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T-ADJUNCTION

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ABSTRACT. We show that the operator which associates the upper topology to a pre-order relation can be extended to the morphisms of a subcategory of **Top**, in such way that it results a functor and it is right adjoint of the respective extension of the operator which associates to each topology its specialization pre-order.

KEY WORDS AND PHRASES. Specialization order, Alexandrov topology, upper topology, adjoint functor, adjoint function.

RESUMEN. Se muestra que el operador que asocia la topología superior a una relación de pre-orden puede extenderse a los morfismos de una subcategoría de **Top**, de tal manera que resulta un funtor y es adjunto a derecha de la respectiva extensión del operador que asocia a cada topología su pre-orden de especialización.

PALABRAS CLAVES. Orden de especialización, topología de Alexandrov, topología superior, funtor adjunto, función adjunta.

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INTRODUCTION

Let X be an ordered set. We say that a topology on X is *(order-) concordant* if the specialization order (which is defined by $x \leq y \Leftrightarrow x \in cl\{y\}$) is the given order relation. Alexandrov studied maximal concordant topologies and called them "discretes" (see [4]). Later, these topologies, which are characterized like such topologies that are closed under arbitrary intersections, were called *Alexandrov topologies* or *quasi-discrete topologies.* For a given order relation

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there is also a minimal concordant topology called the *upper topology*, *the upper interval topology* or *the weak topology*. Concordant topologies are then those topologies which are between the upper topology and the Alexandrov topology of the ordered set. These topologies are studied by Gierz, Keimel, Lawson, Mislove and Scott in their works on continuous lattices collected in [6]. They are also used by Johnstone in his beautiful book on Stone Spaces [8]. Some of the authors who have used these topologies are Erné, Gatzke, Hofmann, Hoffmann, Schwarz, Stralka, Weck and several others.

Formally, we have:
Given a T_0 topology τ on a set X, the specialization order on X associated to τ is defined by

$$\alpha(\tau) = \{(x, y) \mid x \in cl_\tau(y)\},$$

where $cl_\tau(y)$ denotes the closure of $\{y\}$ in the topological space (X, τ). On the other hand, given an order relation R on X, the Alexandrov topology associated to R is the topology $\gamma(R)$, generated by the sets of the form

$$\uparrow_R(x) = \{y \in X \mid (x, y) \in R\},$$

and the upper topology associated to R is the topology $\upsilon(R)$, generated by the sets of the form $X \setminus \downarrow_R(x)$, where

$$\downarrow_R(x) = \{y \in X \mid (y, x) \in R\}.$$

It is known (see for example [6] and [8]) that these operators are related by

$$\alpha(\tau) = R \Leftrightarrow \upsilon(R) \subseteq \tau \subseteq \gamma(R).$$

This relation makes think in an adjunction. It is known, in fact, that γ is the left adjoint of α, if we consider them as functions between the ordered sets $(Top(X), \supseteq)$ and $(Pos(X), \subseteq)$ (see [1]). Here, $Top(X)$ is the set of all topologies on X and $Pos(X)$ is the set of all order relations on X. However, υ cannot be the right adjoint of α, because it is not a monotone operator between these ordered sets.

In this paper we show that the operator υ can be extended to the morphisms of a sub-category of **Top** in such way that it results a functor and it is right adjoint of the respective extension of the operator α. To this end, we generalize the notion of left adjoint function between ordered sets and introduce the notions of *left r-adjoint function* between sets endowed with a binary relation and *left t-adjoint function* between topological spaces. In this way we show that α is a functor with left adjoint γ and right adjoint υ.

1. Basic notions

In this section we recall the notions of adjoint function and adjoint functor. We assume that the definitions of category and functor are known, but the reader can consult for example [2], [3] and [6]. We also introduce the notion of left r-adjoint function between sets endowed with a binary relation and we show that these functions are the morphisms of a category.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathbf{C} \to \mathbf{D}, G : \mathbf{D} \to \mathbf{C}$ be functors. We say that F is left adjoint of G if there is a natural bijection between the sets $[FX; Y]$ and $[X; GY]$ for each object X of $\mathbf{C}$ and each object Y of $\mathbf{D}$ ($[A; B]$ denotes the set of morphisms from A to B in the respective category). In this case we also say that G is right adjoint of F.

Recall now that every ordered set $(X, \leq)$ (i.e. a set X endowed with a binary relation $\leq$ which is reflexive, antisymmetric and transitive) is a category where the objects are the elements of X and the set of morphisms from a to b is $\{(a, b)\}$ if $a \leq b$ and it is empty otherwise. Under this point of view a functor between two ordered sets is a monotone function. Let $f : (X, \leq) \to (Y, \leq)$ and $g : (Y, \leq) \to (X, \leq)$ be monotone functions. We have that f is left adjoint of g if and only if for each $x \in X$ and each $y \in Y$

$$f(x) \leq y \Longleftrightarrow x \leq g(y). \tag{1.1}$$

In this context we say that f and g are adjoint functions. (The adjoint functions are also called "residuated mappings" (see [5]) and "Galois connexions").

If in (1.1) we use the convention

$$z \in \downarrow (a) \ \Leftrightarrow z \leq a$$

we obtain the next proposition:

Proposition 1. Let $f : (X, \leq) \to (Y, \leq)$ a function between ordered sets. The following are equivalent:

a) f is left adjoint (of some function from $(Y, \leq)$ to $(X, \leq)$).

b) For each $y \in Y$ there is $x \in X$ such that $f^{-1}(\downarrow (y)) = \downarrow (x)$.

Definition 1.

a) Let R be a binary relation on the set X. We will denote by $\downarrow_R (x)$ the set of the elements $y \in X$ such that $(y, x) \in R$.

b) Let $f : (X, R) \to (Y, S)$ be a function between two sets endowed with a binary relation. We will say that f is **compatible** if

$$(x, z) \in R \Rightarrow (f(x), f(z)) \in S.$$

c) Let $f : (X, R) \to (Y, S)$ be a function between two sets endowed with a binary relation. We will say that f is **left r-adjoint** if it is compatible and

$$\text{for each } y \in Y \text{ there is } x \in X \text{ such that } f^{-1}(\downarrow_S (y)) = \downarrow_R (x). \tag{1.2}$$

In the context of the ordered sets, f is left r-adjoint if and only if f is left adjoint. For, the set $\{(y,x) \in Y \times X \mid f^{-1}(\downarrow_S (y)) = \downarrow_R (x)\}$ is a function $g : Y \to X$ which is the right adjoint of f.

Proposition 2. Let $f : (X,R) \to (Y,S)$ be a function between two sets endowed with a binary relation. If f satisfies (1.2) and R is transitive then, f is compatible.

Proof. Let $(x,z) \in R$.

$$\begin{aligned} f(z) \in Y \quad &\Rightarrow \quad (\exists w \in X)(f^{-1}(\downarrow_S (f(z))) = \downarrow_R (w)) \\ &\Rightarrow \quad (z,w) \in R \\ &\Rightarrow \quad (x,w) \in R \text{ (by transitivity)} \\ &\Rightarrow \quad (f(x), f(z)) \in S. \ \square \end{aligned}$$

The following two propositions are evident:

Proposition 3. If $f : (X,R) \to (Y,S)$ and $g : (Y,S) \to (Z,T)$ are left r-adjoint functions then, $g \circ f : (X,R) \to (Z,T)$ is a left r-adjoint function.

Proposition 4. If R is a binary relation on the set X then, $1_X : (X,R) \to (X,R)$ is a left r-adjoint function.

Corollary 1. The class of the sets endowed with a binary relation and the left r-adjoint functions is a category where the composition is the usual one.

Remark 1. This category is a sub-category of the category **Gra** of [1] and [2].

2. The functor α.

In this section we present the operator α which associates to each topology on the set X the pre-order of specialization on X, and we show that it can be extended to a functor from the category of the topological spaces to the category of the pre-ordered sets.

Definition 2. Let τ be a topology on the set X. We define

$$\alpha(\tau) = \{(a,b) \in X \times X \mid a \in cl_\tau(b)\},$$

where $cl_\tau(b)$ denotes the closure of $\{b\}$ in the topological space (X,τ).

Example 1. a) If (X,τ) is a T_1 space then, $\alpha(\tau) = \{(x,x) \mid x \in X\}$.

b) If $\tau = \{\phi, X\}$ then, $\alpha(\tau) = X \times X$.

Proposition 5. a) $\alpha(\tau)$ is a pre-order (reflexive and transitive) relation on X.

b) τ is a T_0 topology if and only if $\alpha(\tau)$ is antisymmetric.

Proof. Part a) is evident. For the part b) it is enough to see that τ is a T_0 topology if and only if

$$cl_\tau(b) = cl_\tau(a) \Rightarrow a = b. \square$$

In the context of the T_0 topological spaces $\alpha(\tau)$ is then an order relation on X called the *specialization order* for τ (see [8]).

Definition 3. Following [1] and [2], we denote by **Gra** the category which has as objects the couples (X, R) where X is a set and R is a binary relation on X, and as morphisms the compatible functions. We will denote by **Pros** the full sub-category of **Gra** where the considered relations are the pre-order ones. We will denote by **Top** the category of the topological spaces and continuous functions.

Remark 2. In [3] the category **Gra** is denoted by **Rel.**

The next proposition shows the behavior of the operator α with continuous functions and allows to consider it as a functor.

Proposition 6. If $f : (X, \tau) \to (Y, \mu)$ is a continuous function between the topological spaces (X, τ) and (Y, μ) then $f : (X, \alpha(\tau)) \to (Y, \alpha(\mu))$ is compatible.

Proof.

$$(x, z) \in \alpha(\tau) \Rightarrow x \in cl_\tau(z) \Rightarrow f(x) \in cl_\mu(f(z)) \Rightarrow (f(x), f(z)) \in \alpha(\mu). \square$$

Corollary 2. If we define $\alpha(f) = f$ then, α is a functor from **Top** to **Pros**.

Let R be a pre-order relation on X. Consider the topologies τ on X such that $\alpha(\tau) = R$. It is known that,

$$\alpha(\tau) = R \Leftrightarrow \upsilon(R) \subseteq \tau \subseteq \gamma(R), \tag{2.1}$$

where $\upsilon(R)$ is the *upper topology* associated to R, which is generated by the sets of the form $\downarrow_R (x)$ with $x \in X$, and $\gamma(R)$ is the Alexandrov topology associated to R, which has as open sets the sets M such that

$$x \in M \wedge (x, y) \in R \Rightarrow y \in M.$$

(This result is proved in the literature for order relations, although the antisymmetry of the relation is not used in the argumentations). The equivalence in (2.1) suggests that γ is left adjoint and υ is right adjoint of α. In fact, if we consider $Top(X) = \{\tau \mid \tau \text{ is a topology on } X\}$, ordered by the relation $\supseteq$, and $Gra(X) = \{R \mid R \text{ is a binary relation on } X\}$, ordered by $\subseteq$, we have that $\alpha : Top(X) \rightarrow Gra(X)$ and $\gamma : Gra(X) \rightarrow Top(X)$ are monotone functions and γ is left adjoint of α. Moreover, we can show that, if we define $\gamma(f) = f$ then, γ is a functor from **Gra** to **Top** and α : **Top** $\rightarrow$ **Gra** is right adjoint of γ : **Gra** $\rightarrow$ **Top** (see [1]). However, if we consider υ as a function between the ordered sets $(Gra(X), \subseteq)$ and $(Top(X), \supseteq)$, it is not monotone, as the following example shows, and consequently, it can not be the right adjoint of α:

Example 2. Let R be the usual ordering on $\mathbb{R}$ and let

$$S = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \mid x \leq y\} \cup \{(x, x) \mid x \in \mathbb{I}\},$$

where $\mathbb{Q}$ denotes the set of the rational real numbers and $\mathbb{I}$ denotes the set of the irrational ones. We have $S \subseteq R$, but $\upsilon(S) \not\supseteq \upsilon(R)$.

Remark 3. If X is a finite set the operators υ and γ are equal.

3. The category **Topla**.

In this section we introduce the notion of left t-adjoint function between topological spaces, we consider the respective sub-category of **Top** and we show that the functor α can be restricted to this sub-category in such way that its values are in the category of the pre-ordered sets and the left r-adjoint functions.

Definition 4. Let $f : (X, \tau) \rightarrow (Y, \mu)$ be a function between the topological spaces (X, τ) and (Y, μ). We will say that f is **left t-adjoint** if it is continuous and for each $y \in Y$ there is $x \in X$ such that $f^{-1}(cl_\mu(y)) = cl_\tau(x)$.

Example 3. Let $f : (X, \tau) \rightarrow (Y, \mu)$ a left t-adjoint function:

a) If (Y, μ) is a T_1topological space then, f is a surjective function.

b) If (X, τ) is a T_1topological space then, f is an injective function.

c) If (X, τ) y (Y, μ) are T_1 then, the left t-adjoint functions between them are the continuous bijections.

The following two propositions show that the left t-adjoint functions are the morphisms of a category.

Proposition 7. If $f : (X, \tau) \rightarrow (Y, \mu)$, $g : (Y, \mu) \rightarrow (Z, \eta)$ are left t-adjoint functions then, $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is a left t-adjoint function.

Proof. It is clear that $g \circ f$ is continuous and if $z \in Z$ there exists $y \in Y$ such that $g^{-1}(cl_\eta(z)) = cl_\mu(y)$. For this y there exists $x \in X$ such that $f^{-1}(cl_\mu(y)) = cl_\tau(x)$. Thus,

$$\begin{aligned}(g \circ f)^{-1}(cl_\eta(z)) &= f^{-1}\left(g^{-1}(cl_\eta(z))\right) \\ &= f^{-1}(cl_\mu(y)) \\ &= cl_\tau(x). \ \square\end{aligned}$$

Proposition 8. $1_X : (X, \tau) \to (X, \tau)$ is a left t-adjoint function.

Proof. Straightforward. $\square$

Corollary 3. The class of the topological spaces and the left t-adjoint functions is a sub-category of **Top**.

Definition 5. **Topla** is the category of the topological spaces and the left t-ajoint functions.

TopT$_0$la is the full sub-category of **Topla** where the objects are the T_0 topological spaces.

Prosla is the category where the objects are the pre-ordered sets and the morphisms are the left r-adjoint functions.

Posla is the full sub-category of **Prosla** where the objects are the ordered sets.

The next proposition shows us that the functor α can be restricted to the sub-category **Topla** of **Top** in such way that its values are in the sub-category **Prosla** of **Gra**.

Proposition 9. If $f : (X, \tau) \to (Y, \mu)$ is a left t-adjoint function then, $f : (X, \alpha(\tau)) \to (Y, \alpha(\mu))$ is a left r-adjoint function.

Proof.

$$\begin{aligned}y \in Y &\Rightarrow (\exists x \in X)(f^{-1}(cl_\mu(y)) = cl_\tau(x)) \\ &\Rightarrow [(z, x) \in \alpha(\tau) \Leftrightarrow (f(z), y) \in \alpha(\mu)] \\ &\Rightarrow f^{-1}\left(\downarrow_{\alpha(\mu)}(y)\right) = \downarrow_{\alpha(\tau)}(x). \ \square\end{aligned}$$

Corollary 4. α is a functor from **Topla** to **Prosla.**

To finish this section, we see that we can replace the pre-order relations by the order relations if we work only with T_0 topological spaces.

Proposition 10. Let $f : (X, \tau) \to (Y, \mu)$ be a left t-adjoint function. If (X, τ) is T_0, the binary relation $\{(y, x) \in Y \times X \mid f^{-1}(cl_\mu(y)) = cl_\tau(x)\}$ is a function from Y to X and we will denote it by $f^\star$.

Proof. It is sufficient to observe that if (X, τ) is T_0, then $cl_\tau(x) = cl_\tau(z)$ if and only if $x = z$. □

Proposition 11. If $f : (X, \tau) \to (Y, \mu)$ is a morphism of **TopT$_0$la**, then $f : (X, \alpha(\tau)) \to (Y, \alpha(\mu))$ is left adjoint of $f^\star : (Y, \alpha(\mu)) \to (X, \alpha(\tau))$.

Proof.

$$\begin{aligned} (f(x), y) \in \alpha(\mu) &\Leftrightarrow f(x) \in cl_\mu(y) \\ &\Leftrightarrow x \in f^{-1}(cl_\mu(y)) \\ &\Leftrightarrow x \in cl_\tau(f^\star(y)) \\ &\Leftrightarrow (x, f^\star(y)) \in \alpha(\tau). \ \square \end{aligned}$$

Corollary 5. α is a functor from **TopT$_0$la** to **Posla**.

4. The functor υ.

In this section we introduce the operator υ mentioned in Section 2 and we show that it can be viewed as a functor from the category of the pre-ordered sets and left r-adjoint functions to the category of the topological spaces and the left t-adjoint functions.

Definition 6. Let R be a binary relation on the set X. We define $\upsilon(R)$ as the coarsest topology on X such that the sets of the form $\downarrow_R (x)$ are closed.

Example 4. a) If $R = \{(x, x) \mid x \in X\}$ then, $\upsilon(R)$ is the cofinite topology on X.

b) If R is the usual order on $\mathbb{R}$ then, $\upsilon(R)$ is the topology of the right open tails on $\mathbb{R}$.

c) $\upsilon(R)$ is the discrete topology on X if and only if $R = \{(x, x) \mid x \in X\}$ and X is a finite set.

d) If $R = X \times X$ then, $\upsilon(R) = \{\phi, X\}$.

Proposition 12. Let R be a binary relation on X. The following are equivalent:

a) For each $x \in X$, $\downarrow_R (x) = cl_{\upsilon(R)}(x)$.

b) R is a pre-order relation on X.

Proof. Let us see that a) implies b): If $x \in X$ we have $x \in cl_{\upsilon(R)}(x) = \downarrow_R (x)$, thus $(x,x) \in R$ and R is reflexive. Suppose now that $(x,y),(y,z) \in R$. This means that $x \in cl_{\upsilon(R)}(y)$ and $y \in cl_{\upsilon(R)}(z)$ and consequently, $x \in cl_{\upsilon(R)}(z) = \downarrow_R (z)$. Then, $(x,z) \in R$ and R is transitive.

Let us see now that b) implies a): Let $x \in X$. By the definition of $\upsilon(R)$, $\downarrow_R (x)$ is a closed set, and it contains x because R is reflexive. Consequently, $cl_{\upsilon(R)}(x) \subseteq \downarrow_R (x)$. Suppose that $y \in \downarrow_R (x)$. If $y \notin cl_{\upsilon(R)}(x)$, there exist $x_1, ..., x_n \in X$ such that $y \in \bigcap_{i=1}^{n} (X \setminus \downarrow_R (x_i))$ and $x \notin \bigcap_{i=1}^{n} (X \setminus \downarrow_R (x_i))$. Then there exists i between 1 and n such that $x \in \downarrow_R (x_i)$. By the transitivity of R we have $y \in \downarrow_R (x_i)$ which is absurd. Then, $\downarrow_R (x) \subseteq cl_{\upsilon(R)}(x)$. □

The next proposition shows the behavior of υ with left r-adjoint functions and allows us to consider it as a functor.

Proposition 13. If $f : (X,R) \to (Y,S)$ is a morphism of **Prosla**, then $f : (X, \upsilon(R)) \to (Y, \upsilon(S))$ is a morphism of **Topla**.

Proof.

$$\begin{aligned} y \in Y \quad &\Rightarrow \quad (\exists x \in X) \left(f^{-1}(\downarrow_S (y)) = \downarrow_R (x)\right) \\ &\Rightarrow \quad [z \in \downarrow_R (x) \Leftrightarrow f(z) \in \downarrow_S (y)] \\ &\Rightarrow \quad \left[z \in cl_{\upsilon(R)}(x) \Leftrightarrow f(z) \in cl_{\upsilon(S)}(y)\right] \\ &\Rightarrow \quad f^{-1}\left(cl_{\upsilon(S)}(y)\right) = cl_{\upsilon(R)}(x). \end{aligned}$$

Clearly, f is continuous since the inverse image of a sub-basic closed set is a closed set. □

Corollary 6. If we define $\upsilon(f) = f$, then υ is a functor from **Prosla** to **Topla**.

The behavior of υ is good, because it transforms order relations in T_0 topologies. Consequently, it can be restricted to a functor from **Posla** to **TopT$_0$la**.

Proposition 14. Let R be a pre-order relation on X. The following are equivalent:

a) R is an order relation on X.

b) $\upsilon(R)$ is a T_0 topology on X.

Proof. It is sufficient to remark that for each $x \in X$, $\downarrow_R (x) = cl_{\upsilon(R)}(x)$. □

Corollary 7. υ is a functor from **Posla** to **TopT$_0$la**.

5. α IS LEFT ADJOINT

In this section we show that the functor υ is right adjoint of the functor α if we work with the appropriate categories. Using this adjunction we introduce the notion of "specialized topology".

Theorem 1. The functor α : **Topla** $\rightarrow$ **Prosla** is left adjoint of the functor υ : **Prosla** $\rightarrow$ **Topla**.

Proof. Let (X, τ) be an object of **Topla** and (Y, S) be an object of **Prosla**. We state the following:

$$[(X, \alpha(\tau)), (Y, S)]_{\mathbf{Prosla}} = [(X, \tau), (Y, \upsilon(S))]_{\mathbf{Topla}}.$$

If $f \in [(X, \alpha(\tau)), (Y, S)]_{\mathbf{Prosla}}$ we have to see that $f : (X, \tau) \rightarrow (Y, \upsilon(S))$ is a left t-adjoint function, for which it is sufficient to show that for each $y \in Y$, $f^{-1}(\downarrow_S (y))$ is the closure of some point in (X, τ). But if $y \in Y$, there exists $x \in X$ such that

$$f^{-1}(\downarrow_S (y)) = \downarrow_{\alpha(\tau)} (x) = cl_\tau(x).$$

On the other hand, if $f \in [(X, \tau), (Y, \upsilon(S))]_{\mathbf{Topla}}$, we have to see that $f : (X, \alpha(\tau)) \rightarrow (Y, S)$ is a left r-adjoint function. In fact,

$$\begin{aligned} f^{-1}(\downarrow_S (y)) &= f^{-1}(cl_{\upsilon(S)}(y)) \\ &= cl_\tau(x), \text{ for some } x \in X \\ &= \downarrow_{\alpha(\tau)} (x), \text{for some } x \in X. \ \square \end{aligned}$$

Corollary 8. The functor α : **TopT$_0$la** $\rightarrow$ **Posla** is left adjoint of the functor υ : **Posla** $\rightarrow$ **TopT$_0$la**.

Proposition 15. For every pre-order relation on X we have $\alpha(\upsilon(R)) = R$.

Proof.

$$(a, b) \in \alpha(\upsilon(R)) \Leftrightarrow a \in cl_{\upsilon(R)}(b) \Leftrightarrow a \in \downarrow_R (b) \Leftrightarrow (a, b) \in R. \ \square$$

Proposition 16. For every topology τ on X we have $\upsilon(\alpha(\tau)) \subseteq \tau$.

Proof. Because of the adjunction of Theorem 5.1, $1_X : (X, \tau) \rightarrow (X, \upsilon(\alpha(\tau)))$ is a continuous function, thus $\upsilon(\alpha(\tau)) \subseteq \tau$. $\square$

Example 5. Let X be an infinite set and let τ be a T_1 topology on X and μ be the cofinite topology on X.We have $\upsilon(\alpha(\tau)) = \mu$.

This example shows that $\upsilon(\alpha(\tau))$ is not always equal to τ and gives sense to the following definition:

Definition 7. Let τ be a topology on X. If $\upsilon(\alpha(\tau)) = \tau$ we will say that τ is **specialized**.

Example 6.

1) The cofinite topology on X is specialized.
2) The topology $\{\phi, X\}$ on X is specialized.
3) The only T_1 specialized topology on X is the cofinite one.
4) The topology of the open right tails on $\mathbb{R}$ is specialized.
5) The topology of the open intervals centered in 0 on $\mathbb{R}$ is specialized.
6) The specialized topologies on X are exactly those of the form $\upsilon(R)$ for some pre-order relation on X.

Problem: Characterize topologically the specialized topologies on a set X.

6. α IS ALSO RIGHT ADJOINT

In this section we show that the known adjunction between α and γ can be restricted to the sub-categories introduced in Section 3. Therefore, α is a functor which is right and left adjoint. Additionally we obtain several other adjunctions and isomorphisms between some sub-categories of the treated categories.

We know that $\alpha : \mathbf{Top} \to \mathbf{Gra}$ is right adjoint of $\gamma : \mathbf{Gra} \to \mathbf{Top}$ (see [1]). If we restrict these functors to the sub-categories **Topla** of **Top** and **Prosla** of **Gra**, we obtain the following:

Theorem 2. $\alpha : \mathbf{Topla} \to \mathbf{Prosla}$ is right adjoint of $\gamma : \mathbf{Prosla} \to \mathbf{Topla}$.

Proof. We know that $[(X, \gamma(R)), (Y, \mu)]_{\mathbf{Top}} = [(X, R), (Y, \alpha(\mu))]_{\mathbf{Gra}}$. It is enough to show that $f : (X, \gamma(R)) \to (Y, \mu)$ is left t-adjoint if and only if $f : (X, R) \to (Y, \alpha(\mu))$ is left r-adjoint. For this, we observe that for each $y \in Y$ and each $x \in X$ we have

$$\begin{aligned} cl_{\mu}(y) &= \downarrow_{\alpha(\mu)}(y) \\ cl_{\gamma(R)}(x) &= \downarrow_{R}(x). \ \square \end{aligned}$$

It is known that the topologies of the form $\gamma(R)$ are exactly those which are quasi-discrete (i.e. those which are closed under arbitrary intersections). Then, if we call **Topqdla** the full sub-category of **Topla** where the objects are the quasi-discrete topological spaces and **Topspla** the full sub-category of **Topla** where the objects are the specialized topological spaces, we have the following:

Corollary 9.

a) $\gamma\alpha$: **Topla** $\to$ **Topqdla** is left adjoint of the inclusion functor from **Topqdla** into **Topla**.

b) $\upsilon\alpha$: **Topla** $\to$ **Topspla** is right adjoint of the inclusion functor from **Topspla** into **Topla**.

c) $\gamma\alpha$: **Topqdla** $\to$ **Topspla** and $\upsilon\alpha$: **Topspla** $\to$ **Topqdla** are isomorphisms, where $(\gamma\alpha)^{-1} = \upsilon\alpha$.

d) α : **Topqdla** $\to$ **Prosla** and γ : **Prosla** $\to$ **Topqdla** are isomorphisms, where $\alpha^{-1} = \gamma$.

e) α : **Topspla** $\to$ **Prosla** and υ : **Prosla** $\to$ **Topspla** are isomorphisms, where $\alpha^{-1} = \upsilon$.

7. The fibre of the construct **Topla**

Following the ideas in [2], we introduce and study the order relation on the set of topologies on X, induced by the forgetful functor from the category of the topological spaces and the left t-adjoint functions, to the category of sets.

We will denote by $Topla(X)$ the set of topologies on X ordered by

$$\tau \le \mu \Leftrightarrow 1_X : (X,\tau) \to (X,\mu) \text{ is a morphism of } \textbf{Topla}.$$

Now, we are able to characterize this order relation using the $\supseteq$ order and the function α.

Proposition 17. Let $\tau, \mu \in Topla(X)$.

$$\tau \le \mu \Leftrightarrow \tau \supseteq \mu \wedge \alpha(\tau) = \alpha(\mu).$$

Proof. Suppose that $\tau \le \mu$. Since $1_X : (X,\tau) \to (X,\mu)$ is continuous, we have $\tau \supseteq \mu$. Moreover, since $1_X : (X,\alpha(\tau)) \to (X,\alpha(\mu))$ is left r-adjoint we have $\alpha(\tau) = \alpha(\mu)$. Conversely, suppose that $\tau \supseteq \mu$ and $\alpha(\tau) = \alpha(\mu)$. Therefore $1_X : (X,\tau) \to (X,\mu)$ is continuous and for each $x \in X$, $(1_X)^{-1}(cl_\tau(x)) = cl_\mu(x)$. Consequently, $1_X : (X,\tau) \to (X,\mu)$ is left t-adjoint. □

The equivalence relation determined by α on $Topla(X)$ divides this set in classes in which the given order is the $\supseteq$ order. Moreover, if two topologies are in different classes they are not comparable.

The following proposition is now evident:

Proposition 18. Let R be a pre-order relation on X.

$$\begin{aligned}\alpha^{-1}(R) &= \{\tau \in Topla(X) \mid \gamma(R) \le \tau \le \upsilon(R)\}\\ &= \{\tau \in Top(X) \mid \gamma(R) \supseteq \tau \supseteq \upsilon(R)\}.\end{aligned}$$

Example 7. a) If $R = \{(x,x) \mid x \in X\}$ then, $\alpha^{-1}(R) = \{\tau \in Top(X) \mid \tau$ is $T_1\}$.

b) If $R = X \times X$ then, $\alpha^{-1}(R) = \{\{\phi, X\}\}$.

c) If R is the usual order on $\mathbb{R}$ then, $\alpha^{-1}(R) = \{\tau \in Top(X) \mid \mu \supseteq \tau \supseteq \eta\}$, where μ is the right tails topology on $\mathbb{R}$ and η is the open right tails topology on $\mathbb{R}$.

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