DECISION MAKING WITH DEMPSTER-SHAFER THEORY AND UNCERTAIN INDUCED AGGREGATION OPERATORS

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ABSTRACT

We develop a new approach for decision making with Dempster-Shafer (D-S) theory of evidence. We focus on a problem where the available information is uncertain and it can be assessed with interval numbers. In order to aggregate the information, we suggest the use of different types of uncertain induced aggregation operators such as the uncertain induced ordered weighted averaging (UIOWA) and the uncertain induced hybrid averaging (UIHA) operator. We also develop an application of the new approach in a decision making problem about selection of investments.

I. INTRODUCTION

The Dempster-Shafer (D-S) theory of evidence (Dempster, 1967; Shafer, 1976) provides a unifying framework for representing uncertainty because it includes the situations of risk and ignorance as special cases. For further reading on the D-S theory, we recommend for example (Yager and Liu, 2008).

Usually, when using the D-S theory in decision making, it is assumed that the available information are exact numbers (Engemann et al. 1996; Merigó and Casanovas, 2007; Yager, 1992; 2004). However, this may not be the real situation found in the decision making problem because often, the available information is vague or imprecise and it is necessary to use another approach such as the use of interval numbers.

Going a step further, the aim of this paper is to suggest a new approach for uncertain decision making with D-S theory by using uncertain induced aggregation operators. Then, we will be able to use in the same formulation a unifying framework between ignorance and risk, uncertain information assessed with interval numbers and a reordering process in the aggregation step that uses order inducing variables. We will consider different types of uncertain induced aggregation operators such as the uncertain induced ordered weighted averaging (UIOWA) and the uncertain induced hybrid averaging (UIHA) operator.

This paper is organized as follows. First, we briefly review some basic concepts. Then we introduce the new approach when the information is aggregated with the UIOWA or with the UIHA operator. Next, we present an illustrative example of the new approach in a financial decision making problem. We end the paper summarizing the main results.

II. PRELIMINARIES

In this Section, we briefly review some basic concepts about the interval numbers, the UIOWA and the UIHA operator and the D-S theory.

The interval number is a very useful and simple technique for representing the uncertainty. It has been used in an astonishingly wide range of applications. For further reading, see for example (Moore, 1966). In the literature, we find different types of interval numbers. For example, if we assume a 4-tuple (a_1, a_2, a_3, a_4), that is to say, a quadruplet; we could consider that a_1 and a_4 represents the minimum and the maximum of the interval number, and a_2 and a_3, the interval with the highest probability or possibility, depending on the use we want to give to the interval numbers. Note that $a_1 \leq a_2 \leq a_3 \leq a_4$. If $a_1 = a_2 = a_3 = a_4$, then, the interval number is an exact number and if $a_2 = a_3$, it is a 3-tuple known as triplet. Some basic operations with two triplets A and B are: $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$; $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$; $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$. Note that other operations could be studied, see for example (Moore, 1966).

The uncertain induced OWA (UIOWA) operator was introduced by Xu (2006). It is an extension of the OWA operator (Beliakov et al. 2007; Merigó 2007; Yager, 1988; 1993) that uses the main characteristics of two well known aggregation operators: the induced OWA (Yager and Filev, 1999) and the uncertain OWA operator (Xu and Da, 2003). Then, it uses interval numbers for representing the uncertain information and a reordering process that it is based on order inducing variables. It can be defined as follows:

Definition 1. *Let Ω be the set of interval numbers. An UIOWA operator of dimension n is a mapping UIOWA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then*:

$$UIOWA(\langle u_1,\tilde{a}_1\rangle, \ldots, \langle u_n,\tilde{a}_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the $\tilde{a}_i$ value of the UIOWA pair $\langle u_i, \tilde{a}_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and the $\tilde{a}_i$ are interval numbers.

The uncertain induced hybrid averaging operator is an extension of the hybrid averaging (Xu, 2006; Xu and Da, 2003) that uses the weighted average (WA) and the OWA operator, at the same time. It also uses interval numbers for representing the uncertain information and a reordering process based on inducing variables. It is defined as follows:

Definition 2. *Let Ω be the set of interval numbers. An UIHA operator of dimension n is a mapping UIHA: $\Omega^n \rightarrow \Omega$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then*:

$$UIHA(\langle u_1,\tilde{a}_1\rangle, \ldots, \langle u_n,\tilde{a}_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j is the $\hat{a}_i$ ($\hat{a} = n\omega_i\tilde{a}_i$, $i = 1,2,\ldots,n$) value of the UIHA pair $\langle u_i, \tilde{a}_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $\tilde{a}_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1, and the $\tilde{a}_i$ are interval numbers.

The D-S theory of evidence (Dempster, 1967; Shafer, 1976) provides a unifying framework for representing uncertainty as it can include the situations of risk and ignorance as special cases. Note that the case of certainty is also included as it can be seen as a particular case of risk or ignorance. Since its appearance, the D-S theory has been applied in a wide range of applications (Yager and Liu, 2008).

Definition 3. *A D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for j = 1,…,n, called focal elements and a mapping m, called the basic probability assignment, defined as, m*: $2^X \rightarrow [0, 1]$ *such that*: 1) $m(B_j) \in [0, 1]$; 2) $\sum_{j=1}^{n} m(B_j) = 1$; 3) $m(A) = 0$, $\forall A \neq B_j$.

III. UIOWA AND UIHA OPERATORS IN DECISION MAKING WITH D-S THEORY

A new approach for decision making with D-S theory is possible by using uncertain induced aggregation operators. The main advantages of using this type of aggregation are the possibility of dealing with uncertain information, the possibility of using an aggregation that provides a parameterized family of aggregation operators between the maximum and the minimum, and the possibility of using a general formulation in the reordering of the arguments by using inducing variables. Note that in this paper we will focus on the UIOWA and the UIHA operators, but it is also possible to consider other types of uncertain induced aggregation operators. The motivation for using interval numbers appear because sometimes, the available information is not clear and it is necessary to assess it with another approach such as the use of interval numbers. Although the information is uncertain and it is difficult to take decisions with it, at least we can represent the best and worst possible scenarios. The decision process can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, …, A_q\}$ with states of nature $\{S_1, …, S_n\}$. $\tilde{a}_{ih}$ is the uncertain payoff, given in the form of interval numbers, to the decision maker if he selects alternative A_i and the state of nature is S_h. The knowledge of the state of nature is captured in terms of a belief structure *m* with focal elements $B_1, …, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative which gives the best result to the decision maker. In order to do so, we should follow the following steps:

Step 1: Calculate the uncertain payoff matrix.

Step 2: Calculate the belief function *m* about the states of nature.

Step 3: Calculate the collection of weights, *w*, to be used in the UIOWA aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the interests of the decision maker (Merigó, 2007; Yager, 1988; 1993; 2007; Yager and Filev, 1994).

Step 4: Determine the uncertain payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of *i* and *k*. Hence $M_{ik} = \{a_{ih} \mid S_h \in B_k\}$.

Step 5: Calculate the uncertain aggregated payoff, V_{ik} = UIOWA(M_{ik}), using Eq. (1), for all the values of *i* and *k*.

Step 6: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k) \tag{3}$$

Step 7: Select the alternative with the largest C_i as the optimal.

In some situations, the decision maker could prefer to use another type of uncertain aggregation operator such as the UIHA operator. The main advantage of this operator is that it uses the characteristics of the UWA and the UIOWA in the same aggregation. Then, if we introduce this operator in decision making with D-S theory, we are able to develop a unifying framework that includes in the same formulation probabilities, UWAs and UIOWAs.

In order to use this type of aggregation in D-S framework we should consider that now in *Step* 3, when calculating the collection of weights to be used in the aggregation, we are using two weighting vectors because we are mixing in the same problem the UWA and the UIOWA. In *Step* 5, when calculating the uncertain aggregated payoff, we should use the UIHA operator instead of the UIOWA operator by using Eq. (2).

By choosing a different manifestation in the weighting vector of the UIOWA and the UIHA operator, we are able to develop different families of UIOWA and UIHA operators. For example, it is possible to obtain the UA and the UWA. The UA is found with the UIOWA when $w_j = 1/n$, for all $\tilde{a}_i$ and the UWA if $u_i > u_{i+1}$, for all a_i. The UOWA is obtained when the ordered position of the values of the u_i is the same than j. Note that other families could be used such as the ones explained in (Merigó, 2007, Yager, 1993).

IV. ILLUSTRATIVE EXAMPLE

In the following, we are going to develop an application of the new approach in a decision making problem about the selection of financial strategies. We will use the example considering a wide range of uncertain induced aggregation operators such as the UA, the UWA, the UOWA, the UIOWA and the UIHA operator.

Assume a company is planning its financial strategy for the next year and they consider 5 possible financial strategies to follow: A_1 = Financial strategy 1; A_2 = Financial strategy 2; A_3 = Financial strategy 3; A_4 = Financial strategy 4; A_5 = Financial strategy 5.

In order to evaluate these financial strategies, the company uses a group of experts. They consider that the key factor is the economic situation of the company for the next year. After careful analysis, the experts have considered five possible situations that could happen in the future: S_1 = Very bad, S_2 = Bad, S_3 = Normal, S_4 = Good, S_5 = Very good. Then, depending on the uncertain situations that could happen, the experts establish the uncertain payoff matrix. As the available information about the future benefits of the company is very imprecise, the experts use interval numbers. The results are shown in Table 1.

Table 1. Uncertain payoff matrix

	S_1	S_2	S_3	S_4	S_5
A_1	(10,20,30)	(40,50,60)	(70,80,90)	(40,50,60)	(50,60,70)
A_2	(50,60,70)	(30,40,50)	(20,30,40)	(60,70,80)	(40,50,60)
A_3	(70,80,90)	(40,50,60)	(30,40,50)	(30,40,50)	(40,50,60)
A_4	(30,40,50)	(50,60,70)	(20,30,40)	(50,60,70)	(60,70,80)

After careful analysis of the information, the experts have obtained some probabilistic information about which state of nature will happen in the future. This information is represented by the following belief structure about the states of nature: $B_1 = \{S_2, S_3, S_4\} = 0.3$; $B_2 = \{S_1, S_2, S_5\} = 0.3$; $B_3 = \{S_1, S_2, S_3, S_4\} = 0.4$.

The attitudinal character of the company is very complex because it involves the opinion of different members of the board of directors. Therefore, the experts use order inducing variables for analyzing the attitudinal character of the enterprise. The results are shown in Table 2.

Table 2. Order inducing variables

	S_1	S_2	S_3	S_4	S_5
A_1	30	22	16	35	26
A_2	12	18	24	20	30
A_3	16	11	21	33	25
A_4	30	26	12	18	24

The experts establish the following weighting vectors for both the UWA and the UIOWA operator: $W_3 = (0.3, 0.3, 0.4)$; $W_4 = (0.2, 0.2, 0.3, 0.3)$; $W_5 = (0.1,0.2, 0.2, 0.2, 0.3)$. With this information, we can obtain the aggregated payoffs shown in Table 3.

Table 3. Uncertain aggregated payoffs

	UA	*UWA*	*UOWA*	*UIOWA*	*UIHA*
V_{11}	(50,60,70)	(49,59,69)	(49,59,69)	(52,62,72)	(52,62,72)
V_{12}	(33.3,43.3,53.3)	(35,45,55)	(31,41,51)	(34,44,54)	(40,50,60)
V_{13}	(40,50,60)	(43,53,63)	(37,47,57)	(37,47,57)	(42,51,60)
V_{21}	(36.6,46.6,56.6)	(39,49,59)	(35,45,55)	(36,46,56)	(36,46,56)
V_{22}	(40,50,60)	(40,50,60)	(39,49,59)	(41,51,61)	(37,46.5,56)
V_{23}	(40,50,60)	(40,50,60)	(37,47,57)	(37,47,57)	(32.5,41,49.5)
V_{31}	(33.3,43.3,53.3)	(33,43,53)	(33,43,53)	(34,44,54)	(34,44,54)
V_{32}	(50,60,70)	(49,59,69)	(49,59,69)	(49,59,69)	(44.5,54.5,64.5)
V_{33}	(42.5,52.5,62.5)	(40,50,60)	(40,50,60)	(40,50,60)	(34.5,43,51.5)
V_{41}	(40,50,60)	(41,51,61)	(38,48,58)	(38,48,58)	(38,48,58)
V_{42}	(46.6,56.6,66.6)	(48,58,68)	(45,55,65)	(48,58,68)	(55.5,66,76.5)
V_{43}	(37.5,47.5,57.5)	(37,47,57)	(35,45,55)	(35,45,55)	(34,43,52)

Once we have the aggregated results, we have to calculate the uncertain generalized expected value. The results are shown in Table 4.

Table 4. Uncertain generalized expected value

	UA	*UWA*	*UOWA*	*UIOWA*	*UIHA*
A_1	(41,51,61)	(42.4,52.4,62.4)	(38.8,48.8,58.8)	(40.6,50.6,60.6)	(44.4,54,63.6)
A_2	(39,49,59)	(39.7,49.7,59.7)	(37,47,57)	(37.9,47.9,57.9)	(34.9,44.15,53.4)
A_3	(42,52,62)	(40.6,50.6,60.6)	(40.6,50.6,60.6)	(40.9,50.9,60.9)	(37.35,46.75,56.15)
A_4	(41,51,61)	(41.5,51.5,61.5)	(38.9,48.9,58.9)	(39.8,49.8,59.8)	(41.65,51.4,61.15)

As we can see, depending on the uncertain aggregation operator used, the results and decisions may be different. With the UA, the UOWA and the UIOWA the optimal choice is A_3. And with the UWA and the UIHA, the best result is A_1.

V. CONCLUSIONS

We have studied the D-S theory of evidence in decision making with uncertain information assessed with interval numbers. By using interval numbers, we can represent uncertain situations where the results are not clear. We have also used uncertain induced aggregation operators because it gives more flexibility in the attitudinal character of the decision maker. Mainly, we have focused on the UIOWA and the UIHA operators. Then, we have obtained two new aggregation operators: the BS-UIOWA and the BS-UIHA operator. We have analysed some of the main properties and different particular cases.

We have also developed an application of the new approach in a business decision making problem. We have seen the usefulness of this approach about using probabilities, UWAs and UIOWAs in the same problem. We have also seen that depending on the aggregation operator used, the results and decisions may be different.

REFERENCES

Beliakov, Gleb, Ana Pradera and Tomasa Calvo. Aggregation Functions: A guide for practitioners, Springer-Verlag, Berlin, 2007.

Casanovas, Montserrat and José M. Merigó. "Using fuzzy OWA operators in decision making with Dempster-Shafer belief structure", in: Proceedings of the AEDEM International Conference, Krakow, Poland, 2007, pp. 475-486.

Dempster, Arthur P. "Upper and lower probabilities induced by a multi-valued mapping", Annals of Mathematical Statistics, 38, 1967, 325-339.

Engemann, Kurt J., Holmes E. Miller and Ronald R. Yager. "Decision making with belief structures: an application in risk management", International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 4, 1996, 1-26.

Merigó, José M. Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión empresarial, Unpublished thesis (In Spanish), Department of Business Administration, University of Barcelona, 2007.

Merigó, José M. and Montserrat Casanovas. "Induced aggregation operators in decision making with the Dempster-Shafer belief structure", Working Papers in Economics, 184, University of Barcelona, 2007.

Moore, Ramon E. Interval Analysis, Prentice Hall, Englewood Cliffs, NJ, 1966.

Shafer, Glen A. Mathematical Theory of Evidence, Princeton University Press, Princeton, NJ, 1976.

Xu, Zeshui. "Induced uncertain linguistic OWA operators applied to group decision making", Information Fusion, 7, 2006, 231-238.

Xu, Zeshui and Qing-Li Da. "The Uncertain OWA Operator", International Journal of Intelligent Systems, 17, 2002 569-575.

Xu, Zeshui and Qing-Li Da. "An Overview of Operators for Aggregating the Information", International Journal of Intelligent Systems, 18, 2003, 953-969.

Yager, Ronald R. "On Ordered Weighted Averaging Aggregation Operators in Multi-Criteria Decision Making", IEEE Transactions on Systems, Man and Cybernetics B,'18, 1988, 183-190.

Yager, Ronald R. "Decision Making Under Dempster-Shafer Uncertainties", International Journal of General Systems, 20, 1992, 233-245.

Yager, Ronald R. "Families of OWA operators", Fuzzy Sets and Systems, 59, 1993, 125-148.

Yager, Ronald R. Uncertainty modeling and decision support, Reliability Engineering and System Safety, 85, 2004, 341-354.

Yager, Ronald R. and Dimitar P. Filev. "Induced ordered weighted averaging operators", IEEE Transaction on Systems, Man and Cybernetics, 29, 1999, 141-150.

Yager, Ronald R. and Liping Liu. Classic Works of the Dempster-Shafer Theory of Belief Functions, Springer-Verlag, Berlin, 2008.