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NUMERICAL QUENCHING SOLUTIONS OF LOCALIZED SEMILINEAR PARABOLIC EQUATION

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Abstract. This paper concerns the study of the numerical approximation for the following initial-boundary value problem:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \varepsilon(1-u(0,t))^{-p}, & (x,t) \in (-l,l) \times (0,T), \\ u(-l,t) = 0, \quad u(l,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in (-l,l), \end{cases}$$

where $p > 1$, $l = \frac{1}{2}$ and $\varepsilon > 0$. Under some assumptions, we prove that the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also show that the semidiscrete quenching time in certain cases converges to the real one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

Key Words and Phrases. Semidiscretizations, localized semilinear parabolic equation, semidiscrete quenching time, convergence.

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Resumen. En este artículo se estudia la aproximación numérica para el siguiente problema de valor de frontera inicial:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \varepsilon(1-u(0,t))^{-p}, & (x,t) \in (-l,l) \times (0,T), \\ u(-l,t) = 0, \quad u(l,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in (-l,l), \end{cases}$$

donde $p > 1$, $l = \frac{1}{2}$ y $\varepsilon > 0$. Bajo algunas hipótesis, probamos que la solución de una forma semidiscreta del problema anterior se satisface en un tiempo finito y estimamos su tiempo semidiscreto de enfriamiento. También mostramos que el tiempo de enfriamiento converge en ciertos casos a un real cuando la malla tiende a cero. Finalmente, presentamos algunos experimentos numéricos para ilustrar nuestro análisis.

Palabras claves. Semidiscretizaciones, ecuación parabólica semilineal localizada, tiempo semidiscreto de enfriamiento, convergencia.

1. Introduction

We consider the following initial-boundary value problem:

$$u_t(x,t) = u_{xx}(x,t) + \varepsilon(1-u(0,t))^{-p}, \quad (x,t) \in (-l,l) \times (0,T), \tag{1}$$

$$u(-l,t) = 0, \quad u(l,t) = 0, \quad t \in (0,T), \tag{2}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in (-l,l), \tag{3}$$

where $p > 1$, $l = \frac{1}{2}$, $\varepsilon > 0$, $u_0(x)$ is a function which is bounded and symmetric. In addition, $u_0(x)$ is nondecreasing on the interval $(-l,0)$ and $u_0''(x) + \varepsilon(1 - u(0,t))^{-p} \geq 0$ on $(-l,l)$. Here $(0,T)$ is the maximal time interval on which $\|u(x,t)\|_\infty < 1$, where $\|u(x,t)\|_\infty = \max_{0 \leq x \leq 1} |u(x,t)|$. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, then we have

$$\lim_{t \to T} \|u(x,t)\|_\infty = 1.$$

In this case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u.

The above problem is related to a popular model arising in the study of polarization phenomenon. It also represents a model which is related via transformation to a certain class of physical problem of ignition where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological systems and in chemical reaction diffusion processes in which the reaction takes place only at some local sites. For more physical motivation see for instance [4] and [7].

In this paper, we are interested in the numerical study of the above problem. Let I be a positive integer, and consider the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 2l/I$. We approximate the solution u of (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{d}{dt}U_i(t) = \delta^2 U_i(t) + \varepsilon(1 - U_k(t))^{-p}, \quad 1 \leq i \leq I-1, \quad t \in (0, T_q^h), \tag{4}$$

$$U_0(t) = 0, \quad U_I(t) = 0, \quad t \in (0, T_q^h), \tag{5}$$

$$U_i(0) = \varphi_i \geq 0, \quad 0 \leq i \leq I, \tag{6}$$

where k is the integer part of the number $I/2$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\varphi_0 = 0, \quad \varphi_I = 0, \quad \varphi_i = \varphi_{I-i}, \quad 0 \leq i \leq I, \quad \delta^+\varphi_i > 0, \quad 0 \leq i \leq k-1,$$

$$\delta^+\varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}.$$

Here $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$ with $\|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|$. When the time T_q^h is finite, we say that $U_h(t)$ quenches in a finite time and the time T_q^h is called the quenching time of the solution $U_h(t)$.

The theoretical study of quenching solutions for semilinear parabolic equations has been the subject of investigations of many authors (see [3], [6], [7], [8] and the references cited therein). In particular in [6] and [7], the author has proved that under some assumptions, the solution of (1)–(3) quenches in a finite time and the quenching time is estimated. Here we are interested in the numerical study using the semidiscrete form defined in (4)–(6). We give some assumptions under which the solution of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time. We also show that the semidiscrete quenching time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1] and [6] for the phenomenon of blow-up where the authors have considered the problem (1)–(3) in the case where the reaction term $\varepsilon(1-u(0,t))^{-p}$ is replaced by $(u(x,t))^q$ with $q > 1$ (we say that a solution blows up in a finite time if it attains the value infinity in a finite time). In the same way in [2] the numerical extinction has been studied using some discrete and semidiscrete schemes (we say that a solution u extincts in a finite time if it reaches the value zero in a finite time).

Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, under some hypotheses, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we give a result about the convergence of semidiscrete quenching times in some cases where the quenching occurs. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. The following lemma is a semidiscrete version of the maximum principle.

Lemma 2.1. *Let $a_h(t) \in C^0([0,T], \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ such that*

$$\frac{d}{dt}V_i(t) - \delta^2 V_i(t) + a_k(t)V_k(t) \geq 0, \quad 1 \leq i \leq I-1, \quad t \in (0,T), \tag{7}$$

$$V_0(t) \geq 0, \quad V_I(t) \geq 0, \quad t \in (0,T), \tag{8}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{9}$$

Then we have $V_i(t) \geq 0$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} V_i(t)$. Since for $i \in \{0,...,I\}$, $V_i(t)$ is a continuous function, there exists $t_0 \in [0,T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0,...,I\}$. Assume that $m < 0$. If $i_0 = 0$ or $i_0 = I$, we have a contradiction because of (8). For $i_0 \in \{1,...,I-1\}$, it is not hard to see that

$$\frac{dV_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \leq 0, \tag{10}$$

$$\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \geq 0. \tag{11}$$

Define the vector $Z_h(t) = e^{\lambda t}V_h(t)$ where λ is large enough that $a_k(t_0)V_k(t_0) - \lambda m < 0$. Use (10) and (11) to obtain $\frac{dZ_{i_0}(t_0)}{dt} \leq 0$ and $\delta^2 Z_{i_0}(t_0) \geq 0$, which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) < 0. \tag{12}$$

On the other hand, from (7), we derive the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) \geq 0.$$

Therefore, we have a contradiction because of (12).
Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 2.2. *Let $V_h(t)$, $U_h(t) \in C^1([0,T],\mathbb{R}^{I+1})$ and $f \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that for $t \in (0,T)$*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + f(V_k(t),t) \geq \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_k(t),t),\ 1 \leq i \leq I-1, \tag{13}$$

$$V_0(t) \geq U_0(t), \quad V_I(t) \geq U_I(t), \tag{14}$$

$$V_i(0) \geq U_i(0), \quad 0 \leq i \leq I. \tag{15}$$

Then we have $V_i(t) \geq U_i(t)$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Introduce the vector $Z_h(t) = V_h(t) - U_h(t)$. A direct calculation yields

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + f_y(\theta_k(t),t)Z_k(t) \geq 0,$$

$$Z_0(t) \geq 0, \quad Z_I(t) \geq 0,$$

$$Z_i(0) \geq 0,$$

where θ_k is an intermediate value between U_k and V_k and f_y is the partial derivative of f with respect to the second variable. Since $f \in C^1$ then $f_y(\theta_k(t),t)$ is bounded on $(0,T)$. Use Lemma 2.1 to complete the rest of the proof.
The lemma below shows that when i is between 1 and $I-1$, then $U_i(t)$ is positive where $U_h(t)$ is the solution of the semidiscrete problem.

Lemma 2.3. *Let U_h be the solution of (4)–(6). Then we have*

$$U_i(t) > 0, \quad 1 \leq i \leq I-1.$$

Proof. Let $\alpha = \min_{1\leq i\leq I-1} \varphi_i$ and introduce the vector V_h defined by $V_i = \alpha e^{-\lambda_h t}\sin(i\pi h)$, $0 \leq i \leq I$, where $\lambda_h = \frac{2-2\cos(h\pi)}{h^2}$. It is not hard to see that

$$\frac{dU_i}{dt} - \delta^2 U_i \geq \frac{dV_i}{dt} - \delta^2 V_i = 0,$$

$$U_0(t) = V_0(t), \quad U_I(t) = V_I(t) = 0,$$

$$U_i(0) \geq V_i(0), \quad 1 \leq i \leq I-1.$$

We deduce from Lemma 2.2 that $U_i(t) \geq \alpha e^{-\lambda_h t}\sin(i\pi h)$, $0 \leq i \leq I$. This implies that $U_i(t) > 0$, $1 \leq i \leq I-1$, and the proof is complete.
The following lemma reveals that the solution $U_h(t)$ of the semidiscrete problem is symmetric and $\delta^+ U_i(t)$ is positive when i is between 1 and $k-1$.

Lemma 2.4. *Let U_h be the solution of (4)–(6). Then we have for $t \in (0,T_q^h)$*

$$U_{I-i}(t) = U_i(t),\ 0 \leq i \leq I, \quad \delta^+ U_i(t) > 0,\ 0 \leq i \leq k-1. \tag{16}$$

Proof. Introduce the vector V_h defined as follows $V_i(t) = U_{I-i}(t)$ for $0 \leq i \leq I$. It is not hard to see that $V_h(t)$ is a solution of (4)–(6). It follows from Lemma 2.1 that $V_h(t) = U_h(t)$. Now, define the vector $Z_h(t)$ such that

$$Z_i(t) = U_{i+1}(t) - U_i(t), \quad 0 \leq i \leq k-1,$$

and let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$ but $Z_{i_0}(t_0) = 0$. Without loss of the generality, we assume that i_0 is the smallest integer such that $Z_{i_0}(t_0) = 0$. If $i_0 = 0$ then we have $U_1(t_0) = U_0(t_0) = 0$, which is a contradiction because from Lemma 2.3. $U_1(t_0) > 0$. It is easy to see that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) = 0, \quad if \quad 1 \leq i_0 \leq k-1. \tag{17}$$

On the other hand, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} > 0, \quad 1 \leq i_0 \leq k-2,$$

and we know that if $i_0 = k-1$,

$$\begin{aligned}\delta^2 Z_{k-1}(t_0) &= \delta^2 U_k(t_0) - \delta^2 U_{k-1}(t_0)\\ &= \frac{U_{k+1}(t_0) - 2U_k(t_0) + U_{k-1}(t_0) - U_k(t_0) + 2U_{k-1}(t_0) - U_{k-2}(t_0)}{h^2}.\end{aligned}$$

Since k is the integer part of the number $I/2$, using the fact that the discrete solution is symmetric, we have either $U_{k+1}(t) = U_{k-1}(t)$ or $U_{k+1}(t) = U_k(t)$. In both cases, we find that

$$\delta^2 Z_k(t_0) = \frac{Z_{k-2}(t_0)}{h^2} > 0.$$

The above inequalities imply that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) < 0$, which is a contradiction because of (17) and the proof is complete.
A discrete version of the Green's formula is the following

Lemma 2.5. *Let $U_h(t)$ and $V_h(t)$ two vectors such that $U_0(t) = 0$, $U_I(t) = 0$, $V_0(t) = 0$, $V_I(t) = 0$. Then we have*

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i = \sum_{i=1}^{I-1} hV_i\delta^2 U_i. \tag{18}$$

Proof. A routine calculation yields

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i = \sum_{i=1}^{I-1} hV_i\delta^2 U_i + \frac{V_I U_{I-1} - U_I V_{I-1} + V_0 U_1 - U_0 V_1}{h}$$

and the result follows using the assumptions of the lemma.
Now, let us state a result on the operator δ^2.

Lemma 2.6. *Let $U_h \in C^0([0,T], R^{I+1})$ such that $U_h \geq 0$. Then we have*

$$\delta^2(1-U_i)^p \geq p(1-U_i)^{p-1}\delta^2 U_i(t) \quad for \quad 1 \leq i \leq I-1.$$

Proof. Using Taylor's expansion, we get

$$\delta^2(1-U_i)^{-p} = p(1-U_i)^{p-1}\delta^2 U_i(t) + (U_{i+1}-U_i)^2 \frac{p(p+1)}{2h^2}\theta_i^{p-2}$$
$$+(U_{i-1}-U_i)^2\frac{p(p+1)}{2h^2}\eta_i^{p-2} if \quad 1 \leq i \leq I-1,$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_i and U_{i-1}. The result follows taking into account the fact that U_h is nonnegative.
To end this section, let us give another property of the operator δ^2.

Lemma 2.7. *Let V_h and $U_h \in C^1([0,T], \mathbb{R}^{I+1})$. If*

$$\delta^+(U_i)\delta^+(V_i) \geq 0, \quad and \quad \delta^-(U_i)\delta^-(V_i) \geq 0, \tag{19}$$

then

$$\delta^2(U_iV_i) \geq U_i\delta^2(V_i) + V_i\delta^2(U_i),$$

where $\delta^+(U_i) = \frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1}-U_i}{h}$.

Proof. A straightforward computation yields

$$h^2\delta^2(U_iV_i) = U_{i+1}V_{i+1} - 2U_iV_i + U_{i-1}V_{i-1}$$
$$= (U_{i+1}-U_i)(V_{i+1}-V_i) + V_i(U_{i+1}-U_i) + U_i(V_{i+1}-V_i) + U_iV_i - 2U_iV_i$$
$$+ (U_{i-1}-U_i)(V_{i-1}-V_i) + (U_{i-1}-U_i)V_i + U_i(V_{i-1}-V_i) + U_iV_i,$$

which implies that

$$\delta^2(U_iV_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + U_i\delta^2(V_i) + V_i\delta^2(U_i).$$

Using (19), we obtain the desired result.

3. Quenching solutions

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time.
The statement of our first result on quenching is the following

Theorem 3.1. *Suppose that there exists a constant $A > 0$ and such that the initial data at (4) satisfies*

$$\delta^2\varphi_i + \varepsilon(1-\varphi_i)^{-p} \geq Asin(ih\pi)(1-\varphi_i)^{-p}, \quad 0 \leq i \leq I, \tag{20}$$

and

$$1 - \frac{2\pi^2}{A(p+1)}(1 - \|\varphi_h\|_\infty)^{p+1} > 0. \tag{21}$$

Then the solution $U_h(t)$ of (4)–(6) quenches in a finite time T_q^h which estimated as follows

$$T_q^h \le -\frac{1}{\pi^2} ln(1 - \frac{2\pi^2}{A(p+1)}(1 - \|\varphi_h\|_\infty)^{p+1}).$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the function $J_h(t)$ defined as follows

$$J_i(t) = \frac{d}{dt}U_i(t) - C_i(t)(1 - U_i(t))^{-p}, \quad 0 \le i \le I,$$

where $C_i(t) = Ae^{-\lambda_h t} sin(ih\pi)$, with $\lambda_h = \frac{2-2cos(\pi h)}{h^2}$. A straightforward computation reveals that

$$\begin{aligned} \frac{dJ_i}{dt} - \delta^2 J_i &= \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - (1-U_i)^{-p}\frac{dC_i}{dt} - pC_i(1-U_i)^{-p-1}\frac{dU_i}{dt} \\ &+ \delta^2(C_i(1-U_i)^{-p}). \end{aligned}$$

From Lemma 2.6 and 2.7, the last term on the right hand side of the equality $\delta^2(C_i(1-U_i)^{-p})$ is bounded from below by $(1-U_i)^{-p}\delta^2 C_i + p(1-U_i)^{-p-1}C_i\delta^2 U_i$ due to the fact $\delta^+(C_i)\delta^+(1-U_i)^{-p}$ and $\delta^-(C_i)\delta^-(1-U_i)^{-p}$ are nonnegative because the results of Lemma 2.4 hold for $U_h(t)$ and $C_h(t)$. We deduce that

$$\begin{aligned} \frac{dJ_i(t)}{dt} - \delta^2 J_i(t) &\ge \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - (1-U_i)^{-p}(\frac{dC_i(t)}{dt} - \delta^2 C_i(t)) \\ &- pC_i(t)(1-U_i)^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)). \end{aligned}$$

Using (4) and the fact that $\frac{d}{dt}C_i(t) - \delta^2 C_i(t) = 0$, we find that

$$\begin{aligned} &\frac{dJ_i}{dt} - \delta^2 J_i \ge \varepsilon p(1-U_k)^{-p-1}\frac{dU_k}{dt} - \varepsilon p(1-U_i)^{-p-1}C_i(1-U_k)^{-p-1} \\ \ge\ & \varepsilon p(1-U_k)^{-p-1}(J_k + C_k(1-U_k)^{-p}) - \varepsilon p(1-U_i)^{-p-1}C_i(1-U_k)^{-p-1} \\ \ge\ & \varepsilon p(1-U_k)^{-p-1}J_k + \varepsilon p(1-U_k)^{-p}(C_k(1-U_k)^{-p-1} - C_i(1-U_i)^{-p-1}). \end{aligned}$$

From Lemma 2.4, $U_k \ge U_i$. We also observe that $C_k \ge C_i$. We deduce that

$$\frac{dJ_i}{dt} - \delta^2 J_i \ge \varepsilon p(1-U_k)^{-p-1}J_k, \quad 1 \le i \le I-1.$$

It is not hard to see that $J_0(t) = 0$, $J_I(t) = 0$ and the relation (20) implies that $J_h(0) \ge 0$. It follows from Lemma 2.1 that $J_h(t)$ is nonnegative, which implies that

$$\frac{dU_i}{dt} \geq C_i(1-U_i)^{-p}, \quad 0 \leq i \leq I.$$

Using Taylor's expansion, we find that $\cos(h\pi) \geq 1 - \pi^2 \frac{h^2}{2}$, which implies that $\lambda_h \leq \pi^2$. Obviously $\sin(kh\pi) \geq \frac{1}{2}$. We deduce that

$$\frac{dU_k}{dt} \geq \frac{A}{2} e^{-\pi^2 t}(1-U_k)^{-p}, \quad 0 \leq i \leq I.$$

This inequality may be rewritten as follows

$$(1-U_k)^p dU_k \geq \frac{A}{2} e^{-\pi^2 t} dt. \tag{22}$$

Integrating this inequality over $(0, T_q^h)$, we arrive at

$$\frac{A(1-e^{-\pi^2 T_q^h})}{2\pi^2} \leq \frac{(1-U_k(0))^{p+1}}{p+1},$$

which implies that

$$e^{-\pi^2 T_q^h} \geq 1 - \frac{2\pi^2}{A(p+1)}(1-U_k(0))^{p+1}.$$

Since $\|U_h(0)\|_\infty = U_k(0)$, the restriction on the initial data in (21) implies that the term on the right hand side of the above inequality is positive. Therefore we find that

$$T_q^h \leq -\frac{1}{\pi^2} ln(1 - \frac{2\pi^2}{A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1}).$$

Remark 3.1. *Integrating the inequality in (22) over (t_0, T_q^h), we get*

$$T_q^h - t_0 \leq -\frac{1}{\pi^2} ln(1 - \frac{2\pi^2}{A(p+1)} e^{\pi^2 t_0}(1-\|U_h(t_0)\|_\infty)^{p+1}).$$

The proof of the above theorem allows us to establish the estimation in Remark 3.1 which is crucial to prove the convergence of the semidiscrete quenching time. When the initial data is null, the hypotheses of Theorem 3.1 are satisfied if the parameter ε is large enough. The theorem below also shows that ε is large enough, then the semidiscrete solution quenches in a finite time. In addition, in this case the restriction on ε is better than the one of Theorem 3.1.

Theorem 3.2. *Let $v(0) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)\varphi_i$ and $\lambda_h = \frac{2-2\cos(\pi h)}{h^2}$. Assume that $\varepsilon > \lambda_h \frac{p^p}{(p+1)^{p+1}}$. Then the solution $U_h(t)$ of (4)–(6) quenches in a finite time T_q^h which is estimated as follows*

$$T_q^h \leq \frac{(p+1)^{p+1}(1-v(0))^{p+1}}{\varepsilon(p+1)^{p+1} - \lambda_h p^p}.$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the function $v(t)$ defined as follows

$$v(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h) \sin(i\pi h) U_i(t).$$

Take the derivative of v with respect to t and use (4) to obtain

$$v^{'}(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h) \sin(i\pi h)(\delta^2 U_i(t) + \varepsilon(1 - U_k(t))^{-p}.$$

We observe that $\delta^2 \sin(i\pi h) = -\lambda_h \sin(i\pi h)$. From the above equality and Lemma 2.5, we arrive at

$$v^{'}(t) = -\lambda_h v(t) + \varepsilon(1 - U_k(t))^{-p} \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h) \sin(i\pi h).$$

By a routine calculation, we find that $\sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)$ equals one. Due to the fact that U_k is bigger than $v(t)$, we get

$$v^{'}(t) \geq \varepsilon(1 - v(t))^p (1 - \frac{\lambda_h v(t)}{\varepsilon}(1 - v(t))^p).$$

It is not hard to see that $v(t)(1-v(t))^p$ is bounded from above by $\sup_{0\leq s\leq 1} s(1-s)^p = \frac{p^p}{(p+1)^{p+1}}$. We deduce that

$$v^{'}(t) \geq \varepsilon(1 - \frac{\lambda_h p^p}{(p+1)^{p+1}})(1 - v(t))^{-p},$$

which implies that

$$(1 - v(t))^p dv \geq \varepsilon(1 - \frac{\lambda_h p^p}{(p+1)^{p+1}})dt.$$

Integrating this inequality over $(0, T_q^h)$, we find $T_q^h \leq \frac{(p+1)^{p+1}(1-v(0))^{p+1}}{\varepsilon(p+1)^{p+1} - \lambda_h p^p}$. We conclude that T_q^h is finite and the proof is complete.

Remark 3.2. *Since $U_k(t) = \|U_h(t)\|_\infty$, it is easy to see that*

$$\delta^2 U_k(t) = \frac{U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)}{h^2} \leq 0.$$

Therefore, using (4), we get $\frac{dU_k}{dt} \geq \varepsilon(1 - U_k)^{-p}$ which implies that $(1 - U_k)^p dU_k \geq \varepsilon dt$. Integrating this inequality over $(0, T_q^h)$, we arrive at $T_q^h \geq \frac{(1-\|U_h(0)\|_\infty)^{1+p}}{(p+1)}$. Thus we have a lower bound of the semidiscrete quenching time.

Remark 3.3. *Consider the following semidiscrete scheme*

$$\frac{d}{dt}V_i(t) = \delta^2 V_i(t) + \varepsilon(1 - V_i(t))^{-p}, \quad 1 \le i \le I-1, \quad t \in (0, T_h),$$

$$V_0(t) = 0, \quad V_I(t) = 0, \quad t \in (0, T_h),$$

$$V_i(0) = \varphi_i, \quad 0 \le i \le I,$$

where $(0, T_h)$ *is the maximal time interval on which* $\|V_h(t)\|_\infty < 1$. *We observe that the above scheme is a semidiscretization of the continuous problem below*

$$u_t(x,t) = u_{xx}(x,t) + \varepsilon(1 - u(x,t))^{-p}, \quad (x,t) \in (-l, l) \times (0, T)$$

$$u(-l,t) = 0, \quad u(l,t) = 0, \quad t \in (0, T),$$

$$u(x,0) = u_0(x) \ge 0, \quad x \in (-l, l).$$

Let $U_h(t)$ *be the solution of (4)–(6), we know from Lemma 2.4 that* $U_k \ge U_i$ *for* $1 \le i \le I-1$. *We deduce that*

$$\frac{d}{dt}U_i(t) \ge \delta^2 U_i(t) + \varepsilon(1 - U_i(t))^{-p}, \quad 1 \le i \le I-1, \quad t \in (0, T_q^h),$$

$$U_0(t) = 0, \quad U_I(t) = 0, \quad t \in (0, T_q^h),$$

$$U_i(0) = \varphi_i, \quad 0 \le i \le I.$$

Setting $Z_h(t) = U_h(t) - V_h(t)$, *it is not hard to see that*

$$\frac{d}{dt}Z_i - \delta^2 Z_i - \varepsilon p(1 - \xi_i)^{-p-1} Z_i \ge 0, \quad 1 \le i \le I-1, \quad t \in (0, T_h^*),$$

$$Z_0(t) = 0, \quad Z_I(t) = 0, \quad t \in (0, T_h^*),$$

$$Z_i(0) = 0, \quad 0 \le i \le I,$$

where $T_h^* = \min\{T_h, T_q^h\}$, ξ_i *is an intermediate value between* $U_i(t)$ *and* $V_i(t)$. *Modifying slightly the proof of Lemma 2.1, we find that* $Z_h(t) \ge 0$ *for* $t \in (0, T_h^*)$. *In other words, we have* $U_h(t) \ge V_h(t)$ *for* $t \in (0, T_h^*)$ *and we conclude that* $T_q^h \le T_h$.

4. Convergence of semidiscrete quenching times

In this section, under some assumptions, we show that the semidiscrete quenching time for the solution of the semidiscrete problem converges to the real one when the mesh size goes to zero. In order to prove this result, firstly, we prove the convergence of the semidiscrete scheme by the following theorem on the convergence of the semidiscrete scheme which is crucial for the proof on the convergence of the semidiscrete quenching time.
We denote by $u_h(t) = (u(x_0,t),...,u(x_I,t))^T$.

Theorem 4.1. *Assume that (1)-(3) has a solution $u \in C^{4,1}([0,1]\times[0,T])$ such that $\|u(x,t)\|_{\inf} = \alpha > 0$. Suppose that the initial data at (6) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{23}$$

Then, for h sufficiently small, the problem (4)–(6) has a unique solution $U_h \in C^1([0,T],\mathbb{R}^{I+1})$ such that

$$\max_{0\le t\le T} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0.$$

Proof. Since $u \in C^{4,1}$, there exist two positive constants K and M such that

$$\frac{\|u_{xxxx}\|_\infty}{12} \le K, \quad \|u\|_\infty \le K, \quad \varepsilon p(1-\frac{\alpha}{2})^{-p-1} \le M. \tag{24}$$

The problem (4)–(6) has for each h, a unique solution $U_h \in C^1([0,T_q^h),\mathbb{R}^{I+1})$. Let $t(h)$ the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\alpha}{2} \quad for \quad t \in (0,t(h)). \tag{25}$$

Since the value of the term on the left hand side of the inequality is null when t is equal zero, we deduce that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h),T\}$. By the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \ge \|u(x,t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty \quad for \quad t \in (0,t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \ge \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \quad for \quad t \in (0,t^*(h)). \tag{26}$$

Let $e_h(t) = U_h(t) - u_h(x,t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t^*(h))$,

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) = \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t) + \varepsilon p(1-\xi_i)^{-p-1}e_i(t),$$

where ξ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Using (24) and (26), we arrive at

$$\frac{d}{dt}e_i(t) - \delta^2 e_i(t) \le M|e_i(t)| + Kh^2, \quad 1 \le i \le I-1. \tag{27}$$

Let z_h the vector defined by

$$z_i = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + Kh^2), \quad 0 \le i \le I.$$

A direct calculation yields

$$\frac{d}{dt}z_i - \delta^2 z_i > M|z_i(t)| + Kh^2, \quad 1 \le i \le I-1, \quad t \in (0, t^*(h)),$$

$$z_0 > e_0, \quad z_I > e_I,$$

$$z_i(0) > e_i(0), \quad 0 \le i \le I.$$

It follows from Lemma 2.2 that $z_i > e_i(t)$ for $t \in (0, t^*(h))$, $0 \le i \le I$. By the same way, we also prove that $z_i > -e_i(t)$ for $t \in (0, t^*(h))$, $0 \le i \le I$, which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(M+1)t}\|\varphi_h - u_h(0)\|_\infty + Kh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (25), we obtain

$$(28)\ \frac{\alpha}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Kh^2).$$

Since the term in the right hand side of (28) goes to zero as h tends to zero, we deduce that $\frac{\alpha}{2} \le 0$, which is impossible. Consequently $t^*(h) = T$, and we obtain the desired result.
Now, we are in a position to prove our main theorem of this section.

Theorem 4.2. *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$. Under the assumption of Theorem 4.1, the problem (4)–(6) has a solution $U_h(t)$ which quenches in a finite time T_q^h and $\lim_{h\to 0} T_q^h = T_q$.*

Proof. Letting $\varepsilon > 0$, there exists a positive constant ρ such that

$$(29) -\frac{1}{2\pi^2} ln(1 - \frac{4\pi^2}{A(p+1)} e^{2\pi^2 T_q}(1-x)^{p+1}) \le \frac{\varepsilon}{2} \quad for \quad x \in [1-\rho, 1).$$

Since $\lim_{t\to T_q} \|u(x,t)\|_\infty = 1$, there exist $T_1 < T_q$ and $|T_q - T_1| < \frac{\varepsilon}{2}$ such that $1 > \|u(x,t)\|_\infty \ge 1 - \frac{\rho}{2}$ for $t \in [T_1, T_q)$. From Theorem 4.1, the problem (4)–(6) has a solution $U_h(t)$ such that $\|U_h(t) - u_h(t)\|_\infty < \frac{\rho}{2}$ for $t \in [0, T_2]$ where $T_2 = \frac{T_1+T_q}{2}$. Using triangular inequality, we get $\|U_h(t)\|_\infty \ge \|u_h(t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty \ge 1 - \frac{\rho}{2} - \frac{\rho}{2} \ge 1 - \rho$, for $t \in (0, T_2)$. From Theorem 3.1, $U_h(t)$ quenches at time T_q^h. Using (29), we arrive at $|T_q^h - T_q| \le |T_q^h - T_2| + |T_2 - T_q| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which leads us to the desired result.

5. Numerical results

In this section, we consider the following explicit scheme.

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \varepsilon(1 - U_k^{(n)})^{-p}, \quad 1 \le i \le I-1,$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$U_i^0 = 0, \quad 0 \le i \le I,$$

and the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \varepsilon(1 - U_k^{(n)})^{-p}, \quad 1 \le i \le I-1,$$

$$U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,$$

$$U_i^0 = 0, \quad 0 \le i \le I,$$

where $n \ge 0$, $k = \frac{I}{2}$, $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}$, $\Delta t_n^e = \min\{\frac{h^2}{2}, \Delta t_n\}$, $T^n = \sum_{j=0}^{n-1} \Delta t_j$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $|T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

<u>**First case:** $\varepsilon = 9$.</u>

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.059321	448	-	-
32	0.058758	1736	-	-
64	0.058617	6649	-	2.00
128	0.058582	25353	1	2.02
256	0.058573	96383	4	1.97

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.060274	444	-	-
32	0.058988	1719	-	-
64	0.058674	6579	-	2.04
128	0.058596	25072	2	2.02
256	0.058577	95259	12	2.04

Second case: $\varepsilon = 10$.

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.052837	402	-	-
32	0.052267	1557	-	-
64	0.052125	5966	-	2.01
128	0.052089	22746	1	1.99
256	0.052080	86459	4	2.01

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.053675	398	-	-
32	0.052468	1542	-	-
64	0.052174	5903	-	2.04
128	0.052102	22494	2	2.04
256	0.052083	85450	10	1.93

Third case: $\varepsilon = 11$.

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.047688	364	-	-
32	0.047112	1412	-	-
64	0.046969	5411	-	2.02
128	0.046933	20629	-	2.00
256	0.046924	78404	4	2.01

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit

Euler method

I	T^n	n	CPU time	s
16	0.048435	360	-	-
32	0.047291	1398	-	-
64	0.047013	5353	-	2.05
128	0.046944	20400	2	2.02
256	0.046927	77487	10	2.03

From Remark 3.3, we have seen that the quenching time of the solution of our problem is smaller than the one of the problem where the reaction term is not local. In order to verify this assertion, we do the same experiments when the reaction term is not local and is replaced by $(1 - U_i^{(n)})^{-p}$.

First case: $\varepsilon = 9$.

Table 7: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.061242	462	-	-
32	0.060855	1802	-	-
64	0.060760	6952	-	2.03
128	0.060737	26709	-	2.05
256	0.060731	102312	4	1.95

Table 8: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.062410	458	-	-
32	0.061141	1785	1	-
64	0.060832	6883	1	2.05
128	0.060755	26428	2	2.01
256	0.060736	101187	13	2.03

Second case: $\varepsilon = 10$.

Table 9: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.054197	412	-	-
32	0.053779	1608	-	-
64	0.053677	6199	-	2.04
128	0.053651	23796	1	1.98
256	0.053645	91070	4	2.12

Table 10: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.055242	408	-	-
32	0.054034	1592	-	-
64	0.053740	6137	-	2.05
128	0.053667	23544	2	2.02
256	0.053649	90060	11	2.03

Third case: $\varepsilon = 11$.

Table 11: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU time	s
16	0.048673	372	-	-
32	0.048228	1451	-	-
64	0.048119	5594	-	2.04
128	0.048092	21456	1	2.02
256	0.048086	82050	3	2.17

Table 12: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU time	s
16	0.049615	368	-	-
32	0.048458	1437	-	-
64	0.048176	5537	-	2.04
128	0.048107	21227	2	2.04
256	0.048089	81133	10	1.95

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