,t)$ of the initial Cauchy problem (16).

To study the convergence properties of the developed asymptotic solution, we need one assumption to be made. Let there exists such constant $c>0$, that $\forall k=1,2,\dots$ there holds the following inequality

$$\|L^k f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} \le c^k \cdot \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})}\ . \tag{27}$$

Honestly speaking, this condition is rather rigid and will be discussed further. Nevertheless such a formal assumptions are typical in theory and applied math. The classical example is a Kantorovitch's lemma on the local convergence of Newton method. Everyone knows that there exists a set of conditions to guarantee the convergence, but it is hard to verify them sometimes. Another remark concerns the "width" of this local domain. It could happen that an initial point selected upon this conditions could be treated as a perfect approximation in hardware computations.

Here and in what follows we denote $L^k f_0(q,p) = L(L^{k-1}f_0(q,p))$ — an embedded Liouville operator.

Remark 3.1. *Since the generalized Liouville operator is unbounded, its embedded degree L^k has a restricted definition domain and in general, more restricted than a domain of the initial operator L.*

Proposition 3.1. *Let for the generalized Liouville operator L defined by formula (18) acts according to (19). Inequality (27) and condition $c\tau/s<1$ hold. Then the series $\sum_{k=0}^{\infty} f_k(q,p)\tau^k$ is weakly convergent to the element $f(q,p,\tau)\in L_2(\mathbb{R}^{2n})$ — a solution of the Cauchy problem (23).*

Besides, it holds an estimate

$$\left\|\sum_{k=0}^{\infty} f_k(q,p)\tau^k\right\|_{L_2(\mathbb{R}^{2n})} \le \frac{s}{(1-\tau)(s-c\tau)}\cdot\|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})}. \tag{28}$$

Proposition 3.2. *Let the generalized Liouville operator (18) acts according to (19). The inequality (27) and condition $c/s<1$ are satisfied. Then the series $\sum_{k=0}^{\infty} f_k(q,p)\tau^k$ strongly converges to the element $f(q,p,\tau)\in L_2(\mathbb{R}^{2n})$ — the solution of Cauchy problem (23) and the relation*

$$(f_k,f_n)_{L_2(\mathbb{R}^{2n})} \le \frac{c^2}{(s-c)^2}\cdot\|f_0\|_{L_2(\mathbb{R}^{2n})} = B$$

holds.

Remark 3.2. *If the inequality $c/s < 1$ holds, then the inequality $c\tau < 1$ holds also. Differently speaking, if series $\sum_{k=0}^{\infty} f_k(q,p)\tau^k$ converges strongly to the element $f(q,p,\tau) \in L_2(\mathbb{R}^{2n})$, then it is weakly convergent also.*

Let the orthonormalized function system $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$ be constructed from the linear independent elements of the sequence $\{f_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$ by the Gram-Schmidt orthogonalization process [30]. Below we provide a strong convergence conditions of a series $\sum_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p)$, where

$$a_k(\tau) = \left(\sum_{i=k}^{\infty} f_i \tau^i,\ \Psi_k \right)_{L_2(\mathbb{R}^{2n})} .$$

Proposition 3.3. *Let for generalized Liouville equation (18) acts according to (19). The inequality (27) and condition $c\tau/s < 1$ are fulfilled. Then the series $\sum\limits_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p)$ based upon the an orthonormal system of functions $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$ is strong convergent to the element $f(q,p,\tau) \in L_2(\mathbb{R}^{2n})$ — the solution of the Cauchy problem (23), and*

$$\left\| \sum_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p) \right\|_{L_2(\mathbb{R}^{2n})} \le \left\| \sum_{k=0}^{\infty} f_k(q,p)\tau^k \right\|_{L_2(\mathbb{R}^{2n})} \le$$
$$\le \frac{s}{(1-\tau)(s-c\tau)} \cdot \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} ,$$
$$\sum_{k=0}^{\infty} |a_k(\tau)|^2 \le \frac{s^2}{(1-\tau)^2(s-c\tau)^2} \cdot \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})}^2 .$$

Since $\sum\limits_{k=0}^{\infty} |a_k(\tau)|^2 < +\infty$ and $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$, then according to Riesz-Fisher theorem [25], [3] there exits a unique function $f(q,p,\tau) \in L_2(\mathbb{R}^{2n})$ such that

$$a_k(\tau) = (f, \Psi_k)_{L_2(\mathbb{R}^{2n})},\ \sum_{k=0}^{\infty} |a_k(\tau)|^2 = \|f\|_{L_2(\mathbb{R}^{2n})}^2 .$$

It is well known that $a_k(\tau)$ and $\sum_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p)$ are exactly the coefficients and Fourier series of the function $f(q,p,\tau) \in L_2(\mathbb{R}^{2n})$. On the other hand, the Gram matrix

$$\Phi = \{\Psi_{ij}\} \stackrel{\triangle}{=} (\Psi_i, \Psi_j)_{L_2(\mathbb{R}^{2n})},\ i,j = 0,1,2,\ldots$$

of the sequence $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$ is bounded and positive definite. Then $\sum\limits_{k=0}^{\infty} |a_k(\tau)|^2 < +\infty$, and according to [54] the series $\sum_{k=0}^{\infty} a_k(\tau)\Psi_k(q,p)$ strongly converges to the function $f(q,p,\tau) \in L_2(\mathbb{R}^{2n})$.

The constructed above orthonormal system of functions $\{\Psi_k(q,p)\}_{k=0}^{\infty} \in L_2(\mathbb{R}^{2n})$ is not complete in $L_2(\mathbb{R}^{2n})$. However it is known [30], that every incomplete orthonormal system of functions can be extended to a complete one by associating a number of proper functions. In practice to realize such an extension is fairly complicated. Since the Gram matrix $\Phi = \{\Psi_{ij}\}$ is bounded and positive definite, the countable set $N = \{\Psi_k(q,p)\}_{k=0}^{\infty}$ is an orthonormal basis of the subset $[\bar{\Psi}_k]$ [54], i.e. every element from $[\bar{\Psi}_k]$ is expanded in a unique strong convergent series. Here $[\bar{\Psi}_k]$ is the closure of the linear hull $[\Psi_k]$.

This approach, to be said — recurrent relation (23),(24) also could be used for symbolic processing. But as it will be shown later, there exists an other decomposition with a simpler expression. Moreover, the numerical modeling for the small–time transformation could be done for the known parameter s only. This limitation occurs due to the convergence conditions stated in the above propositions 3.1 and 3.1 where the boundness constant c defines the definition interval for s.

3.3. **Hermite polynomial decomposition.** Since the exact choice of the decomposition basis system of functions could be quite arbitrary, we revised the applicability of the Hermite time–space polynomial decomposition to the problem (16). Hence the basic problem statement could be written as

$$\sum_{k=0}^{\infty} f_k(q,p)H_k'(t) = \sum_{k=0}^{\infty} Lf_k(q,p)H_k(t) \tag{29}$$

with respect to the definition and the properties of the Poisson bracket.

Denote $H_k(x)$ — Hermite polynomial defined by recursive expression

$$H_{k+1}(x) = 2xH_k(x) - 2kH_{k-1}(x)\ ,\ H_0 = 1,\ H_1 = 2x\ ;$$
$$\int_{-\infty}^{+\infty} e^{-x^2} H_k^2(x)\mathrm{d}x = \sqrt{\pi}2^k k!\ .$$

Proposition 3.4. *A solution of the problem (16) in series (29) based on orthogonal Hermite polynomial is*

$$f(q,p,t) = \sum_{k=0}^{\infty} f_k(q,p)H_k(t) \tag{30}$$

$$f_k(q,p) = \frac{1}{2k}Lf_{k-1}(q,p) = \frac{1}{2^k k!}L^k f_0(q,p)\ . \tag{31}$$

Here $L^k f = L(L^{k-1}f)$ is an embedded Liouville operator.

Since Hermite polynomials are orthogonal on $\mathbb{R}$ we can deal with their orthonormal analogs

$$h_k(t) = \frac{H_k(t)}{2^{\frac{k}{2}}\pi^{\frac{1}{4}}\sqrt{k!}}\ ,\ \int_{\mathbb{R}} e^{-t^2} h_k^2(t)\mathrm{d}t = 1\ .$$

The corresponding problem solution (30),(31) becomes

$$f(q,p,t) = \sum_{k=0}^{\infty} \hat{f}_k(q,p) h_k(t)\ . \tag{32}$$

$\hat{f}_k(q,p)$ decomposition coefficients linearly depends on $f_k(q,p)$:

$$\hat{f}_k(q,p) = \frac{1}{\sqrt{2^k k!}} L^k f_0(q,p)\ . \tag{33}$$

To discuss the convergence properties of the (32) decomposition, we need to remind some basic definitions, see [6],[30],[15] and [53] for example.

Definition 3.1. *Let $f(x) \in L_2(V)$. Then the norm in $L_2(V)$ is defined as*

$$\|f\| = \sqrt{(f,f)} = \left[\int_V \omega(x)|f(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}$$

with respect to weight function $\omega(x) > 0$ such that $\int_V |\omega(x)|^2 \mathrm{d}x < \infty$.

Definition 3.2. *Let $f(x), h(x) \in L_2$ be functions and $\omega(x) > 0$ — be real function defined in a domain V. Then*

$$d(f,h) = \|f-h\| = \left[\int_V \omega(x)|f(x)-h(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}$$

is called the distance between functions f and h.

Definition 3.3. *Assume that relation*

$$d^2(s_n, s) = \int_V \omega(x)|f(x)-h(x)|^2 \mathrm{d}x \to 0,\ n \to \infty\ .$$

is fulfilled for the functions $s_0(x), s_1(x), \ldots \in L_2(V)$. Then the sequence $\{s_n\}_{n=0}^{\infty}$ converges to the function $s(x) \in L_2(V)$ in mean.

First of all we have to note that Hermite polynomials do not belong to $L_2(\mathbb{R})$. Nevertheless, assuming $V = \mathbb{R}^{2n} \times \mathbb{R}$, we will be able to obtain some useful convergence estimations in $L_2(\hat{V})$.

Theorem 3.1. *Let the boundness condition (27)*

$$\|L^k f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} \le c^k \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})},\ c > 0$$

holds $\forall k = 1,2,\ldots$. *Then* $\sum_{k=0}^{\infty} \|\hat{f}_k(q,p)\|^2_{L_2(\mathbb{R}^{2n+1})} < \infty$ *and the series* $\sum_{k=0}^{\infty} \hat{f}_k(q,p)h_k(t)$ *converges in mean with respect to weighting function* e^{-t^2} *to the solution* $f(q,p,t) \in L_2(\mathbb{R}^{2n+1})$ *of the problem (16) and*

$$\|s_n(q,p,t) - f(q,p,t)\|_{L_2(\mathbb{R}^{2n+1})} \leq \frac{1}{\sqrt{2^n}} \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} e^{\frac{c^2}{2}} .$$

3.4. **Partial conclusions.** The advantages of the used polynomial decompositions are the relative simplicity of coefficient recurrence relations and their convergence properties both in the infinite initial and finite transformed solution time domains. Assuming the Liouville operator L to be bounded (27), we can choose a proper small time space transformation parameter to guarantee strong or/and weak asymptotic series convergence to the solution of the problem (23) in $\mathbb{R}^{2n}$.

On the other hand, we have a fast growing computational complexity of evaluating embedded Liouville operator even for two and three dimensional generalized vectors q, p. Hence all this techniques could be used mainly for qualitative solution analysis. Nevertheless, once we have some analytic truncated series approximation, we are free to choose any kind of trajectories and impulse vectors dependencies for numerical modeling and/or visualization.

4. Eigen expansion of generalized Liouville operator

The other classical approach to solve Cauchy problem (23) assumes that the expansion of the solution $f(q,p,\tau)$ over the system of eigenfunctions $\{g_k(q,p)\}_{k=0}^{\infty}$ to be found. Here we need the generalized Liouville operator (18) defined on the test functions $C_0^{\infty}(\mathbb{R}^{2n})$. Thus, translating them into the complex Hilbert space $L_2(\mathbb{R}^{2n})$ we are able to fulfill our task. Let $\{g_k(q,p)\}_{k=0}^{\infty}$ be an orthonormal sequence of eigenfunctions of operator (18), complete in $R(L) \subset L_2(\mathbb{R}^{2n})$; $\{\lambda_k\}_{k=0}^{\infty}$ is the corresponding sequence of eigenvalues;

$$\int_{\mathbb{R}^{2n}} g_k(q,p)\bar{g}_n(q,p)\mathrm{d}q\mathrm{d}p = \delta_{kn}$$
$$Lg_k(q,p) = \lambda_k g_k(q,p) \ , \ g_k(q,p) \in D(L) = C_0^{\infty}(\mathbb{R}^{2n}) \ .$$

We will seek the solution of the Cauchy problem (23) in the form of asymptotic expansion

$$f(q,p,\tau) = \sum_{k=0}^{\infty} v_k(\tau)g_k(q,p), \ v_k(\tau) = \int_{\mathbb{R}^{2n}} f(q,p,\tau)g_k(q,p)\mathrm{d}q\mathrm{d}p \ . \tag{34}$$

Let $\tau = 0$. Then (34) becomes

$$f(q,p,0) = f_0(q,p) = \sum_{k=0}^{\infty} v_k(0) g_k(q,p) \ .$$

To find the functions $v_k(\tau)$ we pose the Cauchy problem

$$(1-\tau)\frac{\partial}{\partial \tau}\{v_k(\tau)g_k(q,p)\} = \frac{1}{s}L\{v_k(\tau)g_k(q,p)\} \ ,$$
$$v_k(\tau)|_{\tau=0} = v_k(0) \stackrel{\triangle}{=} v_k^0$$

whose solution has the form

$$v_k(\tau) = v_k^0 \cdot (1-\tau)^{-\lambda_k/s} \ . \tag{35}$$

Formula (35) could be derived directly from (34). In fact, differentiating (34), we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}v_k(\tau) = \frac{1}{s(1-\tau)} \cdot (Lf(q,p,\tau), g_k(q,p))_{L_2(\mathbb{R}^{2n})}.$$

Since there holds an identity

$$(Lf(q,p,\tau), g_k(q,p))_{L_2(\mathbb{R}^{2n})} = v_k(\tau) \cdot (Lg_k(q,p), g_k(q,p))_{L_2(\mathbb{R}^{2n})},$$

then the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}v_k(\tau) = \frac{\lambda_k}{s(1-\tau)} \cdot v_k(\tau)$$

is fulfilled. Taking $v_k(\tau)|_{\tau=0} = v_k^0$, it immediately follows (35). Function $f(q,p,\tau)$ defined by a series (34) with coefficients defined by expression (35) is the solution of the Cauchy problem (23).

A series $\sum_{k=0}^{\infty}(f, g_k)_{L_2(\mathbb{R}^{2n})} \cdot g_k$ will be convergent independently of the terms order. Moreover, the sum is equal to $f(q,p,\tau)$ for $f(q,p,\tau) \in [\bar{\mathfrak{L}}]$, where $\mathfrak{L} = \{g_k(q,p)\}_{k=0}^{\infty}$.

Therefore a series (34) can be differentiated termwise

$$\frac{\partial}{\partial \tau}f(q,p,\tau) = \sum_{k=0}^{\infty} \dot{v}_k(\tau) g_k(q,p), \ \ \dot{v}_k(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}v_k(\tau) \ .$$

Multiplying both sides of previous equality by $(1-\tau)$ and taking into account that

$$(1-\tau)g_k(q,p) \cdot \dot{v}_k(\tau) = \frac{1}{s}v_k(\tau)Lg_k(q,p)$$

we obtain

$$(1-\tau)\cdot\frac{\partial}{\partial\tau}f(q,p,\tau)=\frac{1}{s}\sum_{k=0}^{\infty}v_k(\tau)Lg_k(q,p)=$$

$$\frac{1}{s}L\left\{\sum_{k=0}^{\infty}v_k(\tau)g_k(q,p)\right\}=\frac{1}{s}Lf(q,p,\tau).$$

Here $f(q,p,\tau)|_{\tau=0}=f_0(q,p)$. Thus we derived the formula

$$f(q,p,\tau)=\sum_{k=0}^{\infty}v_k^0\cdot(1-\tau)^{-\lambda_k/s}\cdot g_k(q,p) \tag{36}$$

representing the solution expansion of the generalized Liouville equation with respect to the complete orthonormal system of eigenfunctions $\{g_k(q,p)\}_{k=0}^{\infty}$ of operator L in $R(L)$.

Making the backward time scale substitution we transform (36) into

$$f(q,p,t)=\sum_{k=0}^{\infty}v_k^0\cdot e^{\lambda_k t}\cdot g_k(q,p)\ .$$

Now we have to pay attention for evaluating the eigenfunctions $g_k(q,p)$ and the eigenvalues λ_k of the operator L. To this end we take an element $g_i(q,p)$ and consider the expansion with respect to the orthonormal function system

$$g_i(q,p)=r_{0i}\psi_0(q,p)+r_{1i}\psi_1(q,p)+\ldots+r_{ij}\psi_j(q,p)+\ldots \tag{37}$$

where $r_{ji}=(g_i,\psi_j)_{L_2(\mathbb{R}^{2n})}$. The equality (37) holds for $g_i(q,p)\in[\bar{\psi_k}]$. Since the series on the right of (37) is convergent independently of the terms order, there holds an equality

$$Lg_i(q,p)=r_{0i}L\psi_0(q,p)+r_{1i}L\psi_1(q,p)+\ldots+r_{ji}L\psi_j(q,p)+\ldots$$

or

$$\lambda_i g_i(q,p)=r_{0i}L\psi_0(q,p)+r_{1i}L\psi_1(q,p)+\ldots+r_{ji}L\psi_j(q,p)+\ldots\ . \tag{38}$$

Denote

$$x_{ik}=(g_i,\psi_k)_{L_2(\mathbb{R}^{2n})},\ a_{ik}=(L\psi_i,\psi_k)_{L_2(\mathbb{R}^{2n})}\ . \tag{39}$$

Scalar product of (38) and function $\psi_k(q,p)$ could be written

$$\lambda_i x_{ik}=r_{0i}a_{0k}+r_{1i}a_{1k}+\ldots+r_{ji}a_{jk}+\ldots$$

or

$$\lambda_i x_{ik}=x_{i0}a_{0k}+x_{i1}a_{1k}+\ldots+x_{ij}a_{jk}+\ldots$$

in notations (39). The last relation could be expressed in the standard matrix form

$$\lambda_i X_i=AX_i\ . \tag{40}$$

Thus the expansion coefficients $r_{ji} \triangleq (g_i, \psi_j)_{L_2(\mathbb{R}^{2n})} = x_{ij}$ of the function $g_i(q,p)$ in a series (37) are set by a linear algebraic problem on eigenvalues (40) for infinite matrix A.

Taking some given number of elements in truncated series (34) the problem (40) could be solved for certain finite matrix A. Here λ_i and $g_i(q,p)$ are complex in general, matrix A — non symmetrical one. But under certain suppositions an operator iL, $i^2 = -1$ will be symmetrical and even selfadjoint. In this particular case λ_i are real and $g_i(q,p)$ are imaginary. Assuming this, one can derive an efficient algorithm for solving (40) for a coefficients $a_{jk} \triangleq (L\psi_j, \psi_k)_{L_2(\mathbb{R}^{2n})} = a_{kj}$. Using natural notations for the Gram-Shmidt orthonormalization process we get

$$\begin{aligned} \psi_i(q,p) &= \alpha_0 f_0(q,p) + \alpha_1 f_1(q,p) + \ldots + \alpha_i f_i(q,p) \ , \\ f_i(q,p) &= \gamma_0 \psi_0(q,p) + \gamma_1 \psi_1(q,p) + \ldots + \gamma_i \psi_i(q,p) \ . \end{aligned}$$

Therefore the following chain of identities holds

$$\begin{aligned} L\psi_i(q,p) = \alpha_0 L f_0(q,p) + \alpha_1 f_1(q,p) + \ldots + \alpha_i L f_i(q,p) = \\ = \xi_1 f_1(q,p) + \xi_2 f_2(q,p) + \ldots + \xi_i f_i(q,p) + \xi_{i+1} f_{i+1}(q,p) = \\ = \beta_0 \psi_0(q,p) + \beta_1 \psi_1(q,p) + \ldots + \beta_i \psi_i(q,p) + \beta_{i+1} \psi_{i+1}(q,p) \ , \end{aligned}$$

and if $j > i+1$, then $(L\psi_i, \psi_j)_{L_2(\mathbb{R}^{2n})} = (\psi_i, L\psi_j)_{L_2(\mathbb{R}^{2n})} = 0$. Hence matrix A is a band matrix, or exactly speaking, it has a three diagonal form

$$A = \begin{Vmatrix} a_1 & b_1 & & & 0 & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & a_3 & b_3 & & \\ & & b_3 & a_4 & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & 0 & & & \ddots & \ddots \end{Vmatrix} . \tag{41}$$

The eigenvalue problem for the banded matrices is well studied and proper algorithms could be found in hundreds of books and research articles. The standard numerically stable approach like QR-shift method could be directly applied to the matrix A of the given structure. Fixing the number of series (36) terms we solve problem (40) for the matrix A (41) of some finite dimension. For larger dimensions – the case of special interest in practice, the computational error grows and the elements $\psi_i(q,p)$ loose their orthogonal property in numerics.

Making a short resume, the eigen decomposition technique hardly could be recommended for general practical applications especially for complicated analytical initial distribution functions $f_0(q,p)$ since they generate linear systems of complex valued nonsymmetric matrices turning the truncated solution to be a numerical one. The numerical stability characteristics are uncertain. Also we have a tripled (at least) computational requirements.

5. Hermitian function expansion

Since the standard Hermite polynomial provides only a convergence in mean (see theorem 1.1) and $H_k(t)$, $h_k(t)$ do not belong to $L_2(\mathbb{R})$, we are interested in obtaining some other expansions over a certain set of functions $\{u_k\}_{k=0}^{\infty} \in L_2(\mathbb{R})$. Such functions based on Hermite orthogonal polynomials are usually called Hermitian [53], or associated Hermite functions, see chapter 22, "Orthogonal Polynomials" [1]. Formally they are constructed as a parametric family of orthonormal functions with additional useful properties

$$u_n(t) = \sqrt{\frac{a}{\sqrt{\pi}n!2^n}} H_n(at)e^{-\frac{at^2}{2}},\ 0 < a < \infty \tag{42}$$

$$\int_{\mathbb{R}} u_n(t)u_m(t)\mathrm{d}t = \delta_{n,m}$$

$$\int_{\mathbb{R}} u_n(t)u'_m(t)\mathrm{d}t = a\sqrt{\frac{n+1}{2}}\delta_{n,n+1} - a\sqrt{\frac{n}{2}}\delta_{n,n-1}\ . \tag{43}$$

Here $\delta_{n,m}$ is the Kronecker's delta.

Hence, using the standard expansion of system (16) in time domain (29) with respect to definition (42), (43) we obtain

$$f_{n+1}(q,p) = \frac{1}{a}\sqrt{\frac{2}{n+1}} Lf_n(q,p) + \sqrt{\frac{n}{n+1}} f_{n-1}(q,p)\ ,\ n = 0, 1, \dots \tag{44}$$

$$f_{-1}(q,p) \equiv 0\ .$$

Denote $\nu = \dfrac{\sqrt{2}}{a}$ to simplify the further expressions. Then one can prove by direct computations that

$$f_{2k}(q,p) = \frac{1}{\sqrt{(2k)!}} \sum_{s=0}^{k} \lambda_s^{(2k)} \nu^{2s} L^{2s} f_0(q,p)\ ,\ k = 1, 2, \dots \tag{45}$$

$$f_{2k+1}(q,p) = \frac{1}{\sqrt{(2k+1)!}} \sum_{s=0}^{k} \lambda_s^{(2k+1)} \nu^{2s+1} L^{2s+1} f_0(q,p)\ ,\ k = 1, 2, \dots \tag{46}$$

where $\lambda_s^{(n)} \in \mathbb{N}$ is a coefficient in an embedded Liouville operator power series decomposition.

Combining formal expansions (45),(46) with recurrent coefficient formula (44) and recalling that $(2k-1)!! = (2k-1) \cdot (2k-3) \cdot \ldots \cdot 3 \cdot 1$ one can obtain detailed coefficient dependencies for even elements:

$$
\begin{array}{lll}
\lambda_k^{(2k)} & \equiv 1 \ , & k = 1, 2 \ldots \\
\lambda_0^{(2k)} & \equiv (2k-1)!! \ , & k = 1, 2 \ldots \\
\lambda_{k-1}^{(2k)} & \equiv (2k-1)k \ , & k = 2, 3 \ldots
\end{array}
\tag{47}
$$

and for the odd ones:

$$
\begin{array}{lll}
\lambda_k^{(2k+1)} & \equiv 1 \ , & k = 1, 2 \ldots \\
\lambda_0^{(2k+1)} & \equiv (2k+1)!! \ , & k = 1, 2 \ldots \\
\lambda_{k-1}^{(2k+1)} & \equiv (2k+1)k \ , & k = 2, 3 \ldots
\end{array}
\tag{48}
$$

Since $\{u_k(t)\}_{k=0}^{\infty} \in L_2(\mathbb{R})$ is orthonormal, then using partial coefficient value analysis we can revise the applicability of Hermitian function expansion in the sense of Riesz–Fischer theorem.

$$
\text{If } \sum_{k=0}^{\infty} \|f_k(q,p)\|^2_{L_2(\mathbb{R}^{2n})} < \infty \ , \text{ then } \sum_{k=0}^{\infty} f_k(p,q) u_k(t) \to f(q,p,t)
$$

to the unique function $f(q,p,t) \in L_2(\mathbb{R}^{2n+1})$.
Assume that Liouville operator boundness condition (27) holds:

$$
\|L^k f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} \le c^k \cdot \|f_0(q,p)\|_{L_2(\mathbb{R}^{2n})} \ .
$$

Then we obtain

Theorem 5.1. *Suppose Liouville operator (18) to be bounded (27) and parameter a for the associated Hermite functions (42) $a > c\sqrt{2}$. T hen $\sum_{k=0}^{\infty} \|f_k(q,p)\|^2_{L_2(\mathbb{R}^{2n})} < \infty$ and according to Riesz–Fisher theorem the series are convergent*

$$
\sum_{k=0}^{\infty} f_k(q,p) u_k(t) \to f(q,p,t) \ , \ f(q,p,t) \in L_2(\mathbb{R}^{2n+1})
$$

to a unique function.

Comparing the results stated in propositions 3.1, 3.2, 3.3 and theorems 3.1, 5.1 there arise some limitations for its analytical/numerical applications as expansion series:

(1) The formal boundness assumption (27) is sufficient and necessary condition to ensure the convergence of the expansion series for small–time space method or Hermite–based decompositions;
(2) To ensure the applicability of Riesz–Fisher theorem, one should exactly know the value of the bounding constant c. This value is needed for evaluating the correct small–time parameter s or Hermite associated function parameter a;

(3) Supposing the Liouville operator boundness constant c to be finite unknown number, we can guarantee only the convergence in mean for standard Hermite polynomial decomposition.

Nevertheless, the Hermitian function decomposition could be extremely useful for approximate evaluation of the constant c. We mean that having some analytically expressed partial decomposition sums with a formal parameter a involved, one can make a set of numerical experiments to revise if a sequence of partial sums is convergent in a model space D. The smallest value keeping the sum convergent could be assumed to be the right parameter a providing the correspondent constant c approximation.

6. Asymptotical treatment of Camassa–Holm equation

One of the newly discovered partial differential equations of great practical importance is the so called Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{49}$$

(see [9],[16],[28] for detailed explanation). This equation also could be written in general as

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \ ,$$

dealing with completely integrable dispersive shallow water equation. κ is a constant, related to the critical shallow water wave speed. Limiting $\kappa \to 0$, see (49), the equation possesses solitons solutions that have peaks where the first derivatives are discontinuous. Such solutions dominate for $\kappa = 0$. Remarkably, the multisoliton solution is obtained by simply superimposing the single peakon solutions and solving for the evolution of their amplitudes and the positions of their peaks as a completely integrable finite dimensional Hamiltonian system. So, the solution is expressed as a superposition of an arbitrary N number of peakons (or peaked solutions)

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x - q_j(t)|} \ .$$

The peaks positions $q_j(t)$ and their momentum $p_j(t)$ satisfy an associated dynamical system taking the canonical Hamiltonian form

$$\dot{p}_j = -\frac{\partial h}{\partial q_j}, \ \dot{q}_j = -\frac{\partial h}{\partial p_j}, \tag{50}$$

$$h = \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k e^{-|q_j - q_k|} \ . \tag{51}$$

As it was shown in [9], this is completely integrable finite-dimensional system, with a $n \times n$ Lax pair. Hence the study of the system dynamics is still a question of a general interest.

To discover the system (50),(51) properties in a descriptive form we take $n \equiv 2$. Then a particular form of a Hamiltonian (51) becomes

$$h = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 p_2 e^{-|q_1 - q_2|} \ . \tag{52}$$

$$\begin{array}{rlrl} \frac{\partial h}{\partial q_1} = & -\mathrm{sign}(q_1 - q_2) p_1 p_2 e^{-|q_1-q_2|} \ , & \frac{\partial h}{\partial p_1} = & p_1 + p_2 e^{-|q_1-q_2|}; \\ \frac{\partial h}{\partial q_2} = & \mathrm{sign}(q_1 - q_2) p_1 p_2 e^{-|q_1-q_2|} \ , & \frac{\partial h}{\partial p_2} = & p_2 + p_1 e^{-|q_1-q_2|}. \end{array} \tag{53}$$

To simplify the case studied first we take the Liouville operator (18) irrespective the influence of nonpotential forces. Then the operator is written as a standard Poisson bracket $Lf = [h, f]$

$$[h, f] = \left(\frac{\partial h}{\partial p_1}\frac{\partial f}{\partial q_1} - \frac{\partial h}{\partial q_1}\frac{\partial f}{\partial p_1}\right) + \left(\frac{\partial h}{\partial p_2}\frac{\partial f}{\partial q_2} - \frac{\partial h}{\partial q_2}\frac{\partial f}{\partial p_2}\right) \ ,$$

or with respect to (53)

$$\begin{aligned} Lf = {} & \mathrm{sign}(q_1 - q_2) p_1 p_2 e^{-|q_1-q_2|} \left(\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial p_2}\right) + \\ & + p_1 \left(\frac{\partial f}{\partial q_1} + \frac{\partial f}{\partial q_2} e^{-|q_1-q_2|}\right) + p_2 \left(\frac{\partial f}{\partial q_2} + \frac{\partial f}{\partial q_1} e^{-|q_1-q_2|}\right) \ . \end{aligned} \tag{54}$$

While expression (54) is easy handled by computer analytical evaluations, we introduce one basic property of the generalized Liouville operator $L\cdot$.

Lemma 6.1. *Let the generalized Liouville operator be defined as (18)*

$$L\cdot = [H(q,p), \cdot] - \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^*(q,p)\cdot\} \ .$$

Then $L\{f \cdot g\} = f \cdot Lg + g \cdot Lf$.

Proof. According to the definition of Poisson bracket $[h, f \cdot g] = f[h, g] + g[h, f]$ and

$$\begin{aligned} L\{f \cdot g\} &= [H, f \cdot g] - \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^*(f \cdot g)\} = \\ &= f[H, g] - f\sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^* g\} + g[H, f] - g\sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^* f\} = \\ &= f\left([H, g] - \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^* g\}\right) + g\left([H, f] - \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\{Q_i^* f\}\right) = fLg + gLf \ . \end{aligned}$$

□

Based on lemma (6.1) we can introduce two highly applicable in computer analytical calculations formulas, described by lemmas 6.2 and 7.1.

Lemma 6.2. *Suppose*

$$f_0 \cdot g_i = L^i f_0 = L L^{i-1} f_0 = L\{f_0 \cdot g_{i-1}\} \ .$$

Here $f_0(q,p)$ *and* $g_i(q,p)$, $i = 1,2,\ldots$, $g_0(q,p) \equiv 1$. *Hence*

$$g_i(q,p) = L g_{i-1}(q,p) + g_{i-1}(q,p) g_1(q,p) \ . \tag{55}$$

Proof. According to lemma 6.1 and this lemma assumptions

$$\begin{aligned} L\{f_0(q,p) \cdot g_{i-1}(q,p)\} &= f_0(q,p) L g_{i-1}(q,p) + g_{i-1}(q,p) L f_0(q,p) = \\ &= f_0(q,p) L g_{i-1}(q,p) + g_{i-1}(q,p) f_0(q,p) g_1(q,p) = \\ &= f_0(q,p)[L g_{i-1}(q,p) + g_{i-1}(q,p) g_1(q,p)] \end{aligned}$$

and

$$g_i(q,p) = L g_{i-1}(q,p) + g_{i-1}(q,p) g_1(q,p) \ .$$

□

Taking as an initial distribution function

$$f_0(q,p,r,d) = e^{-\frac{1}{2}\left(p_1^2 + p_2^2\right) + r p_1 p_2 + d \ln\left(e^{-|q_1 - q_2|} - 1\right)} \ , \tag{56}$$

we have

$$\begin{array}{llll} \frac{\partial f_0}{\partial q_1} = & -\mathrm{sign}(q_1 - q_2) f_0(q,p) \frac{d e^{-|q_1-q_2|}}{e^{-|q_1-q_2|} - 1} \ , & \frac{\partial f_0}{\partial p_1} = & (r p_2 - p_1) f_0(q,p); \\ \frac{\partial f_0}{\partial q_2} = & \mathrm{sign}(q_1 - q_2) f_0(q,p) \frac{d e^{-|q_1-q_2|}}{e^{-|q_1-q_2|} - 1} \ , & \frac{\partial f_0}{\partial p_2} = & (r p_1 - p_2) f_0(q,p) \ . \end{array} \tag{57}$$

Using straightforward formula (56) the definition domain is limited:

$$e^{-|q_1-q_2|} - 1 \geq 0 \ \Rightarrow \ q_1 \equiv q_2 \ .$$

Nevertheless, transforming

$$d \ln\left(e^{-|q_1-q_2|} - 1\right) \equiv \ln\left(e^{-|q_1-q_2|} - 1\right)^d$$

and supposing d to be integer or rational number $d = m/n$ given, one have

$$\begin{cases} d = 2k, \quad k = 1,2,\ldots d & \text{is integer,} \\ n = 2k, \quad k = 1,2,\ldots d & \text{is rational.} \end{cases}$$

On the other hand, initial distribution function (56) also could be written as

$$f_0(q,p,r,d) = \left(e^{-|q_1-q_2|} - 1\right)^d e^{-\frac{1}{2}\left(p_1^2 + p_2^2\right) + r p_1 p_2} \ ,$$

and this representation is defined everywhere in $\mathbb{R}^4$ for $d \geq 1$. Hence the limitation for d to be even is not necessary.

The similar or more generic distributions of such type are typical in thermodynamics or physical chemistry [36] with some constants r and $d = \pm 2k$, for example. The most common case corresponds to constant $d > 0$. Namely, since $q = (q_1, q_2)$ is a generalized coordinate vector and assuming impulse vector p to be bounded, we have

$$\lim_{q_1 \to q_2} f(q,p,r,+2k) = \lim_{q_1 \to q_2} e^{-\frac{1}{2}\left(p_1^2+p_2^2\right) + rp_1p_2 + k\ln\left(e^{-|q_1-q_2|}-1\right)^2} = 0$$

meaning that particles are not "attracted" to a certain trajectory. Taking $d < 0$ we obtain

$$\lim_{q_1 \to q_2} f(q,p,r,-2k) = +\infty$$

that could correspond to a description of stiff polymer–type molecular compounds.
Thus, keeping both parameters r, d in mind, further we just use a $f_0(q,p)$ or f_0 notations instead of complete the formula simplification $f(q,p,r,d)$.

7. Particular analysis of the bidimensional Camassa-Holm equation

Using Hamiltonian definition (52) for bidimensional Camassa–Holm equation

$$\frac{1}{2}\left(p_1^2+p_2^2\right) + p_1p_2e^{-|q_1-q_2|}$$

without nonpotential forces applied, one can see, that the $e^{-|q_1-q_2|}$ term will permanently arise in all embedded orders of Liouville operator (54). A straightforward computation with respect to (54),(57) gives

$$Lf_0(q,p) = \operatorname{sign}(q_1-q_2)f_0(q,p)e^{-|q_1-q_2|}(p_1-p_2)(p_1p_2(r+1)-d)\ . \tag{58}$$

Using (58), lemma 6.2 can be rewritten as

Lemma 7.1. *Suppose that*

$$f_0e^{-|q_1-q_2|}\cdot g_i = L^if_0 = LL^{i-1}f_0 = L\{f_0e^{-|q_1-q_2|}\cdot g_{i-1}\}\ ,$$

$f_0(q,p)$ *and* $g_i(q,p)$, $i = 2, 3, \ldots$. *The initial coefficient being*

$$g_1(q,p) = \operatorname{sign}(q_1-q_2)(p_1-p_2)(p_1p_2(r+1)-d)\ .$$

Then

$$\begin{aligned} g_i(q,p) &= Lg_{i-1}(q,p) + \operatorname{sign}(q_1-q_2)(p_1-p_2)g_{i-1}(q,p)\times \\ &\times\left[(p_1p_2(r+1)+(1-d))e^{-|q_1-q_2|}-1\right]\ . \end{aligned} \tag{59}$$

Proof. According to lemma 6.1,6.2 and this lemma assumptions

$$
\begin{aligned}
&L\{f_0(q,p)e^{-|q_1-q_2|}\cdot g_{i-1}(q,p)\} = \\
&= f_0(q,p)e^{-|q_1-q_2|}Lg_{i-1}(q,p) + g_{i-1}(q,p)L\{f_0(q,p)e^{-|q_1-q_2|}\} = \\
&= f_0(q,p)e^{-|q_1-q_2|}Lg_{i-1}(q,p) + g_{i-1}(q,p)\left[f_0(q,p)Le^{-|q_1-q_2|} + e^{-|q_1-q_2|}Lf_0(q,p)\right] = \\
&= f_0(q,p)e^{-|q_1-q_2|}\Big[Lg_{i-1}(q,p) + +\mathrm{sign}(q_1-q_2)(p_1-p_2)(1-e^{-|q_1-q_2|})g_{i-1}(q,p) \\
&+\mathrm{sign}(q_1-q_2)(p_1-p_2)(p_1p_2(r+1)-d)g_{i-1}(q,p)e^{-|q_1-q_2|}\Big] = \\
&= f_0(q,p)e^{-|q_1-q_2|}\,[Lg_{i-1}(q,p) + \mathrm{sign}(q_1-q_2)(p_1-p_2)g_{i-1}(q,p)\times \\
&\times\Big((p_1p_2(r+1)-d)e^{-|q_1-q_2|} + (1-e^{-|q_1-q_2|})\Big)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
g_i(q,p) = Lg_{i-1}(q,p) &+ \mathrm{sign}(q_1-q_2)(p_1-p_2)g_{i-1}(q,p)\times \\
&\times\left[(p_1p_2(r+1)-(d+1))e^{-|q_1-q_2|}+1\right] \ .
\end{aligned}
$$

□

Taking $n = 2$ introduces a five-dimensional model space described by a composite model vector q_1, q_2, p_1, p_2, t which can not be visualized without making additional suppositions:

- We assume the impulse vector $p = (p_1, p_2) = (z_1(s), z_2(s))$ to be parametrically defined by some functions $z_i(s)$, where s is an impulse coordinate parameter. According to physical since we can keep $z_i(s) = k_i s$, $k_i \in \mathbb{R}$.
- Coordinate vector $q = (q_1, q_2)$ is assumed to be parametrically defined in ordinary geometric sense.

For simplicity we deal only with a time t dependent periodic trajectories defined over period $t \in [0, 6\pi]$. Under this assumptions the 5-dimensional initial model is reduced to 2-dimensional model (s, t). The same approach could be applied to an arbitrary model space dimension n.
Since the motion trajectories could be either opened ($\|q\| \to \infty$ for some $t \geq 0$) or closed ($\|q\| < \infty,\ \forall t,\ t \geq 0$), we introduce three periodic trajectories.

To show the great difference of the classical Camassa-Holms equation with and without nonpotential forces involved, we consider the simplest case, when

$$
Q_i^* = q_i + \Delta p_i,\ i = 1, \ldots, n\ . \tag{60}
$$

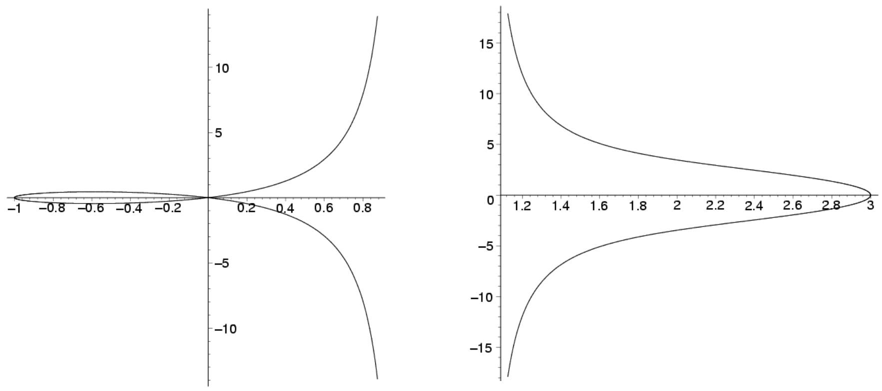

Figure 1: Open trajectory $(x-a)^2(x^2+y^2)-l^2x^2=0$, $a=1$, $l=2$. Left and right branches respectively.

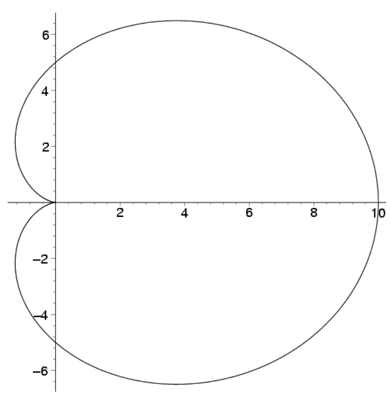

Figure 2: Closed trajectory: Cardioid $(x^2+y^2)(x^2+y^2-2ax)-a^2y^2=0$, $a=5$;

The definition (60) corresponds to the case of the moving particle, for example. The Δ coefficient values could be different, being $\Delta<1$ for the particle motion at sublightspeed velocities.

The principle problem for visualizing partial decomposition sums of Liouville operator is a high computational cost for evaluating analytical representation of the embedded Liouville operator. It is even higher when the nonpotential forces are involved. The authors used **Maple** package to get these partial sums for further impulse and trajectory parameter substitution.

In all cases dimensionless parameter d has been taken $d=2$.

8. Numerical modelling and visualization

According to the peakons solution form of Camassa-Holm equation we are expecting to see the peaked solutions without a respect of trajectory form.

8.1. **Open trajectory, without nonpotentional forces.** To represent partial decomposition figures (3)–(7) in structured manner we put the left column to correspond the left branch $-\frac{\pi}{2},\ \frac{\pi}{2}$ of the curve (1) taken for $t = [\frac{3\pi}{2},\ \frac{5\pi}{2}]$. The right column corresponds to the left branch $[\frac{\pi}{2},\ \frac{3\pi}{2}]$ of the curve (1) taken for $t = [\frac{5\pi}{2},\ \frac{7\pi}{2}]$.

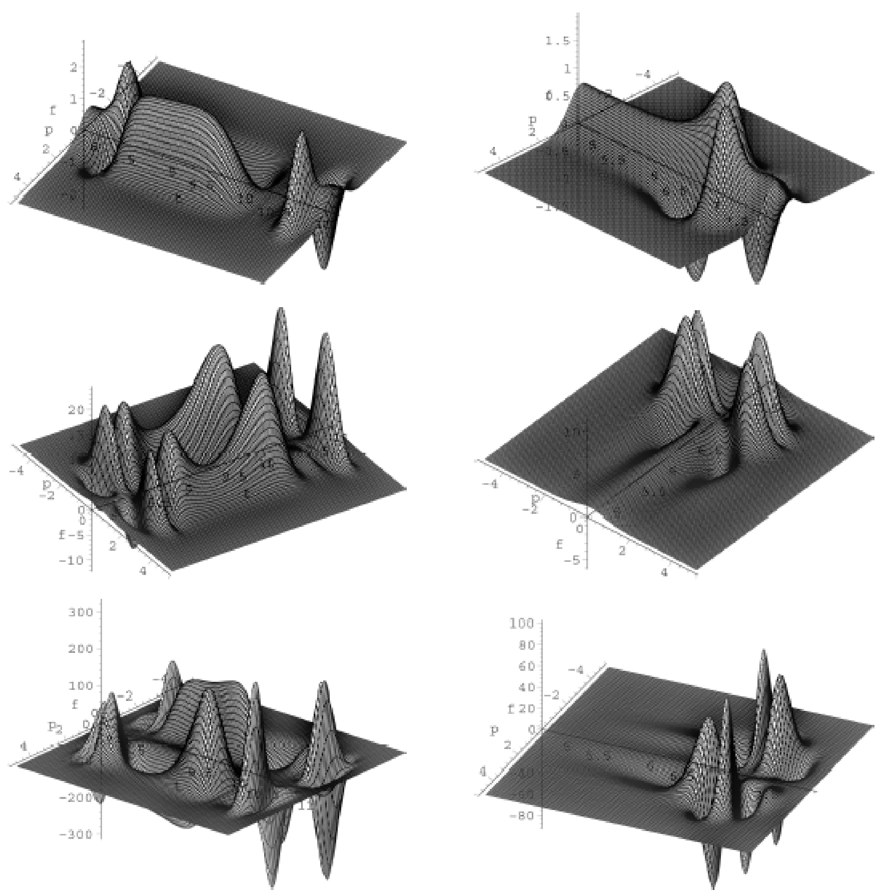

Figure 3: Open trajectory 1. $r = 0.1$, $p = (s, -s)$. Two — four coefs decomposition from above to bottom respectively

A short look at figure 3 shows that a solution form vary, taking a stable character starting from four coefficient decomposition: figure 3, images at bottom and figures 4–7. Most clearly it could be seen for the right trajectory branch taking the **bell**, or the peakon form. The form of the solution for the left branch is more complex:

- Involving more embedded Liouville operators in decomposition produces a better separation of the two clearly seen peakons.
- The separate higher peakon at the right corresponds to the "open" part of trajectory ($q_1 > 0$, Fig. 1) while the smaller one on the left corresponds to the "closed" part ($q_1 < 0$, Fig. 1).
- The size of this "closed" peakon diminishes by adding new coefficients to the partial sum decomposition.

Comparing the left branch solutions for all provided partial sums one sees that peakons become well separated only starting form eight coefficient decomposition, figure 6. Until that moment partial sums describe the qualitative character solution only partially or, for the decompositions up to the fifth coefficient involved — even incorrectly.

8.2. **Open trajectory, nonpotential forces involved.** To represent partial decomposition figures (8)–(10) we use the same representation scheme using slightly shorter visualization intervals, avoiding hardware computation faults: $t = [\frac{3\pi}{2} + 0.00001,\ \frac{5\pi}{2} - 0.00001]$ and $t = [\frac{5\pi}{2} + 0.00001,\ \frac{7\pi}{2} - 0.00001]$.
Besides, regarding this figures we can see, that a "macroscopic" view can not show any remarkable differences since the $\delta(\cdot)$–type behavior dominate.
To make a complete observation we have to note that a fast grow at the ends of the interval makes impossible to view the properties of the solution in the internal points, far from the borders of periodic segment. Thus we provide two additional figures, which clearly show, that inside the left branch definition interval we have the other peakon, very similar of those at figure 10 (for a complete view see page 158).

8.3. **Closed trajectories, nonpotential forces involved.** Clearly, that in the case of the closed trajectory (2) there are no discontinuities at points $\pm\frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$ and we can find more detailed figures. To provide a better function representation, the images are made for different time intervals, specified in the figures captions.
Comparing figures with 2–4 members of particle decomposition (figures 12) we see them better separated with a major number of coefficients involved. That makes it necessary to revise the partial sums of higher order.
As in a previous subsection, the form of the solution given by a partial sum depends on the number of the coefficients, making the symmetry odd or even. But their form still could be approximated by an overlapping "bell", or peakon.

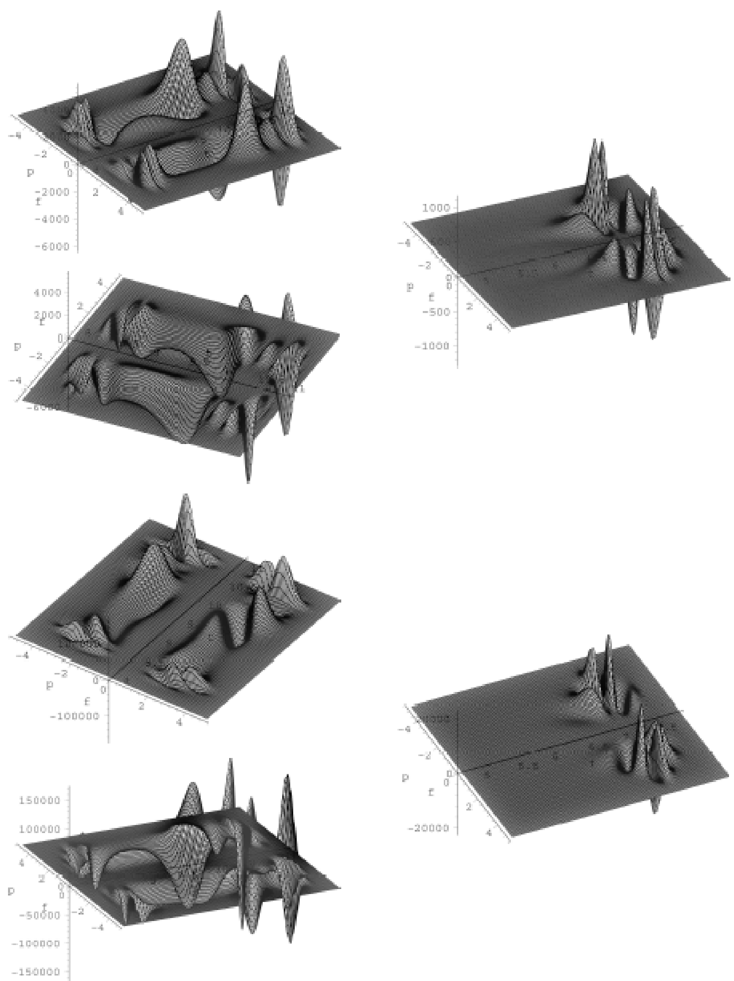

Figure 4: Open trajectory 1. $r = 0.1$, $p = (s, -s)$. Five and six coefs decomposition

Figures 13b and 13c clearly show that using only two or three decomposition coefficients hardly could be enough for this trajectory. And only taking into account figure 14 we can see the stabilization of the distribution function form.

8.4. **Closed trajectory, without nonpotential forces.** To compare the form of the solution curve with and without nonpotential forces, we divided

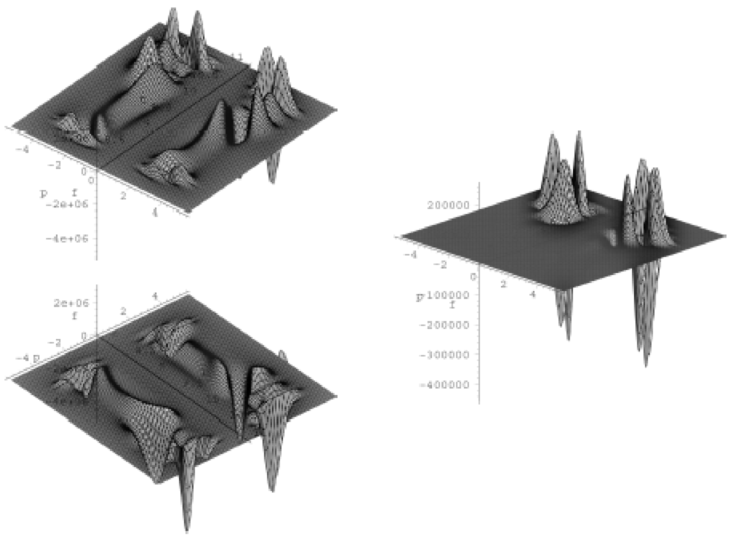

Figure 5: Open trajectory 1. $r = 0.1$, $p = (s, -s)$. Seven coefs decomposition

the corresponding figures into two parts, depending on the trajectory form pp. 166–168.

9. Conclusions

Based on the analysis, provided in modeling section, we see that a partial sum decompositions being made for Hermitian polynomials should contain at least 6–7 coefficients to provide more or less truthful qualitative convergence characteristics, especially for the models with a non-potential forces applied. It means, that a proposed approach was validated and could be recommended for the qualitative study of the solutions for the generalized Liouville equations.

Since the evaluation of the truncated series is done by symbolic computer calculations, one can revise the properties of the solution along the arbitrary trajectories and potentials. Since the series convergence is guaranteed for bounded Liouville operator, one can use this approach even without checking this condition to verify if the problem solution "converge" to a certain form, or not. Such a possibility is extremely important for the engineers, physicians, chemists and applied mathematicians who are not too comfortable with the underlying theory issues. They could start a preliminary research using the operator definition and the initial distribution function f_0 only. Besides, they can check

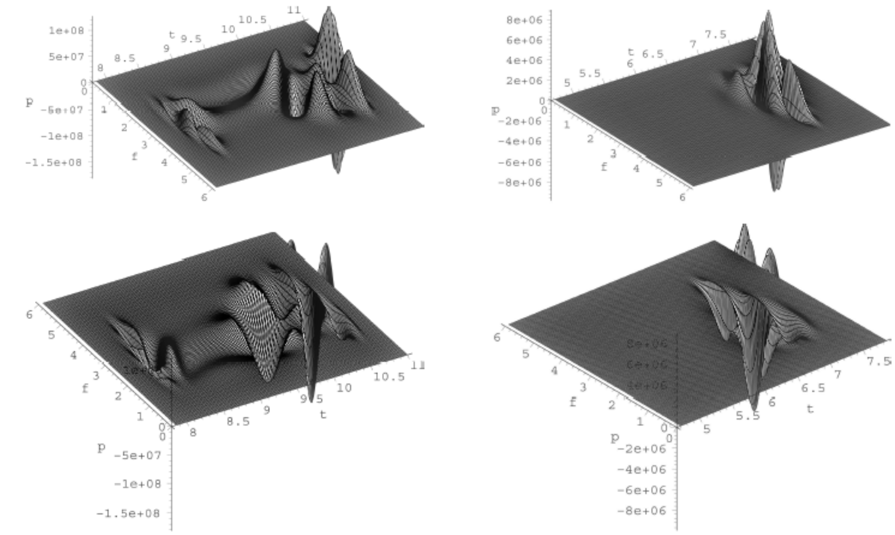

Figure 6: Open trajectory 1. $r = 0.1$, $p = (s, -s)$. Eight coefs decomposition. Left and right branches. View from the above is on the top, view form the below is at bottom

only the required domains/trajectories and time segments being inspired by some physical or natural assumptions.

Things to do

To extend the obtained results we are highly interested in exchange the simplest linear form of non-potential forces with a complex one, like $Q_i^* = \frac{p_i}{q_1^2+q^2}$, $i = 1, 2$. Such a statement is analogous to a "two bodies" problem that makes it a point of great practical interest.

Acknowledgments

We kindly thanks all the referees and our college Natalia Andrianova for their valuable support and useful remarks.

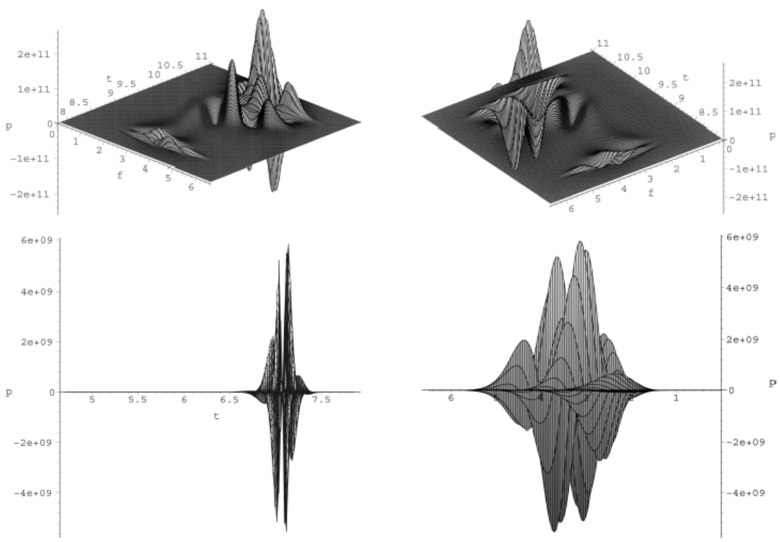

Figure 7: Open trajectory 1. $r = 0.1$, $p = (s, -s)$. Ten coefs decomposition. Left branch on the top: view from the above upside, view form the below — downside. Right branch at bottom: Side view is on the left, front view is at right

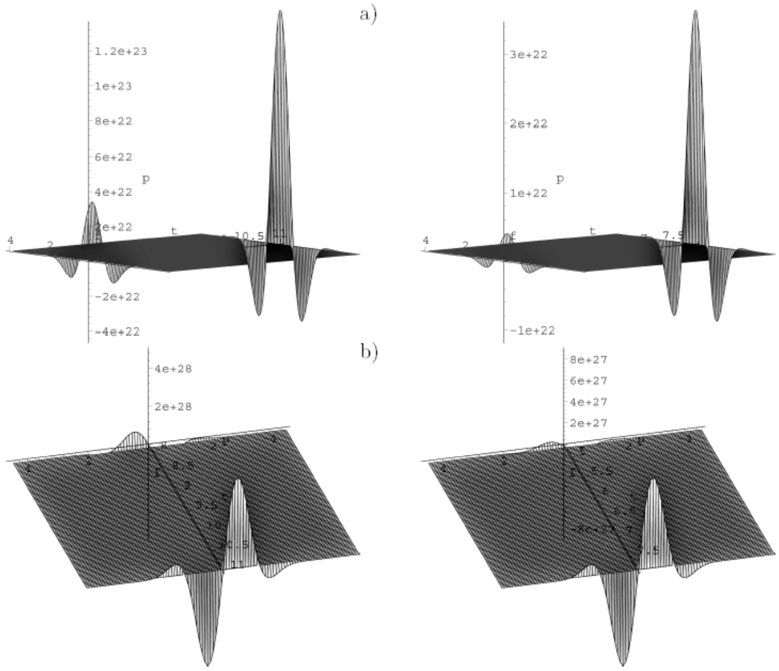

Figure 8: Open trajectory 1. $\Delta = 0.2, r = 0.1$, $p = (s, -s)$. a) five coefs decomposition; b) six coefs decomposition

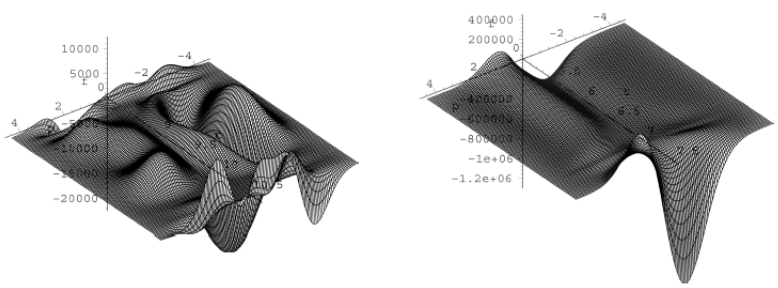

Figure 9: Open trajectory 1. $\Delta = 0.2$, $r = 0.2$, $p = (s, \frac{s}{10})$ six coefs decomposition. Detailed view for intervals $[\frac{5\pi}{2} + \frac{\pi}{10}, \frac{7\pi}{2} - \frac{\pi}{10}]$, $[\frac{3\pi}{2} + \frac{\pi}{10}, \frac{5\pi}{2} - \frac{\pi}{10}]$

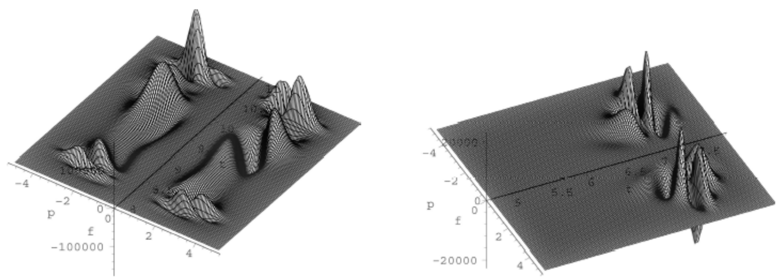

Figure 10: Open trajectory 1 without nonpotential forces. $r = 0.1$, $p = (s, -s)$. Six coefs decomposition. Intervals $[\frac{5\pi}{2}, \frac{7\pi}{2}]$, $[\frac{3\pi}{2}, \frac{5\pi}{2}]$

a) b)

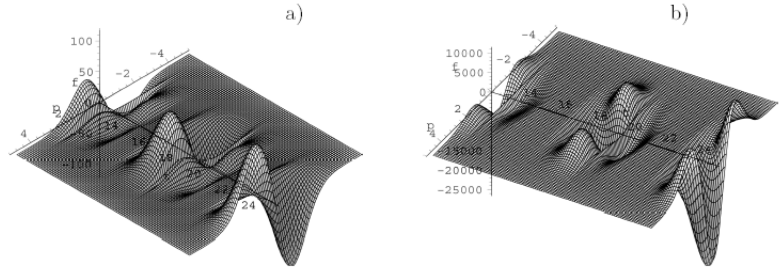

Figure 11: Closed trajectory 2a. $\Delta = 0.2, r = 0.3$, $p = (s, \frac{s}{10})$. a) two coefs decomposition; b) three coefs decomposition; a,b) $[4\pi, 8\pi]$

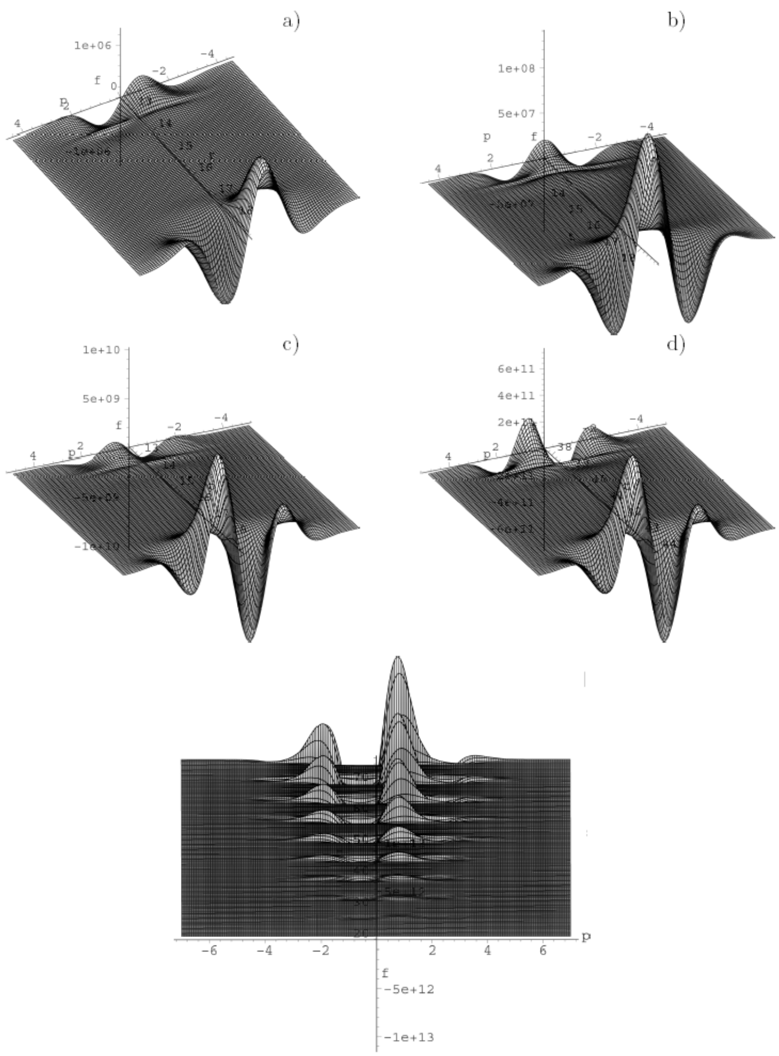

Figure 12: Closed trajectory 2a. $\Delta = 0.2, r = 0.3$, $p = (s, \frac{s}{10})$. a) four coefs decomposition; b) five coefs decomposition; c,d) six coefs decomposition; a–c) $[4\pi, 6\pi]$; d) $[12\pi, 14\pi]$; Six coefs overall view, $[6\pi, 24\pi]$ - at the bottom

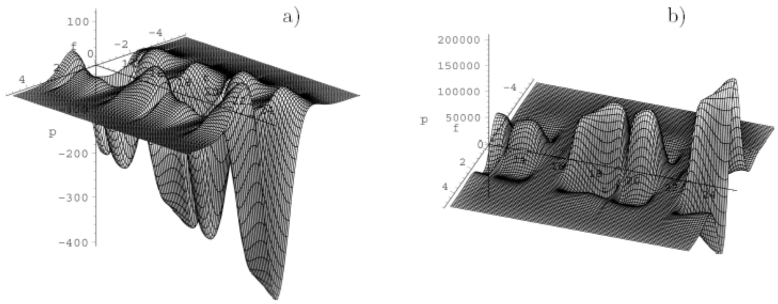

Figure 13: Closed trajectory 2a. $\Delta = 10, r = 0.01$, $p = (s, -\frac{s}{5})$. a) two coefs decomposition; b) three coefs decomposition; b,c) $[4\pi, 8\pi]$

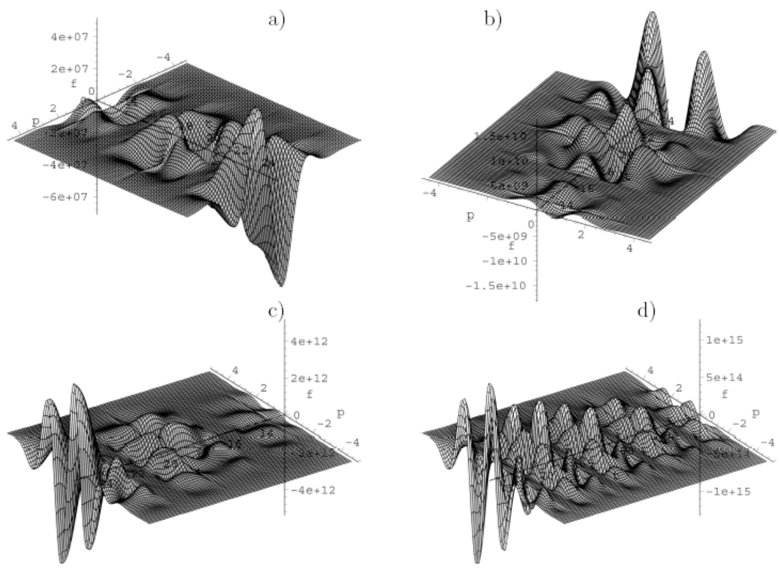

Figure 14: Closed trajectory 2a. $\Delta = 10, r = 0.01$, $p = (s, -\frac{s}{5})$. a) four coefs decomposition; b) five coefs decomposition; c,d) six coefs decomposition; a–c) $[4\pi, 8\pi]$; d) $[16\pi, 24\pi]$

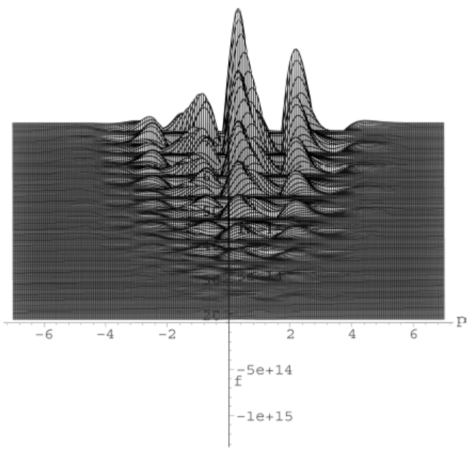

Figure 15: Closed trajectory 2a. $\Delta = 10, r = 0.01,\ p = (s, -\frac{s}{5})$. Six coefs overall view, $[6\pi, 24\pi]$

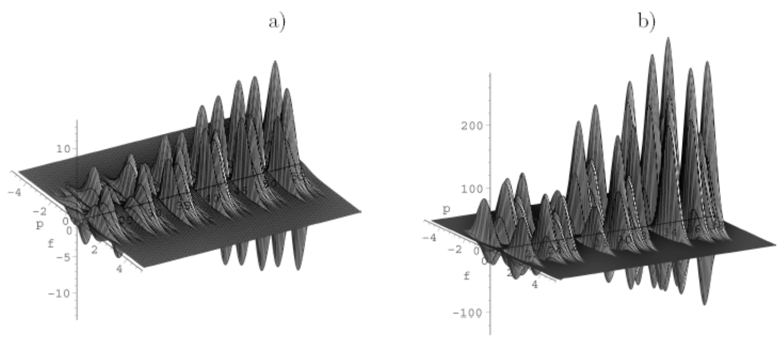

Figure 16: Closed trajectory 2a. $r = 0.01,\ p = (s, -s)$. a) two coefs decomposition on the top $[6\pi, 18\pi]$; b) three coefs decomposition at the bottom $[6\pi, 12\pi]$

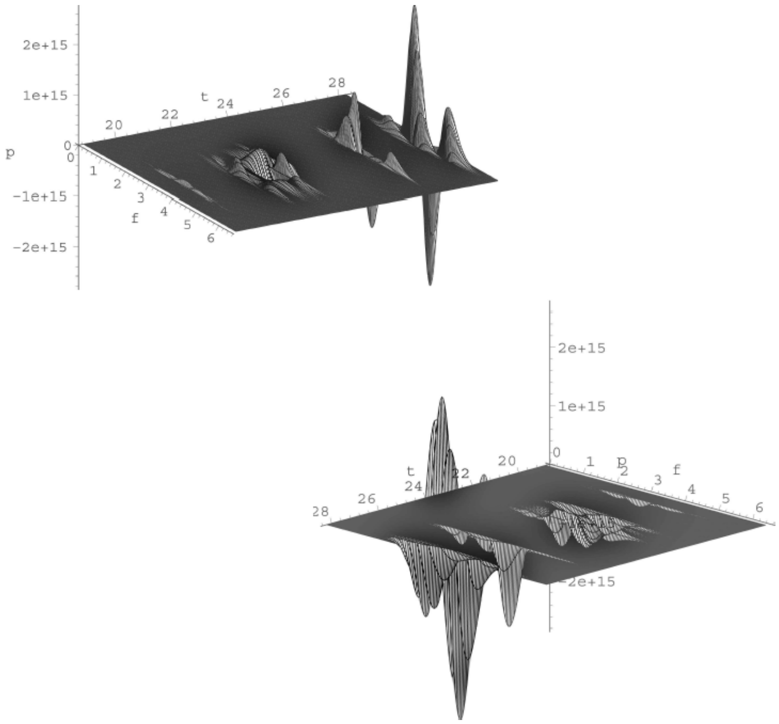

Figure 17: Closed trajectory 2a. $r = 0.01$, $p = (s, -s)$. Ten coefs decomposition, $[6\pi, 12\pi]$. View from the above on the top, view from the below at bottom

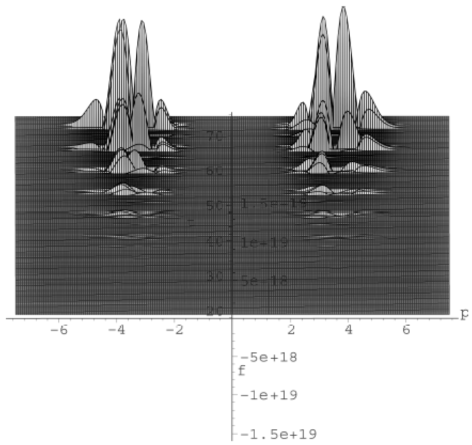

Figure 18: Closed trajectory 2a. $r = 0.01$, $p = (s, -s)$. Ten coefs decomposition. Overall view $t \in [6\pi, 24\pi]$

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