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Simulation of High Available Fault-Tolerant Systems by Simulating Stiff and Large Markov Chains

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1. Introduction

A fault-tolerant system is characterized by its capability to return automatically to an operational state after a fault. When such system is highly available, the failure and recovery rates have very different orders of magnitude. Consequently, the continuous time Markov chain (CTMC) used to study the behavior of this system is generally stiff and often have a large state space. So the evaluation of stationary quantitative performances, by simulation, comes back to simulate the stationary behavior of a stiff and large Markov chain. With a standard Monte Carlo simulation, we are confronted to a time complexity problem. There are many variance reduction techniques which answer, each in it's manner, to this problem. The Importance Sampling (IS) is a variance reduction technique which is well adapted to the simulation of the embedded discrete time Markov chain (EMC). This technique encourage the rare event (i.e. rare transitions of the EMC) and consequently allows a reduction in the time of simulation with respect to the standard Monte Carlo simulation. But for a stiff and large EMC, the simulation spends a very long time before reaching a state from where there is a possible rare transition. So that the simulation turns a long time before the intervention of the IS technique.

In this paper, we propose a new approach, based on the IS and on a distance technique, which reduces considerably this time.

The paper is organized as follows : In section 2, we define the model and the stationary performance measure to simulate. The philosophy and the strategy of IS is presented in section 3. Our new approach is discussed in section 4. In section 5, we study a numerical example which shows the usefulness of the proposed approach. A summary is given in the last section.

2. Model and stationary performance

We consider an irreducible CTMC $X = \{X_t \ , \ t \in \mathbb{R}^+\}$ with finite state space E and transition rate matrix $A = (a_{i,j})$. The set E can be written as $E = O \cup F$ where O and F are the operational and the nonoperational state space respectively. We define

$$\partial O_F = \{i \in O \ / \ \exists j \in F \ : \ p_{ij} > 0\}$$

where

$$p_{ij} = \frac{a_{i,j}}{\mid a_{i,i} \mid} 1\!\!1_{\{i \neq j\}}(i,j) \quad , \quad i,j \in E$$

and $P = (p_{ij})_{i,j \in E}$ is the transition matrix of the EMC of X and $1\!\!1_A(x)$ is the indicator function of the set A. We denote the EMC by $Z = \{Z_n \ ; \ n \in \mathbb{N}\}$. The main aim of this work is to simulate the stationary performance measure of the model X by simulating Z. We assume that X starts in the perfect state i_0 (i.e. all components of the system are operational). Letting the sequence (τ_i) such that :

$$\tau_0 = \inf\{n > 0 \ \mid \ Z_n = i_0\}$$

$$\tau_i = \inf\{n > \tau_{i-1} \ \mid \ Z_n = i_0\} \quad , \quad i > 0$$

then the process Z is regenerative and each τ_i is a regeneration point for Z.Now, we denote by $(\pi_i)_{i \in E}$ the stationary probability distribution of X and consider that a realization is a sequence of states which starts at i_0 and finishes when Z becomes to i_0. Our focus is concentrated on the simulation of the quantity $\mathbb{E}[f(Z)]$, where f is a function of state and the notation $\mathbb{E}[.]$ is used for expectation. Assuming that $\mathbb{E}\left[\mid f(Z) \mid\right] < +\infty$, this quantity can be expressed as a ratio of expectations. This can be seen in the following proposition.

Proposition. *If* $\sum_{j \in E} \mid f(j) \mid \pi_j < +\infty$ *then*

$$\sum_{j \in E} f(j)\pi_j = \frac{\mathbb{E}\left[\sum_{k=0}^{\tau_0 - 1} f(Z_k)h(Z_k)\right]}{\mathbb{E}\left[\sum_{k=0}^{\tau_0 - 1} h(Z_k)\right]}$$

where $h(i) = \frac{1}{|a_{ii}|}$ *is the holding time in state* i *and* $(Z_0, ..., Z_{\tau_0 - 1})$ *is the first regenerative cycle of* Z.

The proof, which is based on the regenerative property of Z, is given in Chung (1960).

3. Importance Sampling (IS)

For simplicity, we present the philosophy of IS through the simulation of the quantity $\theta = \mathbb{E}[f(U)]$ where U is a random variable with probability density φ_u and f is a function of state, such that $\mathbb{E}[|f(U)|] < +\infty$: Let V another random variable with probability density φ_v such that :

$$\varphi_v(x) = 0 \quad if \quad \varphi_u(x) = 0$$

$$\varphi_v(x) > 0 \quad if \quad \varphi_u(x) > 0$$

The quantity θ can be written :

$$\theta = \mathbb{E}[f(U)] = \mathbb{E}[f(V)R(V)] \quad where \quad R(x) = \frac{\varphi_u(x)}{\varphi_v(x)}$$

It is proved in Nasroallah (1991) that for x such that $R(x) << 1$, the simulation of $\mathbb{E}[f(V)R(V)]$ leads to a reduction variance with respect to the simulation of $\mathbb{E}[f(U)]$. It means that if we take φ_v such that $\varphi_v(x) > \varphi_u(x)$ (i.e. we increase the chance of occurrence of x by sampling from φ_v when it is rarely sampled from φ_u), then we reduce the variance of the estimator of $\theta = \mathbb{E}[f(V)R(V)]$.
Now let us give some remarks and how IS procedure is used, in Conway and Goyal (1987), to simulate the model presented in section 2.

Remarks. 1. IS is useful only when there are rare events in simulation.

2. The IS procedure, as used in Conway and Goyal (1987), is applied only when the simulated model Z enters in a state $i \in \partial O_F$.

3. If θ can be written as a ratio of expectations, like in the proposition above,then IS can be used to simulate the numerator and the denominator independently. It is shown in Heidelberger et al. (1987) that this technique is better than IS, and it is called MSIS (Measure Specific Importance Sampling).

The stiffness of the model X implies that the fault state space F is rarely visited by Z. So that when Z is in a state $i \in \partial O_F$ and if p is the probability that Z makes a transition to F, the IS strategy is based on the choice of a new probability α, with $\alpha > p$, which favors the occurrence of the rare event (i.e. transition to F). In this case, the estimation is weighted by the ratio

p/α. It is important to note that IS is applied only when the model Z enters in a state $i \in \partial O_F$. But, a large and stiff Markovian model stays in $O \backslash \partial O_F$, where $O \backslash \partial O_F = \{i \in O \ ; \ i \notin \partial O_F\}$, for a large number of transitions before it reaches ∂O_F from where IS is used. For reducing this number of transitions without affecting the estimates, we propose the SDT algorithm.

4. Simulation with a Distance Technique (SDT)

The idea of SDT is to apply the MSIS procedure from any state $i \in O$ and not only from $i \in \partial O_F$. The probability α is then adapted to this situation. It is computed from an analogy with a birth and death process. Let present now some preliminaries :

If I and J are two sub-sets of E, we define the distance from I to J by the minimum number of transitions to reach J from I. This distance is denoted $d(I, J)$. If we consider the sequence $(F_k)_{k \geq 0}$ defined by $F_k = \{i \in O \ / \ d(\{i\}, F) = k\}$ (i.e. F_k is the set of states which are at distance k far from F), then it is simple to see that for an irreducible Markov chain, it exists $m \in \mathbb{N}$ such that $E = \bigcup_{k=0}^{m} F_k$ where $O = \bigcup_{k=1}^{m} F_k$ and $F = F_0$.

Now consider F_k like a state of a birth and death process for which we fix the transition probability from F_k to F_{k-1} equal to α for $k = 1, ..., m$. The transition probability from F_k to F_{k+1} is then equal to $1 - \alpha$ for $k = 1, ..., m - 1$. With this assumption, the transition graph of Z can be seen as a transition graph of a birth and death process with an absorbed state as described in Figure 1.

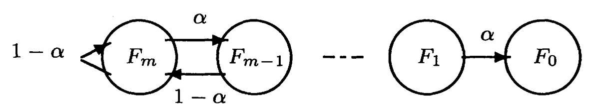

Figure 1 : Transition rate Diagram of a birth and death process with an absorbed state.

F is considered as an absorbed state because when in F, we stop the reduction variance procedure until the model comes back to the regenerative state i_0. A stiff model returns quickly to i_0.

We take α as the minimum chance for Z to make transition in a state which decreases the distance between the current state and F. This choice increases the chance of occurrence of paths visiting F. So if $i \in F_k$ is the current state, then the transition probability from i to F_{k-1} is taken to be

$$p'_i = \max(\alpha, \sum_{j \in F_{k-1}} p_{i,j})$$

Now using balanced equations for the process in Figure 1, we have

$$x_0 = 0, \quad x_k = \alpha x_{k+1} + (1-\alpha)x_{k-1} + 1, \quad 1 \le k \le m-1$$

where x_k is the minimum number of transitions to reach $F_0 = F$ from the state F_k. Thus for a given x_m, we can approximate α by numerical analysis procedures. It is clear that x_m depends on the stiffness of the model and on Card(O). By taking a simple form of x_m, for example $x_m = \gamma\text{Card}(O)$ where γ is a scalar depending on the stiffness of the model, SDT will produce a significative variance reduction when the model is large. The illustration of the capabilities of SDT approach can be shown through an example of simulation of a fault-tolerant system which is highly available and large.

5. Numerical example

Let consider a fault-tolerant system which is modelled by a CTMC. His transition rate diagram is given in Figure 2.

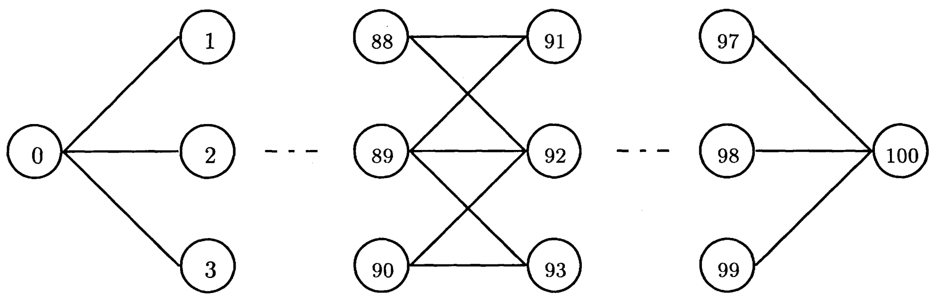

Figure 2 : Transition rate diagram of a CTMC with 101 states

We have Card(E) = 101, $O = \{0, 1, ..., 90\}$, $F = \{91, 92, ..., 100\}$ and $\partial O_F = \{88, 89, 90\}$. When in O, the failure and repair rates are λ_O and μ_O respectively. In F, these rates are λ_F and μ_F respectively. Between ∂O_F and F, the failure and repair rates are λ and μ respectively.

The performance measure which we simulate is the steady-state unavailability of the model (i.e. the probability of being in F : $\theta = \sum_{i \in F} f(i)\pi_i$, where $f(i) = \mathbb{1}_F(i)$).

We take $\lambda_O = 0.5$, $\mu_O = 1$, $\lambda_F = 0.5$, $\mu_F = 2$, $\lambda = 10^{-2}$ and $\mu = 3$. For these values, $\theta = 3 \times 10^{-12}$ which is computed by a SUN computer with a SPARC processor.

This model is simulated by the two algorithms : MSIS and SDT. For MSIS we

have fixed 0.999 as the new probability to favors the rare event. This value is taken to be the best one in Heidelberger et al. (1987). The parameter γ, for SDT, is taken to be equal to 10.
Results and summary statistics are illustrated in Table 1 for MSIS and Table 2 for SDT as follows : For each table, the first column contains the CPU time, the second one contains an estimation $\bar{\theta}$ of θ, scaled by the factor 10^{12}, and the last one contains the half-width of 99% confidence interval (*hwci*) scaled by the factor 10^{13}.

CPU time	$\bar{\theta} \times 10^{12}$	$hwci \times 10^{13}$
30645	51.21	1319.2
46212	33.72	868.7
67230	23.57	607.12
100899	15.73	405.23

Table 1 : Evolution of the unavailability estimation $\bar{\theta}$
and the *hwci* with respect to the CPU time for the MSIS algorithm.

CPU time	$\bar{\theta} \times 10^{12}$	$hwci \times 10^{13}$
969	3.08	1.10
1445	3.10	1.56
1931	3.10	1.28
2396	3.09	1.15
2839	3.10	1.03
28297	3.09	0.34

Table 2 : Evolution of the unavailability estimation $\bar{\theta}$
and the *hwci* with respect to the CPU time for the SDT algorithm.

We remark in Table 2 that for moderate CPU times, we get good estimations of θ with acceptable confidence intervals. For a simple comparison between MSIS and SDT, we see for example the first line in Table 1 and the last line in Table 2 : for 28297 CPU, we get a good estimation $\bar{\theta} = 3.09 \times 10^{-12}$ with $hwci = 0.34 \times 10^{-13}$ by the SDT algorithm, and for 30000 CPU, we get

$\bar{\theta} = 51.21 \times 10^{-12}$ with $hwci = 1319.2 \times 10^{-13}$ by the MSIS algorithm. The improvement factor in confidence interval ($hwci$[MSIS]/$hwci$[SDT]) is then greater than 3915.

6. Summary

In this paper, we have presented an algorithm called SDT which is based on the MSIS technique. It allows to simulate stationary performances of a large class of Markovian models (i.e. stiff and large Markovian models). This new approach, which takes account of the Markov chain graph's structure, reduces significantly the time spent by simulation in the operational state (i.e. sub-set of state space).

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