THE INDUCED GENERALIZED HYBRID AVERAGING OPERATOR AND ITS APPLICATION IN FINANCIAL DECISION MAKING

José M. Merigó Lindahl, jmerigo@ub.edu, Universidad de Barcelona

Montserrat Casanovas Ramón, mcasanovas@ub.edu, Universidad de Barcelona

RESUMEN

Se presenta el operador induced generalized hybrid averaging (IGHA). Es un nuevo operador de agregación que generaliza la agregación híbrida (HA) a través de utilizar medias generalizadas y variables de ordenación inducidas. Con esta formulación, se obtiene una amplia gama de operadores de medias tales como el HA inducido (IHA), el HA inducido geométrico (IHGA), el HA inducido cuadrático (IHQA), etc. Obsérvese que el operador OWA y la media ponderada (WA) están incluidos como casos particulares del operador HA. Por eso, con esta generalización se puede obtener una amplia gama de WA, OWA y OWA inducidos (IOWA) como el operador OWA generalizado inducido (IGOWA), el operador OWA generalizado (GOWA), etc. También se presenta una generalización mayor al operador IGHA a través de utilizar medias cuasi-aritméticas que denominamos como el operador Quasi-IHA. Finalmente, también se desarrolla un ejemplo ilustrativo del nuevo modelo en un problema de toma de decisiones financieras. Su principal ventaja radica en la amplia gama de casos particulares disponibles lo cual ofrece al decisor una mejor visión del problema en cuestión.

Palabras clave: Toma de decisiones; Operadores de agregación; Medias híbridas; Medias generalizadas.

ABSTRACT

We present the induced generalized hybrid averaging (IGHA) operator. It is a new aggregation operator that generalizes the hybrid averaging (HA) by using generalized means and order inducing variables. With this formulation, we get a wide range of mean operators such as the induced HA (IHA), the induced hybrid quadratic averaging (IHQA), the HA, etc. The ordered weighted averaging (OWA) operator and the weighted average (WA) are included as special cases of the HA operator. Therefore, with this generalization we can obtain a wide range of aggregation operators such as the induced generalized OWA (IGOWA), the generalized OWA (GOWA), etc. We further generalize the IGHA operator by using quasi-arithmetic means. Then, we get the Quasi-IHA operator. Finally, we also develop an illustrative example of the new approach in a financial decision making problem. The main advantage of the IGHA is that it gives a more complete view of the decision problem to the decision maker because it considers a wide range of situations depending on the operator used.

Keywords: Decision making; Aggregation operators; Hybrid averaging; Generalized means.

1. INTRODUCTION

In the literature, we find a wide range of aggregation operators for aggregating the information. A very common aggregation method is the ordered weighted averaging (OWA) operator (Yager, 1988). It has been used in an astonishingly wide range of applications (Beliakov et al. 2007; Calvo et al. 2002; Merigó, 2007; Merigó and Casanovas, 2007; Yager, 1992; 1993; Yager and Kacprzyk, 1997). The main advantage of this operator is that it provides a parameterized family of aggregation operators that includes the maximum, the minimum and the average, as special cases.

An interesting extension of the OWA operator is the generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004). It generalizes the OWA operator by using generalized means (Dujmovic, 1974; Dyckhoff and Pedrycz, 1984). Then, it includes all the special cases of the OWA operator and a lot of other extensions such as the ordered weighted geometric averaging (OWG) operator, the ordered weighted quadratic averaging (OWQA) operator, etc. Note that the GOWA can be further generalized (Beliakov, 2005) by using quasi-arithmetic means (Hardy et al., 1934, Kolmogoroff, 1930; Nagumo, 1930). The result is the Quasi-OWA operator (Fodor et al., 1995). Recently, Merigó and Gil-Lafuente (2007) have suggested an extension of the GOWA operator that uses order inducing variables in a similar way as the induced OWA (IOWA) operator (Yager, 2003; Yager and Filev, 1999). This operator has been called the induced generalized OWA (IGOWA) operator. It provides a wider generalization than the GOWA because it includes the GOWA as a special case, but it also includes a wide range of induced aggregation operators such as the IOWA, the induced OWG, the induced OWQA, etc. This operator has also been further generalized by using quasi-arithmetic means (Merigó and Gil-Lafuente, 2007) and it is known as the Quasi-IOWA operator. Other generalizations of the OWA operator are found in (Merigó and Casanovas, 2007b; 2007c; Wang and Hao, 2006). Note that these generalizations have a different meaning than those developed in (Schaefer and Mitchell, 1999).

A further interesting aggregation operator is the hybrid averaging (HA) operator (Xu and Da, 2003). It is an aggregation operator that uses the weighted average (WA) and the OWA operator in the same formulation. The HA operator has been studied by several authors (Merigó, 2007; Xu, 2004; 2006). Another interesting extension of the HA operator is the one that uses a more general attitudinal character by using order inducing variables. It is known as the induced HA (IHA) operator. In the HA operator, it is also possible to generalize it by using generalized means. Then, we get the generalized hybrid averaging (GHA) operator. This generalization includes a wide range of mean operators such as the HA, the hybrid geometric averaging (HGA), the hybrid quadratic averaging (HQA), etc. Note that in this case, it is also possible to generalize it by using quasi-arithmetic means. The result is the Quasi-HA operator.

Going a step further, we see that it is also possible to develop a generalization of the HA operator that uses order inducing variables. We will call this aggregation operator, the induced generalized hybrid averaging (IGHA) operator. The main advantage of this operator is that it provides a wider generalization of the GHA because it provides a more complete attitudinal character by using inducing variables. The IGHA operator includes the GHA as a particular case. Therefore, all the particular cases of the GHA are also included in this generalization. It also provides with other types of means such as the IHA, the induced HGA, the induced HQA, the IGOWA operator, etc.

We further generalize it by using quasi-arithmetic means and we obtain, as a result, the Quasi-IHA operator. Finally, we will also develop an illustrative example of the new aggregation operator. We will focus on a financial decision making problem about the selection of investments. The main advantage of using the IGHA in decision making problems is that it gives a more complete view of the decision problem to the decision maker. Then, the decision maker will be able to consider a wide range of scenarios and select the one that it is in accordance with its interests.

In order to do this, this paper is organized as follows. In Section 2, we briefly review some basic concepts such as the IHA and the IGOWA operator. Section 3 presents the IGHA operator. Section 4 analyzes different families of IGHA operators. In Section 5, we present an illustrative example of the new approach in a financial decision making problem. Finally, in Section 6 we summarize the main conclusions of the paper.

2. PRELIMINARIES

In this Section, we briefly describe the main concepts of the IGOWA operator and the IHA operator.

2.1. IGOWA OPERATOR

The IGOWA operator was introduced in (Merigó and Gil-Lafuente, 2007) and it represents a generalization of the IOWA operator by using generalized means. Then, it is possible to include in the same formulation, different types of induced operators such as the IOWA operator or the induced OWG (IOWG) operator. It can be defined as follows.

Definition 1. *An IGOWA operator of dimension n is a mapping IGOWA:* $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *then*:

$$IGOWA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{1}$$

where b_j *is the* a_i *value of the IGOWA pair* $\langle u_i,a_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable,* a_i *is the argument variable and* λ *is a parameter such that* $\lambda \in (-\infty, \infty)$.

As we can see, if $\lambda = 1$, we get the IOWA operator. If $\lambda = 0$, the IOWG operator and if $\lambda = 2$, the IOWQA operator. Note that it is possible to further generalize the IGOWA operator by using quasi-arithmetic means. The result is the Quasi-IOWA operator.

2.2. INDUCED HYBRID AVERAGING OPERATOR

The induced HA (IHA) operator is an extension of the HA operator that uses order inducing variables. The HA operator (Xu and Da, 2003) is an aggregation operator that uses the WA and the OWA in the same formulation. Then, in the IHA operator it is possible to consider in the same problem, a complex attitudinal character of the decision maker and its subjective probability.

Definition 2. *An IHA operator of dimension n is a mapping* $IHA: R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that the sum of the weights is* 1 *and* $w_j \in [0,1]$, *then*:

$$IHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j *is the* $\hat{a}_i$ *value* ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), *of the IHA pair* $\langle u_i,a_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the* a_i, *with* $\omega_i \in [0, 1]$ *and the sum of the weights is* 1.

From a generalized perspective of the reordering step, we can distinguish between the descending IHA (DIHA) operator and the ascending IHA (AIHA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DIHA and w^*_{n-j+1} the jth weight of the AIHA operator. Different families of IHA operators are found by using a different manifestation in the weighting vector such as the step-IHA operator, the window-IHA operator, the median-IHA operator, the centered-IHA operator, etc. (Merigó, 2007).

3. THE INDUCED GENERALIZED HYBRID AVERAGING OPERATOR

The IGHA operator is a generalization of the IHA operator by using generalized means. It includes in the same formulation the weighted generalized mean and the IGOWA operator. It also uses order inducing variables in the reordering process. Then, this operator includes the WA, the OWA, the IOWA and the IOWG operator as special cases. It is defined as follows.

Definition 3. *An IGHA operator of dimension n is a mapping* $IGHA: R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that the sum of the weights is* 1 *and* $w_j \in [0,1]$, *then*:

$$IGHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{3}$$

where b_j *is the* $\hat{a}_i$ *value* ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), *of the IHA pair* $\langle u_i,a_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the* a_i, *with* $\omega_i \in [0, 1]$ *and the sum of the weights is* 1, *and* λ *is a parameter such that* $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, we can distinguish between the descending IGHA (DIGHA) operator and the ascending IGHA (AIGHA) operator. Note that they can be used in situations where the highest value is the best result and in situations where the lowest value is the best result. But in a more efficient context, it is better to use one of them for one situation and the other one for the other situation. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DIGHA and w^*_{n-j+1} the jth weight of the AIGHA operator. As we can see, the main difference is that in the AIGHA operator, the elements b_j (j= 1, 2, …, n) are ordered in an increasing way: $b_1 \leq b_2 \leq \ldots \leq b_n$ while in the DIGHA (or IGHA) they are ordered in a decreasing way.

The IGHA operator is commutative, monotonic and idempotent. It is commutative because any permutation of the arguments has the same evaluation. That is, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = IGHA(\langle u_1,d_1\rangle, \langle u_2,d_2\rangle\ldots, \langle u_n,d_n\rangle)$, where $(d_1,\ldots, d_n)$ is any permutation of the arguments $(a_1,\ldots, a_n)$. It is monotonic because if $a_i \geq d_i$, for all a_i, then, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) \geq IGHA(\langle u_1,d_1\rangle, \langle u_2,d_2\rangle\ldots, \langle u_n,d_n\rangle)$. It is idempotent because if $a_i = a$, for all a_i, then, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = a$.

Another interesting issue when analysing the IGHA operator is the problem of ties in the reordering step. In order to solve this problem, we recommend the policy developed by Yager and Filev (1999) where they replace each argument of the tied IOWA pairs by their average. For the IGHA operator, instead of using the arithmetic mean, we will replace each argument of the tied IGHA pairs by its generalized mean. Then, depending on the parameter λ, we will use a different type of mean to replace the tied arguments.

As it is explained in (Yager and Filev, 1999) for the IOWA operator, when studying the order inducing variable of the IGHA operator, we should note that the values used can be drawn from a space such that the only requirement is to have a linear ordering. Then, it is possible to use different kinds of attributes for the order inducing variables that permit us, for example, to mix numbers with words in the aggregations (Zadeh, 1996). Note that in some situations it is possible to use the implicit lexicographic ordering associated with words such as the ordering of words in dictionaries (Yager and Filev, 1999).

4. FAMILIES OF IGHA OPERATORS

In this Section, we will analyze different types of IGHA operators. We will distinguish between two general classes: those found in the weighting vector W and those found in the parameter λ.

4.1. ANALYSING THE WEIGHTING VECTOR *W*

By using a different manifestation of the weighting vector in the IGHA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the hybrid maximum, the hybrid minimum, the generalized mean (GM), the weighted generalized mean (WGM) and the IGOWA operator.

The hybrid maximum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{a_i\}$, then, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = \text{Max}\{a_i\}$. The hybrid minimum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{a_i\}$, then, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = \text{Min}\{a_i\}$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any λ, $IGHA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle) = b_k$, where b_k is the the a_i value of the IGHA pair $\langle u_i,a_i\rangle$ having the *k*th largest u_i. The GM is found when $w_j = 1/n$, and $\omega_i = 1/n$, for all a_i. The WGM is obtained when $w_j = 1/n$, for all a_i. The GOWA is found when $\omega_i = 1/n$, for all a_i, and the ordered position of u_i is the same than the ordered position of b_j such that b_j is the *j*th largest of a_i.

Following a similar methodology as it has been developed in (Merigó, 2007; Yager, 1993), we could study other particular cases of the IGHA operators such as the step-IGHA, the window-IGHA, the olympic-IGHA, the centered-IGHA operator, the S-IGHA operator, the median-IGHA, the E-Z IGHA, the nonmonotonic IGHA operator, etc.

For example, when $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-IGHA operator. Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, and the initial position of the highest u_i is also the initial position of the highest a_i, then, the window-IGHA is transformed in the hybrid maximum. If $m = 1$, $k = n$, and the initial position of the lowest u_i is also the initial position of the lowest a_i, then, the window-IGHA becomes the hybrid minimum.

If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic-IGHA. Note that if $n = 3$ or $n = 4$, the olympic-IGHA is transformed in the median-IGHA and if $m = n - 2$ and $k = 2$, the window-IGHA becomes the olympic-IGHA. Also note that the olympic-IGHA is transformed in the olympic hybrid generalized average if $w_p = w_q = 0$, such that $u_p = \text{Max}_i\{a_i\}$ and $u_q = \text{Min}_i\{a_i\}$, and for all others $w_j = 1/(n - 2)$.

Another type of aggregation that could be used is the E-Z IGHA weights that it is based on the E-Z OWA weights (Yager, 2006). In this case, we should distinguish between two classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n. Note that the E-Z IGHA weights becomes the E-Z GHA weights for the first class if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the *j*th largest of a_i, from $j = 1$ to k. And for the second class, the E-Z IGHA weights becomes the E-Z GHA weights if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the *j*th largest of a_i, from $j = n - k + 1$ to n.

A further type that could be used is the median-IGHA operator. In this case, we should distinguish between two cases. If n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the argument a_i with the $[(n + 1)/2]$th largest u_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest u_i. Note that it is also possible to use the weighted IGHA median. We select the argument a_i that has the *k*th largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5. Note that if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, then, the IGHA median and the weighted IGHA median become the GHA median and the weighted GHA median, respectively.

Another family of aggregation operators that could be used in the IGHA operator is the centered-IGHA weights. This type of operator has been suggested by Yager (2007) for the OWA operator. Following the same methodology, we could define a centered-IGHA operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-IGHA operator. Note that the generalized mean is an example of this particular case of centered-IGHA operator. Another particular situation of the centered-IGHA operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered-IGHA operator. For this situation, we find the median-IGHA as a particular case.

A further interesting family is the S-IGHA operator based on the S-OWA operator (Yager, 1993; Yager and Filev, 1994). It can be divided in three classes, the "orlike", the "andlike" and the generalized S-IGHA operator. The "orlike" S-IGHA operator is found when $w_p = (1/n)(1 - \alpha) + \alpha$, $u_p = \text{Max}\{a_i\}$, and $w_j = (1/n)(1 - \alpha)$ for all $j \neq p$ with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we get the weighted generalized mean and if $\alpha = 1$, we get the hybrid maximum. The "andlike" S-IGHA operator is found when $w_q = (1/n)(1 - \beta) + \beta$, $u_q = \text{Min}\{a_i\}$, and $w_j = (1/n)(1 - \beta)$ for all $j \neq q$ with $\beta \in [0, 1]$. Note that in this class, if $\beta = 0$ we get the weighted generalized mean and if $\beta = 1$, we get the hybrid minimum. Finally, the generalized S-IGHA operator is obtained when $w_p = (1/n)(1 - (\alpha + \beta) + \alpha$, with $u_p = \text{Max}\{a_i\}$; $w_q = (1/n)(1 - (\alpha + \beta) + \beta$, with $u_q = \text{Min}\{a_i\}$; and $w_j = (1/n)(1 - (\alpha + \beta)$ for all $j \neq p,q$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-IGHA operator becomes the "andlike" S-IGHA operator and if $\beta = 0$, it becomes the "orlike" S-IGHA operator.

Finally, note that other families could be studied such as the Gaussian IGHA weights, the nonmonotonic-IGHA operator, etc. For more information, see (Merigó, 2007).

4.2. ANALYSING THE PARAMETER λ

If we analyze different values of the parameter λ, we obtain another group of particular cases such as the usual IHA operator, the induced hybrid geometric averaging (IHGA) operator, the induced hybrid harmonic averaging (IHHA) operator and the induced hybrid quadratic averaging (IHQA) operator.

When $\lambda = 1$, we get the IHA operator.

$$IGHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{4}$$

From a generalized perspective of the reordering step we can distinguish between the DIHA operator and the AIHA operator. Note that if $w_j = 1/n$, for all a_i, we get the WA and if $\omega_j = 1/n$, for all a_i, we get the IOWA operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the arithmetic mean (AM).

When $\lambda = 0$, the IGHA operator becomes the IHGA operator.

$$IGHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \prod_{j=1}^{n} b_j^{\,w_j} \tag{5}$$

In this case, it is also possible to distinguish between descending (DIHGA) and ascending (AIHGA) orders. Note that if $w_j = 1/n$, for all a_i, we get the WGM and if $\omega_j = 1/n$, for all a_i, we get the IOWG operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the geometric average (GA).

When $\lambda = -1$, we get the IHHA operator.

$$IGHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}} \tag{6}$$

In this case, we get the descending IHHA (DIHHA) operator and the ascending IHHA (AHHA) operator. Note that if $w_j = 1/n$, for all a_i, we get the weighted harmonic mean (WHM) and if $\omega_j = 1/n$, for all a_i, we get the induced ordered weighted harmonic averaging (IOWHA) operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the harmonic mean (HM).

When $\lambda = 2$, we get the IHQA operator.

$$IGHA(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^2\right)^{1/2} \tag{7}$$

In this case, we get the descending IHQA (DIHQA) operator and the ascending IHQA (AIHQA) operator. If $w_j = 1/n$, for all a_i, we get the WQM and if $\omega_j = 1/n$, for all a_i, we get the induced OWQA (IOWQA) operator. If $w_j = 1/n$, and $\omega_j = 1/n$, for all a_i, then, we get the quadratic mean (QM).

Note that we could analyze other families by using different values in the parameter λ. Also note that it is possible to study these families individually. Then, we could develop for each case, a similar analysis as it has been developed in Section 3 and 4.1 where we study different properties and families of the aggregation operator.

5. QUASI-IHA OPERATOR

Going a step further, it is possible to generalize the IGHA operator by using quasi-arithmetic means in a similar way as it was done for the IGOWA operator (Merigó and Gil-Lafuente, 2007). The result is the Quasi-IHA operator which is a hybrid version of the Quasi-OWA (Fodor et. al., 1995) and the Quasi-IOWA operator (Merigó and Gil-Lafuente, 2007). It can be defined as follows.

Definition 4. *A Quasi-IHA operator of dimension n is a mapping QIHA:* $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that the sum of the weights is* 1 *and* $w_j \in [0,1]$, *then*:

$$\text{Quasi-IHA}(\langle u_1,a_1\rangle,\ldots,\langle u_n,a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(b_{(j)}\right)\right) \tag{8}$$

where b_j *is the* $\hat{a}_i$ *value* ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), *of the Quasi-IHA pair* $\langle u_i,a_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the* a_i, *with* $\omega_i \in [0, 1]$ *and the sum of the weights is* 1.

As we can see, we replace b^{λ} with a general continuous strictly monotone function *g(b)*. In this case, the weights of the ascending and descending versions are also related by $w_j = w^*_{n-j+1}$, where w_j is the *j*th weight of the Quasi-DIHA and w^*_{n-j+1} the *j*th weight of the Quasi-AIHA operator.

Note that all the properties and particular cases commented in the IGHA operator, are also included in this generalization. For example, we could study different families of Quasi-IHA operators such as the Quasi-IOWA, the Quasi-WA, the Quasi-step-IHA, the Quasi-window-IHA, the Quasi-median-IHA, the Quasi-olympic-IHA, the Quasi-centered-IHA, etc.

6. ILLUSTRATIVE EXAMPLE

In the following, we are going to develop an illustrative example of the new approach in a decision making problem. We will study an investment selection problem where an investor is looking for an optimal investment. Note that other decision making applications could be developed such as the selection of financial products (Merigó and Gil-Lafuente, 2007), the selection of human resources (Merigó, 2007), etc.

We will analyze different particular cases of the IGHA operator such as the AM, the WA, the OWA, the IOWA, the HA, the IHA, the IHQA, etc.

Assume an investor wants to invest some money in an enterprise in order to get high profits. Initially, he considers five possible alternatives.

- A_1 is a computer company.

- A_2 is a chemical company.
- A_3 is a food company.
- A_4 is a car company.
- A_5 is a TV company.

In order to evaluate these investments, the investor uses a group of experts. This group of experts considers that the key factor is the economic environment of the economy. After careful analysis, they consider five possible situations for the economic environment: S_1 = Negative growth rate, S_2 = Growth rate near 0, S_3 = Low growth rate, S_4 = Medium growth rate, S_5 = High growth rate. The expected results depending on the situation S_i and the alternative A_k are shown in Table 1.

Table 1: Payoff matrix

	S_1	S_2	S_3	S_4	S_5
A_1	50	70	40	80	30
A_2	30	60	50	90	50
A_3	60	40	30	80	40
A_4	20	70	70	50	50
A_5	70	30	40	60	40

In this problem, the experts assume the following weighting vector: $W = (0.1, 0.2, 0.2, 0.2, 0.3)$. Due to the fact that the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the experts use order inducing variables to express it. The results are represented in Table 2.

Table 2: Order inducing variables

	S_1	S_2	S_3	S_4	S_5
A_1	7	9	6	5	8
A_2	4	3	6	8	7
A_3	2	8	4	3	6
A_4	5	6	9	2	7
A_5	8	4	3	6	5

With this information, we can aggregate the expected results for each state of nature in order to take a decision. In Table 3, we present different results obtained by using different types of IGHA operators.

Table 3: Aggregated results

	AM	WA	OWA	HA	IOWA	IHA	IHQA
A_1	54	52	49	46.5	55	63.5	73.77
A_2	56	58	50	50.5	56	61	65.34
A_3	50	48	45	43	52	59	65.19
A_4	52	55	47	48.5	47	46.5	48.70
A_5	48	45	44	42	49	53.5	55.87

If we establish an ordering of the alternatives, a typical situation if we want to consider more than one alternative, then, we get the following results shown in Table 4.

Table 4. Ordering of the investments

	Ordering		*Ordering*
AM	$A_2 \succ A_1 \succ A_4 \succ A_3 \succ A_5$	*IOWA*	$A_2 \succ A_1 \succ A_3 \succ A_5 \succ A_4$
WA	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$	*IHA*	$A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$
OWA	$A_2 \succ A_1 \succ A_4 = A_5 \succ A_3$	*IHQA*	$A_1 \succ A_2 \succ A_3 \succ A_5 \succ A_4$
HA	$A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$		

As we can see, depending on the aggregator operator used, the ordering of the investments may be different. Then, it is clear that each particular case of the IGHA may lead to different results and decisions. Obviously, the decision maker will select the particular case that it is in accordance with its interests.

6. CONCLUSIONS

In this paper we have presented the IGHA operator. It is a generalization of the OWA operator that uses the characteristics of three well known aggregation operators: the HA, the GOWA and the IOWA operator. Therefore, this operator uses a unifying framework between the WA and the OWA, generalized means and order inducing variables, in the same formulation. We have studied some of the main properties of this new aggregation operator. We have further generalized it by using quasi-arithmetic means. Then, we have obtained the Quasi-IHA operator.

We have also presented a numerical example of the new approach. We have developed a financial decision making problem about the selection of investments. The main idea behind this aggregation operator is that it includes a wide range of particular cases. Then, depending on the particular case used, the results and decisions may be different.

In future research, we expect to develop further extensions by adding new characteristics in the problem such as the use of uncertain information in the problem represented in the form of interval numbers, fuzzy numbers, linguistic variables, etc. We will also consider other business decision making problems such as human resource management, strategic management, etc.

ACKNOWLEDGEMENTS

We would like to thank the anonymous referees for their valuable comments that have improved the quality of the paper.

REFERENCES

Beliakov, G., 2005. Learning Weights in the Generalized OWA Operators, *Fuzzy Optimization and Decision Making* 4: 119-130.

Beliakov, G., A. Pradera, T. Calvo, 2007. *Aggregation Functions: A guide for practicioners*, Springer-Verlag, Berlin.

Calvo, T., G. Mayor, R. Mesiar, 2002. *Aggregation Operators: New Trends and Applications*, Physica-Verlag, New York.

Dujmovic, J., 1974. Weighted conjunctive and disjunctive means and their application in system evaluation, *Publikacije Elektrotechnickog Faculteta Beograd, Serija Matematika i Fizika,* 483: 147-158.

Dyckhoff, H., W. Pedrycz, 1984. Generalized means as model of compensative connectives, *Fuzzy Sets and Systems* 14: 143-154.

Fodor, J., J.L. Marichal, M. Roubens, 1995. Characterization of the ordered weighted averaging operators, *IEEE Transactions on Fuzzy Systems* 3: 236-240.

Hardy, G.H., J.E. Littlewood, G. Pólya, 1934. *Inequalities*, Cambridge University Press, Cambridge.

Karayiannis, N., 2000. Soft Learning Vector Quantization and Clustering Algorithms Based on Ordered Weighted Aggregation Operators, *IEEE Transactions on Neural Networks* 11: 1093-1105.

Kolmogoroff, A.N., 1930. Sur la notion de la moyenne, *Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez.* 12: 388-391.

Merigó, J.M., 2007. *Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión empresarial*, Unpublished thesis (In Spanish), Department of Business Administration, University of Barcelona.

Merigó, J.M., M. Casanovas, 2007a. Induced aggregation operators in decision making with the Dempster-Shafer belief structure, *Working Papers in Economics 184*, University of Barcelona.

Merigó, J.M., M. Casanovas, 2007b. The fuzzy generalized OWA operator, In *Proceedings of the conference SIGEF 2007*, pp. 504-517, Poiana-Brasov, Romania.

Merigó, J.M., M. Casanovas, 2007c. The uncertain generalized OWA operator and its application in the selection of financial strategies, In *Proceedings of the international conference AEDEM 2007*, pp. 547-556, Krakow, Poland.

Merigó, J.M., A.M. Gil-Lafuente, 2007. The induced generalized OWA operator, In *Proceedings of the conference EUSFLAT 2007* volume 2, pp. 463-470, Ostrava, Czech Republic.

Nagumo, M., 1930. Uber eine klasse der mittelwerte, *Japanese Journal of Mathematics* 6: 71-79.

Schaefer, P.A., H.B. Mitchell, 1999. A generalized OWA operator, *International Journal of Intelligent Systems*, 14: 123-143.

Wang, J.H., J. Hao, 2006. A new version of 2-tuple fuzzy linguistic representation model for computing with words, *IEEE Transactions on Fuzzy Systems*, 14: 435-445.

Xu, Z.S., 2004. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information Sciences* 166: 19-30.

Xu, Z.S., 2006. A Note on Linguistic Hybrid Arithmetic Averaging Operator in Multiple Attribute Group Decision Making with Linguistic Information, *Group Decision and Negotiation* 15: 593-604.

Xu, Z.S., Q.L. Da, 2003. An Overview of Operators for Aggregating the Information, *International Journal of Intelligent Systems* 18: 953-969.

Yager, R.R., 1988. On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making, *IEEE Transactions on Systems, Man and Cybernetics*, B 18: 183-190.

Yager, R.R., 1992. On generalized measures of realization in uncertain environments, *Theory and Decision* 33: 41-69.

Yager, R.R., 1993. Families of OWA operators, *Fuzzy Sets and Systems* 59: 125-148.

Yager, R.R., 1996. Quantifier guided aggregation using OWA operators, *International Journal of Intelligent Systems* 11: 49-73.

Yager, R.R., 2003a. E-Z OWA weights, In *Proceedings of the 10th International Fuzzy Systems Association (IFSA) World Congress*, Istanbul, Turkey, pp. 39-42.

Yager, R.R., 2003b. Induced aggregation operators, *Fuzzy Sets and Systems* 137: 59-69.

Yager, R.R., 2004. Generalized OWA Aggregation Operators, *Fuzzy Optimization and Decision Making* 3: 93-107.

Yager, R.R., 2007. Centered OWA operators, *Soft Computing* 11: 631-639.

Yager, R.R., D.P. Filev, 1994. Parameterized "andlike" and "orlike" OWA Operators, *International Journal of General Systems* 22: 297-316.

Yager, R.R., D.P. Filev, 1999. Induced ordered weighted averaging operators, *IEEE Transactions on Systems, Man and Cybernetics* 29: 141-150.

Yager, R.R., J. Kacprzyk, 1997. *The Ordered Weighted Averaging Operators: Theory and Applications*, Kluwer Academic Publishers, Norwell, MA.