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FINANCIAL MARKETS INCOMPLETENESS AND INEQUALITY OVER THE LIFE-CYCLE

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Resumen

Este artículo aborda la relevancia de la incertidumbre contemporánea bajo mercados financieros incompletos para explicar desigualdad de riqueza y consumo cuando las preferencias son homogéneas. Se explora el rol del ahorro ilíquido (ahorro que no puede ser utilizado para amortiguar shocks contemporáneos) generado por la incertidumbre contemporánea, y un costo para liquidar los ahorros, en la generación de diferencias en las decisiones de ahorro entre inicialmente-ricos e inicialmente-pobres en un modelo de ciclo de vida resuelto numéricamente. Se observa que la existencia de premio por riesgo incrementa significativamente la capacidad de los hogares de acumular riqueza para financiar su consumo, incluso bajo alta aversión al riesgo. Se encuentra que existe desigualdad en el consumo creciente en el ciclo de vida entre grupos y que este resultado es reforzado por el acceso a activos riesgosos y altos valores del coeficiente de aversión. De forma crítica, la posibilidad de endeudamiento contra los ahorros ayuda a impedir sendas de consumo con desigualdad creciente, que es el efecto ahorro ilíquido bajo incertidumbre contemporánea. Los resultados pueden ayudar a entender por qué los pobres ahorran tan poco, y por lo tanto apoyan programas sociales que garantizan un consumo mínimo para aquellos bajo un cierto umbral de riqueza de modo de permitirles alcanzar a los individuos más ricos.

Abstract

This paper addresses the relevance of contemporary uncertainty under incomplete markets in explaining wealth and consumption inequality when preferences are homogeneous. We explore the role of illiquid saving (saving that cannot be used to buffer contemporary shocks) generated by contemporary uncertainty, and a cost of liquidating saving, in generating differences in saving behaviour between initially-rich and initially-poor in a numerically solved life-cycle model. We observe that the existence of a risk premium significantly increases the ability of the households to accumulate wealth to finance their consumption, even under high risk aversion. We find that there is increasing consumption inequality across the life-cycle between groups, and that this result is boosted by access to risky assets and higher values of the coefficient of risk aversion. Critically, the possibility to borrow against saving helps to prevent increased consumption inequality paths, which is the effect of illiquid saving under contemporary uncertainty. Our findings may help us to understand why the poor save so little, and therefore support social programmes that guarantee a minimum consumption for those below a certain level of wealth to allow them to catch up with wealthier individuals.

1   Introduction

Market incompletenesses are present in financial markets in various different forms. One of the most relevant ones is the inability to insure against some types of risk, implying that individuals have to cope with uncertainties on their own. Decision making under uncertainty has received considerable attention in recent years in the study of consumption, saving, wealth accumulation and portfolio choice. This paper introduces contemporary uncertainty as a novel feature of market incompleteness in a standard life-cycle model. The main objective is to assess the relevance of contemporary uncertainty in explaining wealth and consumption inequalities under homogeneous preferences.

The timing of the saving decisions and the timing of the realisation of uncertain events determine whether contemporary uncertainty is present or not. When saving has to be decided before uncertainty about contemporary consumption is resolved then it becomes illiquid, being useless for buffering a bad shock at present. Thus, when markets are incomplete, illiquid saving generates an incentive to decrease the amount of savings. The relative impact of uncertain shocks on overall consumption determines the size of the effect of illiquid saving on the saving rate. Thus, by exploring the effects of contemporary uncertainty and illiquid saving, this paper also contributes to explaining the very low saving rates of the poor.

The standard life-cycle model implies different incentives for saving: patience, risky assets (risk premium), labour income uncertainty and retirement. The discount rate relative to the return rate determines how patient the individuals are. The greater the patience, the greater the saving and wealth accumulation, where the individuals prefer increasing consumption paths. The possibility of investing in risky assets, with a corresponding risk premium, generates less patience given that the expected return rate becomes higher. However, the expected return rate depends on the portfolio decision of the agent. Labour income uncertainty generates forward looking precautionary saving, creating a buffer stock for smoothing future consumption in the presence of future shocks to income. In parallel, under contemporary uncertainty, precautionary behaviour induces less saving. Finally, the existence of a retirement period where pension replaces only a fraction of permanent labour income generates the desire to accumulate wealth in order to smooth consumption during the retirement period and the labour period. By iso-

lating the different saving motives in our life-cycle model, we are able to determine how relevant is the illiquid saving effect on the differences in consumption paths between individuals who differ only in their initial wealth.

Our main result is that the introduction of contemporary uncertainty to a parameterised life-cycle model increases the gap in expected consumption paths between individuals who differ only in their initial wealth. We find that, when risky assets are introduced (with a corresponding risk premium), especially under high risk aversion, the divergence between consumption paths is exacerbated. Access to stock markets would then be critical in making inequality increase even more. We also find that, allowing the financial markets to be more complete, in the sense that individuals can liquify their saving at a certain cost, can contribute to eliminate the effect of increasing inequality over the life-cycle.

Illiquid saving generates a demand for borrowing to buffer a negative shock at present after the saving has been decided. This effect is larger for those who are more affected, in relative terms, by the negative shocks. Borrowing in our model requires the individual to pay an interest rate which is higher than the return rate, creating a wedge between the saving return and the borrowing cost. Since low-wealth individuals will have a higher demand for borrowing than high-wealth individuals, the former will end up borrowing more at higher interest rates. This higher demand for borrowing by low-wealth individuals may help us to understand the credit cards puzzle, where individuals hold assets and at the same time they borrow on their credit cards at higher interest rates.

In addition to this introduction, section 2 briefly reviews the literature relating to uncertainty, saving, portfolio and borrowing. In section 3 we set up a life-cycle model to analyse the effects of contemporary uncertainty on wealth accumulation and consumption inequality. Section 4 extends the basic model to allow borrowing against illiquid saving. Section 5 explores possible extensions of the model. Section 6 concludes.

2 Saving under Uncertainty, Portfolio and Borrowing in the Literature

In studying saving behaviour, one of the key issues is to understand differences in saving rates among rich and poor individuals. As we mentioned above, the standard life-cycle model implies at least four different incentives for saving: patience, risky assets, labour income uncertainty and retirement. Thus, determining the relative importance of these saving motives has been at the centre of the attention. We now make a general survey of the discussion of saving motives in the literature.

Patience has been always a critical point. Some authors have made straight assumptions and some others have preferred to attempt estimations of the discount rate parameter. For example, Deaton (1991) assumes there is impatience, and Attanasio, Banks, Meghir, and Weber (1999) prefer to estimate discount rates from the Euler equations. In any case, the existence of a risk premium can increase the relative patience of the agents.

Future labour income uncertainty generates precautionary saving motives, which have been studied in the context of the life-cycle for a few years. For example, Zeldes (1989) led the use of numerical methods to solve non-linear problems which included uncertainty and constant relative risk aversion (CRRA) utility function. Deaton (1991) focused on the interaction of the precautionary saving motive and liquidity constraints to explain some stylised facts about saving. Caballero (1990, 1991), preferred to use constant absolute risk aversion preferences (CARA), and, combined with some assumptions about the distribution of the shocks, obtained analytical solutions for precautionary saving. Carroll (1994) tested empirically the predictions of the model where those who face more future income uncertainty consume less, finding support for precautionary saving. Hubbard, Skinner and Zeldes (1994, 1995) augmented the life-cycle model with precautionary saving to better fit saving behaviour in the US economy.

The existence of a retirement period, in which pension replaces only a fraction of permanent labour income, generates the desire to accumulate wealth in order to smooth consumption between the labour period and the retirement period. Different pension schemes may generate different incentives to save for retirement. Compulsory or voluntary contributions, the tax system, the replacement rate,

and other factors, are among the critical elements to consider when analysing retirement. A fairly recent discussion of an international comparison of household savings behaviour, focused on retirement savings in seven countries, can be found in Borsch-Supan (2001) and Jappelli (2001).

Apart from the saving motives which were established above, some authors like Bertaut and Haliassos (1997), Carroll (2000), and Dynan, Skinner and Zeldes (2004), have also included bequest motives. Without bequest motives, wealth accumulation only serves the purpose of future consumption. The introduction of bequests has been considered particularly important in understanding high saving rates at a late age, when otherwise saving is theoretically predicted to be rather low. What expectations the individual has about the performance of his offspring are critical in determining the level of bequests. For example, Dynan et al. consider complete mean reversion of incomes, so that richer (poorer) individuals would expect their offspring to be worse (better) off than them, so that they leave a larger (smaller) bequest.

In an uncertain environment, the result of the decision process will look similar to a random-walk process. Individuals will make their decisions according to how much wealth and labour income they have and they expect to have. Since transitory and permanent shocks may affect their wealth and labour income levels, their consumption behaviour will change accordingly. This makes the consumption process to display an increasing variance over the life-cycle. Deaton and Paxson (1994) predict increasing within-cohort consumption inequality under the Permanent Income Hypothesis (PIH), which follows the random-walk feature of income. With that prediction in mind, they account for the increasing inequality within cohorts in the US. On the other hand, Krueger and Perri (2005) have analysed the fact that consumption inequality has increased less than income inequality in the past years in the US. They claim that new trends in income inequality are not reflected in consumption inequality because of the expansion of the use of non-collateralised credit.

The existence of risky assets, with a corresponding risk premium, has been studied in depth in the literature for its implications in portfolio choice. The existence of a risk premium increases significantly the ability of the households to accumulate wealth in order to finance their consumption. The attitude towards holding risky

assets depends on individual risk aversion preferences, background risks, relative importance of risk exposure, future access to risky assets and the ability to borrow at present and in the future. The life-cycle model allows us to explore who holds risky assets and when they hold them, and how that eventually affects consumption paths.

Consumption and portfolio choice under uncertainty has been studied since Merton (1969, 1971) and Samuelson (1969). There are a number of recent studies on portfolio decisions. The theoretical effects of background risk and risk aversion on portfolio choices has been studied by Gollier and Pratt (1996), Guiso, Jappelli and Terlizzese (1996), and Guiso and Paiella (2003).[1] The interrelation between saving decisions and portfolio decisions has received the attention of Quadrini (1999), Quadrini (2000), Carroll (2000, 2002) and Gentry and Hubbard (2004). While Carroll is trying to understand the portfolios of the rich by including wealth in their utility function so that risk aversion declines as wealth increases, the others focus on entrepreneurial activities that require a certain amount of wealth, thus inducing higher saving rates for those interested in becoming entrepreneurs.

Since holding of risky assets are theoretically predicted to be much larger than what is observed, many studies have included fixed entry costs or transaction costs in their models in order to match the stylised facts (Campbell, Cocco, Gomes and Maenhout, 2001; Paiella, 2001; Haliassos and Michaelides, 2003; Alan, 2005; Campbell and Viceira, 2005; Gomes and Michaelides, 2005). The correlation between labour income risk and portfolio risk has also been studied in the literature, but with ambiguous results. On the one hand, Viceira (2001) claims that there is a significant effect in generating less risky portfolios. On the other hand, Haliassos and Michaelides (2003) argue that the positive correlation between earnings shocks and stock returns is unlikely to provide an empirically plausible solution to the portfolio specialisation puzzle.

Some authors have focused on the time horizon as a critical element to take into account when analysing attitudes towards risk. Gollier (2002a) argues theoretically that liquidity constraints reduce the implicit time horizon of the consumer

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[1] Letendre and Smith (2001) attempt to quantify the size of the effect of background risk on portfolios, concluding that it is smaller than the effect on saving rates and may be difficult to detect empirically.

and thereby increase his risk aversion, so that the negative relationship between initial wealth and the implicit time horizon in which the consumer will be liquidity constrained (this is because they have more wealth to use instead of borrowing) provides an argument for decreasing aversion to risk on wealth. Gollier and Zeckhauser (2002) state that, under CRRA preferences, the time horizon only matters when there are incomplete markets (uninsurable risks).

Portfolio choice over the different stages of the life-cycle has been another subject of study in the literature. For example, Bodie, Merton and Samuelson (1992) model labour supply flexibility and portfolio choice in a life cycle model. They find that the ability to vary labour supply ex post induces the individual to assume greater risks in his investment portfolio ex ante. They also find that, at any given age in the life-cycle, the riskier is an individual's human capital, the lower will be his financial investment in risky assets. More recently, Polkovnichenko (2004) uses habit formation to distinguish between young and middle-aged people, arguing that the portfolio share allocated to stocks increases with wealth, implying that younger individuals would have more conservative portfolios. Cocco, Gomes and Maenhout (2005) claim that labour income substitutes for riskless asset holdings, so that the optimal share invested in equities is roughly decreasing over life.

There is no full agreement on what portfolio choice should be through the life-cycle. Then, we will finally quote Gollier (2005, pp.17), who attempts to conciliate the different points of view from a theoretical perspective by stating that "... there is no universal answer to the question of whether younger households should be less risk-averse. Its answer depends upon individual characteristics such as for example the riskiness of the household's human capital, the intensity of potential liquidity constraints faced by it, the degree of flexibility of the household's labour supply, or the quality of the household's knowledge of the functioning of financial markets."

Since we consider borrowing in our model as requiring the individual to pay an interest rate higher than the return rate, creating a wedge between saving return and borrowing cost, we focus on reviewing some studies of borrowing that consider this feature. Life-cycle models which incorporate a wedge for the borrowing cost can be found in Davis, Kübler and Willen (2002), Davis, Kübler and Willen (2005), and Krueger and Perri (2005). On the other hand, there are a number of

studies that propose a debt puzzle, usually referred to as the 'credit cards puzzle', where individuals hold assets and at the same time they borrow on their credit cards at higher interest rates. See for example Laibson, Repetto and Tobacman (2000), Bertaut and Haliassos (2002), Haliassos and Reiter (2003), Bertaut and Haliassos (2004).

We do not incorporate any type of social or partial insurance in our model, so that the only possibility of smoothing consumption in our model is by wealth accumulation and borrowing. The literature suggests that private insurance is low and is achieved mainly through durable goods and borrowing, collateralised and non-collateralised. Attanasio and Davis (1996) report a large correlation between real wages and consumption in the US, implying low between-group consumption insurance. Blundell and Preston (1998) find similar results for Britain. Blundell, Pistaferri and Preston (2004) find some partial insurance of permanent income shocks with more insurance possibilities for the college educated and those nearing retirement. They claim that adding durable expenditures to the consumption measure suggests that durable replacement is an important insurance mechanism, especially for transitory income shocks. Krueger and Perri (2005) propose a model of mutual insurance and conclude that individuals keep their consumption profiles stable with more volatile income by expanding the use of noncollateralised credit. Gruber and Martin (2003) include illiquid durable goods (housing) in a life-cycle model to find the existence of a trade-off between a household's ability to borrow against their durable stock and precautionary saving.

Finally, the effects on saving of social programmes is twofold. On the one hand, as we mentioned above, social insurance programmes are used by Hubbard *et al.* and Dynan *et al.* to show that low-income households smooth their consumption through them, hence they have no incentive to accumulate wealth. On the other hand however, Carmichael and Dissou (2000), using an endogenous growth model, claim that health insurance programmes can increase illiquid investment opposite to liquid saving, where the former contributes to growth.

3   A Life-Cycle Model with Contemporary Uncertainty and Portfolio Choice (without Borrowing)

We have stated that the critical effect of contemporary uncertainty is the effect on savings. Let's summarise first the different reasons for saving in the life-cycle framework.

1. *Patience*: Saving because the discount rate is lower than the return rate; individuals may be patient, preferring increasing paths for consumption (individuals smooth the marginal utility of consumption, not consumption itself).

2. *Risk premium*: Access to financial markets where individuals can buy risky assets that exhibit higher expected return rates may make individuals less patient in relative terms. This is because the average return rate increases with respect to the discount rate.

3. *Future labour income uncertainty*: Saving for a "rainy day"; precautionary saving when there is future uncertainty that generates the creation of a buffer stock through saving.

4. *Retirement*: Saving for retirement when there is a decrease in labour income flow in the retirement period because the retirement scheme covers only a fraction of permanent labour income.

Patience affects both wealthy and non-wealthy individuals, generating a similar pattern of change in saving behaviour. Hence, both wealthy and non-wealthy individuals will prefer increasing consumption paths.

Risk premium generates different patterns of saving depending on the portfolio decisions. If the portfolio decision is biased towards the risky asset, the opportunity cost of not saving today is larger, making saving more attractive. Although the wealthy may hold a larger amount of risky assets, their portfolios are biased towards the risk-free assets because their consumption flow depends highly on the return of their saving. On the contrary, less wealthy individuals may choose a riskier portfolio just because it does not affect them too much since their consumption flow depend mainly on their labour income. Thus, less wealthy individuals would benefit relatively more, in terms of wealth accumulation and consumption,

from access to risky assets with a higher expected return. Observe that, although the less-wealthy individuals have a relatively larger background risk that tempers their portfolio risk taking,[2] the fact that their consumption does not depend significantly on their portfolio choice makes them have riskier portfolios. Note as well that contemporary uncertainty does not generate a background risk for the portfolio decision, because contemporary uncertainty only affects present consumption, not future consumption.

Labour income uncertainty that generates the creation of a buffer stock affects more those who are relatively more affected by the uncertainty, namely those whose consumption flow depends more critically on labour income, i.e., the less wealthy. Thus, less wealthy individuals will have more incentives to save because of the precautionary motive of future labour income uncertainty.

Retirement income being lower than permanent labour income generates the incentive to save towards the retirement period. This effect is more relevant if consumption flow depends primarily on pension income flow. Hence, those who are rich enough that their labour income is not so important in their consumption, may not have sufficient incentives to create additional wealth for their retirement period: they may only consume a bit less of their wealth during each period.

What are the reasons for reducing savings in the presence of contemporary uncertainty? If saving is illiquid and cannot be used to buffer contemporary shocks, then a bad shock can make present consumption drop to zero (or close enough to zero to be highly undesirable). Even a small probability of an "extremely bad" shock can make saving very undesirable (the individual can be "left destitute").

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[2]See Gollier and Pratt (1996) for a formal analysis of the effect of background risk on risk taking.

3.1 The Model

The individual's problem is to maximise his utility function at each point in time t. Additive preferences are assumed, so that the problem is

$$\max_{\{s_t, \phi_t\}} \quad U(c_t, W_t) = \sum_{s=0}^{T-t} \beta^s E_t \left[u(c_{t+s}) \right] \qquad (1)$$

subject to

$$c_t \quad = \quad W_t + \tilde{y}_t - \tilde{x}_t - s_t \qquad (2)$$

$$W_{t+1} \quad = \quad s_t \cdot (\phi_t \tilde{R}_{t+1} + (1 - \phi_t) R_f), \qquad (3)$$

where β is the discount factor, W_t is wealth in period t (assets), $\tilde{y}_t$ is labour income shock in period t, $\tilde{x}_t$ is expenses shock, s_t is saving in period t (which has to be decided *before* uncertainty on $\tilde{x}_t$ is resolved), R_f is the risk free asset return rate which is assumed to be constant over time, $\tilde{R}_t$ is the uncertain return rate of the risky asset at time t, and ϕ_t is the proportion of the savings allocated to the risky asset in each period t. Consumption c_t must be always non-negative, imposing an implicit terminal condition that the individual must ensure non-negative consumption in the last period (period T) by maintaining enough wealth even for the worst scenario (lowest possible labour income $\underline{\tilde{y}}_T$, and highest possible expenses shock $\overline{\tilde{x}}_T$). Then, $W_T \geq \underline{\tilde{y}}_T - \overline{\tilde{x}}_T$.

The timing of the model is as follows:

1. At the beginning of period t, labour income uncertainty is resolved (the individual knows $\tilde{y}_t$). This defines information set I_{t_0}.

2. Saving and portfolio choices, s_t and ϕ_t, have to be made.

3. At *interim* period (sub-period) t_1, expenses shock uncertainty is resolved (the individual knows $\tilde{x}_t$). This defines the information set I_{t_1}.

4. At the end of sub-period t_1 consumption occurs.

This framework implies that the only possibility of transferring resources from one period to another is by saving, which becomes illiquid after it has been chosen. There is no liquid asset to buy, not even at a zero return rate. Hence, "to keep the money under the mattress" is not allowed.

This structure of illiquid saving implies that saving has a high cost when there is contemporary uncertainty. It reflects that decisions such as savings have to be made under different sorts of uncertainty that are resolved at different stages. One possible interpretation is a case where the money is depreciated completely if it is kept in hand from one period to another. This could be thought to be the situation faced by an individual in an economy with an extremely high inflation rate. Then, the individual is forced to decide savings before he knows the realisation of some contemporary uncertainties. From another perspective, this framework could be interpreted simply as a reduced form for savings, where individuals simply do not want to liquidate their savings to buffer some contemporary shocks. This saving commitment might arrive because individuals have some sort of conflict between long-run and short-run preferences, such as in the models of hyperbolic discounting proposed by Laibson *et al.* (2000) or models of "accountant-shopper" proposed by Haliassos and Reiter (2003).

Formally, the problem is to solve the Bellman equation:

$$V(W_t) \quad = \quad \max_{\{s_t, \phi_t\}} \{E_{t_0}[u(c_t) + \beta V(W_{t+1})]\} \tag{4}$$

where V is the value function.

The first order conditions (FOCs) are:

$$\frac{\partial(\cdot)}{\partial s_t} \quad = \quad E_{t_0}[u'(c_t) \cdot (-1)] + \beta E_{t_0}[V'(W_{t+1}) \cdot (\phi_t \tilde{R}_{t+1} + (1 - \phi_t)R_f)] = 0$$

$$E_{t_0}[u'(c_t)] = \beta \cdot E_{t_0}[V'(W_{t+1})(\phi_t \tilde{R}_{t+1} + (1 - \phi_t)R_f)]. \tag{5}$$

and

$$\frac{\partial(\cdot)}{\partial \phi_t} \quad = \quad \beta E_{t_0}[V'(W_{t+1})s_t \tilde{R}_{t+1} - V'(W_{t+1})s_t R_f] = 0$$

$$E_{t_0}[V'(W_{t+1})\tilde{R}_{t+1}] = R_f E_{t_0}[V'(W_{t+1})]. \tag{6}$$

The envelope condition is:

$$\frac{\partial V(W_t)}{\partial W_t} \quad = \quad V'(W_t) = E_{t_0}[u'(c_t)]$$

then

$$V'(W_{t+1}) = E_{t_0+1}[u'(c_{t+1})]. \tag{7}$$

Using the envelope condition (7) in (5) and (6), and applying the law of iterated expectations, we obtain the Euler equation and the portfolio allocation equation, respectively:

$$E_{t_0}[u'(c_t)] = \beta \cdot E_{t_0}[u'(c_{t+1})(\phi_t \tilde{R}_{t+1} + (1 - \phi_t)R_f)] \tag{8}$$

$$E_{t_0}[u'(c_{t+1})\tilde{R}_{t+1}] = R_f E_{t_0}[u'(c_{t+1})]. \tag{9}$$

3.2 Solving the Model

We solve the model numerically backwards for 60 periods accounting from ages 80 to 20. This allows us to include a retirement period where labour income becomes certain, being a fraction of the last permanent income. The information set in t_0 defines the state variable $(W_t + y_t)$ which determines the control variables s_t and ϕ_t. We use isoelastic preferences (CRRA) so that $u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$, where ρ is the relative risk aversion index.

We solve the first order conditions for the last period and then we solve backwards recursively. Since we have two control variables, we solve the model in two steps. First, we solve the portfolio decision (equation (6)) as a function of absolute saving, obtaining a function $\phi_t(s_t)$ (we actually obtain only values of function $\phi_t(s_t)$ for different nodes in a grid). Then, using function $\phi_t(s_t)$, by means of interpolation, we solve the saving problem (equation (5)). We use linear interpolation and spline interpolation. More details of the algorithm used and the scheme for simulations are given in Appendix A.

The parameters used in solving the different models are summarised in Table 1. We have a benchmark model to which we add the different features that generate saving: patience, risky asset, labour income uncertainty and retirement. For each of those models we solve for the cases with and without uncertainty. We finally solve the models, with all the features that generate saving to analyse the overall effect, and with all but the risky asset. This allows us to analyse the effect of the risky asset holding on top of the other features.

The parameters that are used are standard in the literature. We discuss them

in some detail because they are critical for the magnitude of the results we obtain. The risk aversion index has been widely analysed. Gollier (2002b, pp.31) shows that it is reasonable to believe that relative risk aversion is between 1 and 4.[3] We use 1.5 in our benchmark model, following Attanasio and Weber (1995). We choose 5% for the return rate. This is considerably higher than the return rate used in the literature for developed countries.[4] However, it reflects much better the case of developing countries, which have much more scope for long run high interest rates. Also, in developing countries differences in wealth are substantial and a large proportion of the population start their working life having no wealth at all. We choose 5% for the discount rate, reducing it to 1% for the case of patience.[5] The risk premium is set to 5% with a standard deviation of 16%. Gollier (2002b, pp.35) states that "...Historically, the equity premium has been around 6 percent per year over the century in the U.S. markets. The standard deviation of yearly U.S. stock returns over the same period is 16 percent."[6] We normalise the model using permanent labour income. So, labour income is normalised to 1, and transitory shocks have a standard deviation of 17%.[7] For the sake of simplicity, we do not have any permanent income shocks in our model. Although this may be seen as unrealistic, it allows us to focus on the pure effect of saving rates over consumption paths, without any noise that may arise from permanent shocks.

The replacement rate at retirement is set to 0.75.[8] Finally, we set the worst negative shock to 0.725 with a probability of 5% to allow consumption to be very low in the worst case.[9] Notice that the probability of having a bad income shock simultaneously with the worst negative shock ($\tilde{x}_t$), is less than 1%. However, this is enough to make high saving unappealing for those with no wealth.

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[3]Hubbard, Skinner and Zeldes (1995) use risk a aversion coefficient equal to 3, Castañeda, Díaz-Giménez and Ríos-Rull (2003) use 1.5, Viceira (2001) use different values between 2 to 12, Davis, Kübler and Willen (2005) between 0.5 and 8.

[4]Hubbard *et al.* use a risk-free return rate equal to 3%, Viceira use 2% Davis *et al.* use 2%, Low (2005) uses 1.6%.

[5]Hubbard *et al.* use a discount rate equal to 3%, Viceira uses 10%, Castañeda *et al.* use 8%, Davis *et al.* use 5%, Low uses 3%.

[6]Viceira uses an excess return of 4.2% with a standard deviation of 18%, while Davis *et al.* use 8% for the excess return and 15% for the standard deviation.

[7]Viceira uses 10%, Davis *et al.* use different values from 15% to 21%.

[8]Dynan *et al.* use a replacement rate equal to 60% and 75%, Davis *et al.* use from 20% to 100% (80% in the baseline model).

[9]Dynan *et al.* use health shocks of one quarter of retirement income with a probability of 10%.

Table 1: Parameters for Solving the Life-Cycle Model

		Models				
		Benchmark	Patience	Risky Asset	Labour Income Uncert.	Retirement
(i)	Risk Aversion Index	1.5				
(ii)	Discount rate	0.05	0.01			
(iii)	Risk-free return rate	0.05				
(iv)	Risk premium	0				
(v)	Risky Asset shocks	-		{0.82, 1.10, 1.38}		
(vi)	Risky Asset shocks probabilities	-		{0.16, 0.66, 0.16}		
(vii)	Labour Income shocks	{1}			{0.75, 1.00, 1.34}	
(viii)	Labour Income shocks probabilities	{1}			{0.16, 0.66, 0.16}	
(ix)	Replacement rate at retirement	1				0.75
(x)	Contemporary Shocks					
	(a) *Contemporary Certainty*					
	Shocks	{0}				
	Shocks probabilities	{1}				
	(b) *Contemporary Uncertainty*					
	Shocks	{0, 0.1, 0.725}				
	Shocks probabilities	{0.70, 0.25, 0.05}				

Notes: Contemporary certainty and uncertainty shocks and probabilities apply to all models.

3.3 Main Findings

After solving the model we run 10,000 simulations for different types of individuals in order to obtain expected profiles for the different variables. The results of the model without risky asset with contemporary certainty and with contemporary uncertainty are presented in Figures 1 and 2 respectively. In both cases we have included patience, labour income uncertainty and retirement as saving motives. We have chosen five different levels of initial wealth: 0.0001, which is virtually zero; 0.01, which is 10% of the normalised permanent labour income; 0.5, which is 50% of the normalised permanent labour income; 1, which is equal to the normalised permanent labour income; and 5, which is five times the normalised permanent labour income. We will observe in the graphs that paths for initial wealth equal to 0.0001 and equal to 0.01 overlap almost perfectly. As an order of magnitude, it is interesting to note that lifetime permanent labour income in the simplest case with no retirement is $\sum_{t=20}^{t=80} \frac{y_t}{(1+r)^{t-20}} \approx 19.98$ for r=5%.

As we expected, we have humped shaped wealth accumulation paths for all the different levels of initial wealth. Consumption paths are increasing, reflecting both the patience feature and the result of the precautionary saving feature, on top of the desire to accumulate wealth for retirement. This result is in line with the argument of Dynan, Skinner and Zeldes (2002), who claim that accumulated wealth can serve multiple purposes.[10] Saving paths are increasing as wealth accumulates and they start to decrease around age 50. However, saving rates[11] of those with

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[10]Actually, Dynan *et al.* argue that it is useless to attempt to distinguish precautionary saving from saving for bequests on an ex ante basis. However, the argument is equally valid for retirement accumulation.

[11]sr_t is saving rate defined as savings over cash-in-hand when making the saving decision, i.e., $sr_t = \frac{s_t}{W_t + y_t}$.

low-initial-wealth are much lower than those with high-initial-wealth for the first 20 years of the life-cycle.

When comparing contemporary certainty versus contemporary uncertainty (Figure 1 versus Figure 2), interesting results emerge. First, under contemporary uncertainty, all groups accumulate less wealth as a result of lower saving rates at the beginning of the life-cycle. However, saving rates, for low-initial-wealth groups are much more affected by contemporary uncertainty, remaining significantly different for the first 30 years of the life-cycle. The result is that consumption paths tend to widen. We will discuss these differences in consumption later on.

Figure 3 and Figure 4 present the results of the models with access to risky asset with contemporary certainty and with contemporary uncertainty respectively. Wealth accumulation paths are hump shaped, but risk premium allow them to accumulate much higher levels of wealth (compared to the case without risky asset) because of higher expected return rates. Actually, at the top of the hump, with risky asset the accumulation of wealth reaches almost double what it is without risky asset. Consumption paths are increasing as they benefit significantly from accumulated wealth. Saving rates are much lower for low-initial-wealth groups for the first 10 years. This is significantly less time than in the case without risky asset, which is explained by the fact that the saving rate depends on the level of wealth (higher wealth implies a higher saving rate, although the effect is decreasing), so that the ability to rapidly accumulate wealth diminishes the initial effect.

Portfolio choice shows risky asset specialisation for low levels of wealth, when wealth is not yet seen as the major consumption financier. Observe that portfolio specialisation ends when wealth is larger than the present value of future lifetime labour income, which is around 20. The higher the level of wealth, the more conservative the portfolio becomes. Recalling Gollier and Pratt (1996), this is the tempering effect of background risk in action. The fact that low-initial-wealth groups exhibit riskier portfolios does not mean that they hold more risky assets. On the contrary, it means that they have less wealth, so that the flow of labour income is more relevant for them.

When comparing contemporary certainty versus contemporary uncertainty (Fig-

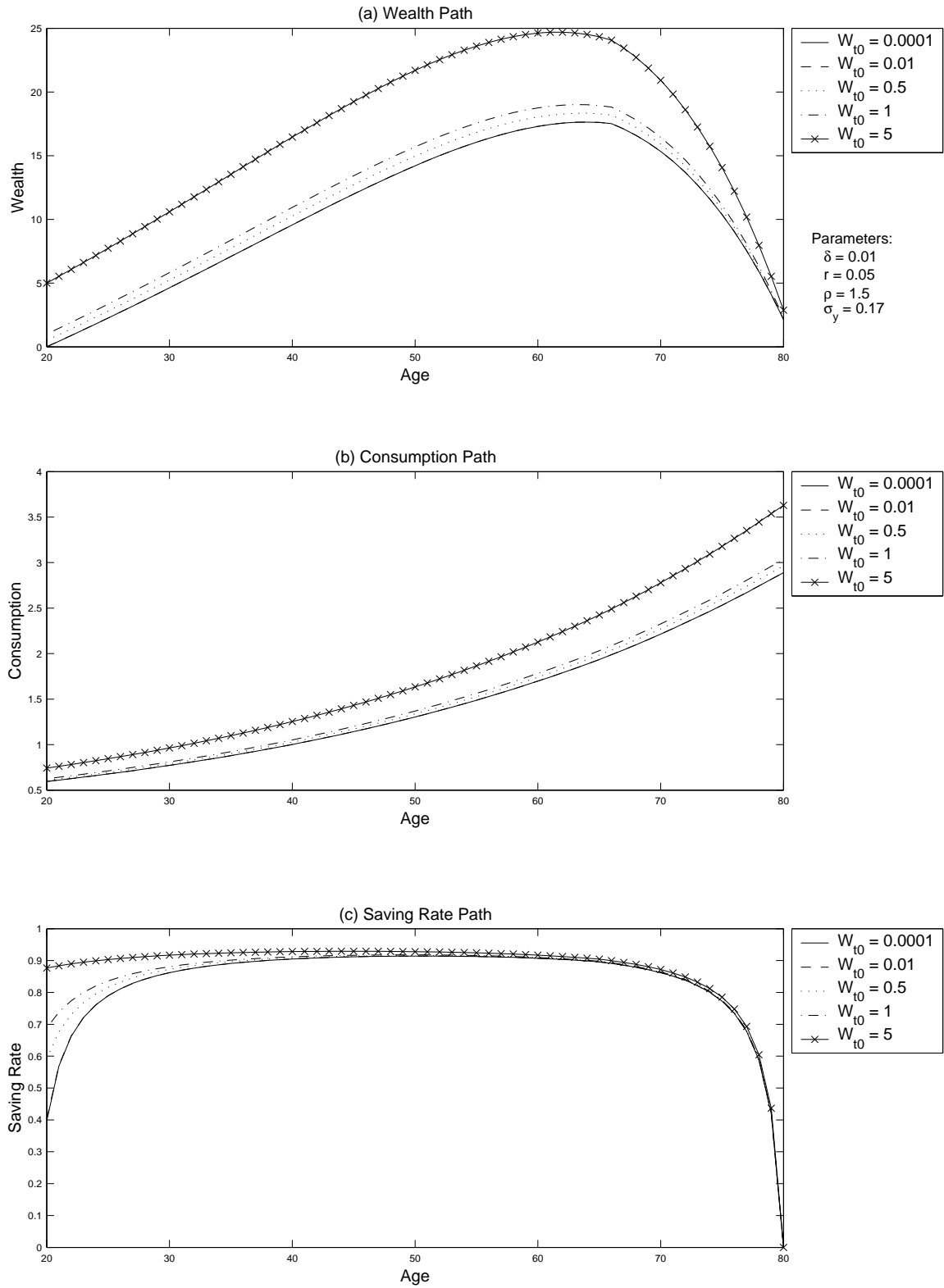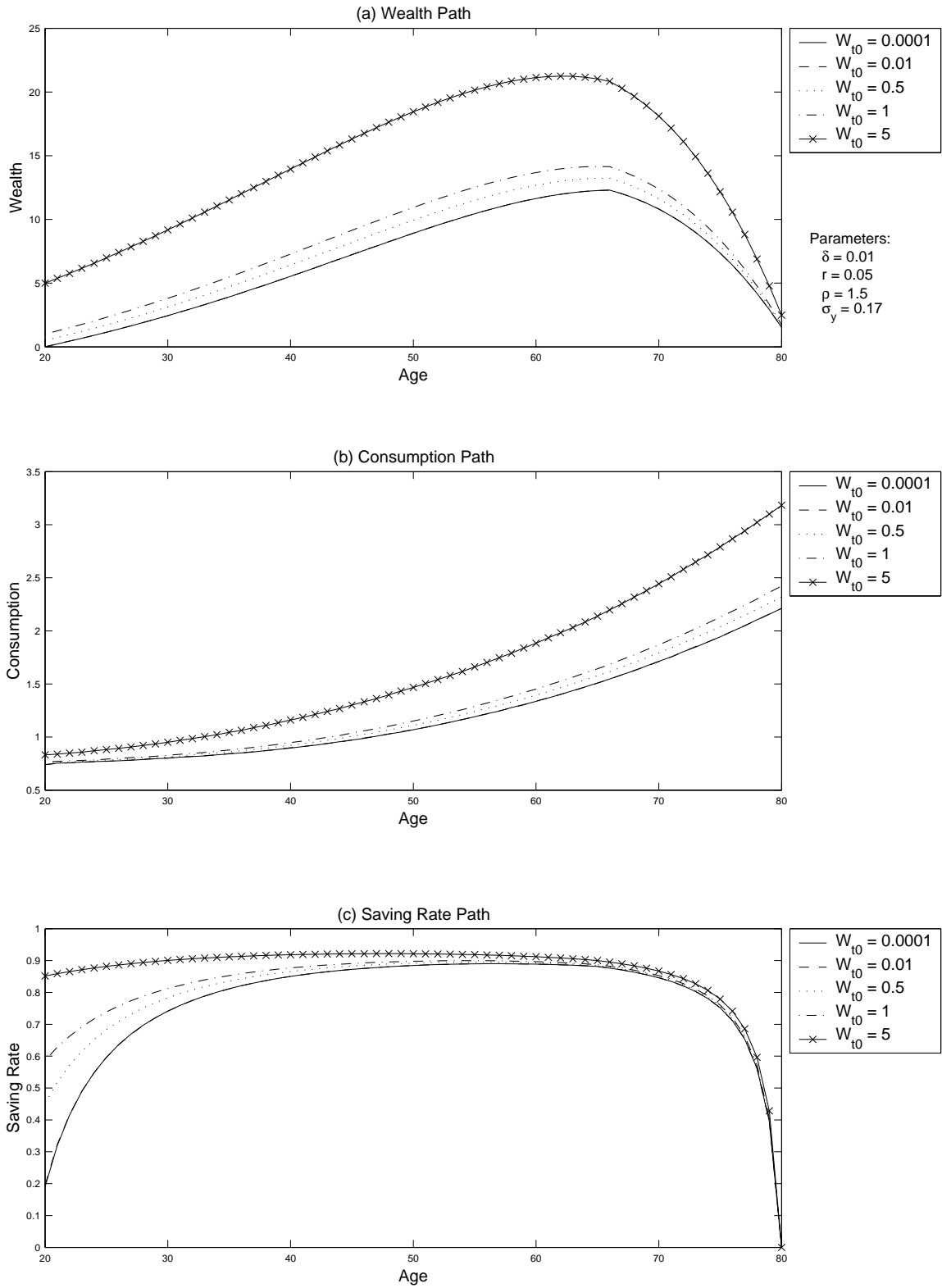Figure 1: Contemporary Certainty, no Risky Asset

Figure 2: Contemporary Uncertainty, no Risky Asset

(a) Wealth Path

(b) Consumption Path

(c) Saving Rate Path

ure 3 versus Figure 4), we observe patterns that are similar to those in the case without risky asset. First, all groups accumulate less wealth as a result of lower saving rates at the beginning of the life-cycle. However, saving rates for low-initial-wealth groups are much more affected by contemporary uncertainty, remaining significantly different for the first 20 years of the life-cycle (only 10 years under contemporary certainty). The result is that consumption paths tend to widen more.

We now turn to the analysis of differences in consumption. It is worth observing that consumption inequality within cohorts increases through time constantly because of similar-to-random walk behaviour. This is analysed in detail in Deaton and Paxson (1994). Nevertheless, our focus is on expected consumption for those who are initially identical in all but their initial wealth endowment. Let's first summarise some facts from the individual addition of the saving motives to the model. First, patience generates that individuals prefer an increasing consumption path, which requires them to generate a humped shaped path of wealth to finance it. Second, risky asset access, with a corresponding risk premium, is incentive enough for higher saving rates to generate increasing consumption paths and humped shaped wealth paths. Third, labour income uncertainty generates precautionary wealth accumulation only for the very poor, because the labour income uncertainty is not relevant enough for those who finance their consumption mainly out of wealth (the wealthy). Fourth, retirement generates wealth accumulation only for the very poor because they will need to create a stock in order to finance the decrease in income in the retirement period. The wealthier just have to save a bit more during their lifetime, but do not have to create additional wealth.

Table 2 presents the results of the differences in consumption as a result of life-cycle decisions. We focus on the comparison between two groups of individuals. The first group have initial wealth $W_{t=20} = 5$ (the wealthy), and the second group have initial wealth $W_{t=20} = 0.0001$ (the very poor), i.e., virtually nothing. The groups of individuals differ in initial wealth, otherwise they are identical, sharing the same preferences and labour income paths.[12] The inequality indicator that we present is the fraction of excess consumption of the high-initial-wealth over the

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[12]We have chosen flat income profiles to isolate the effects of wealth accumulation and contemporary uncertainty, stressing the relevance of the latter.
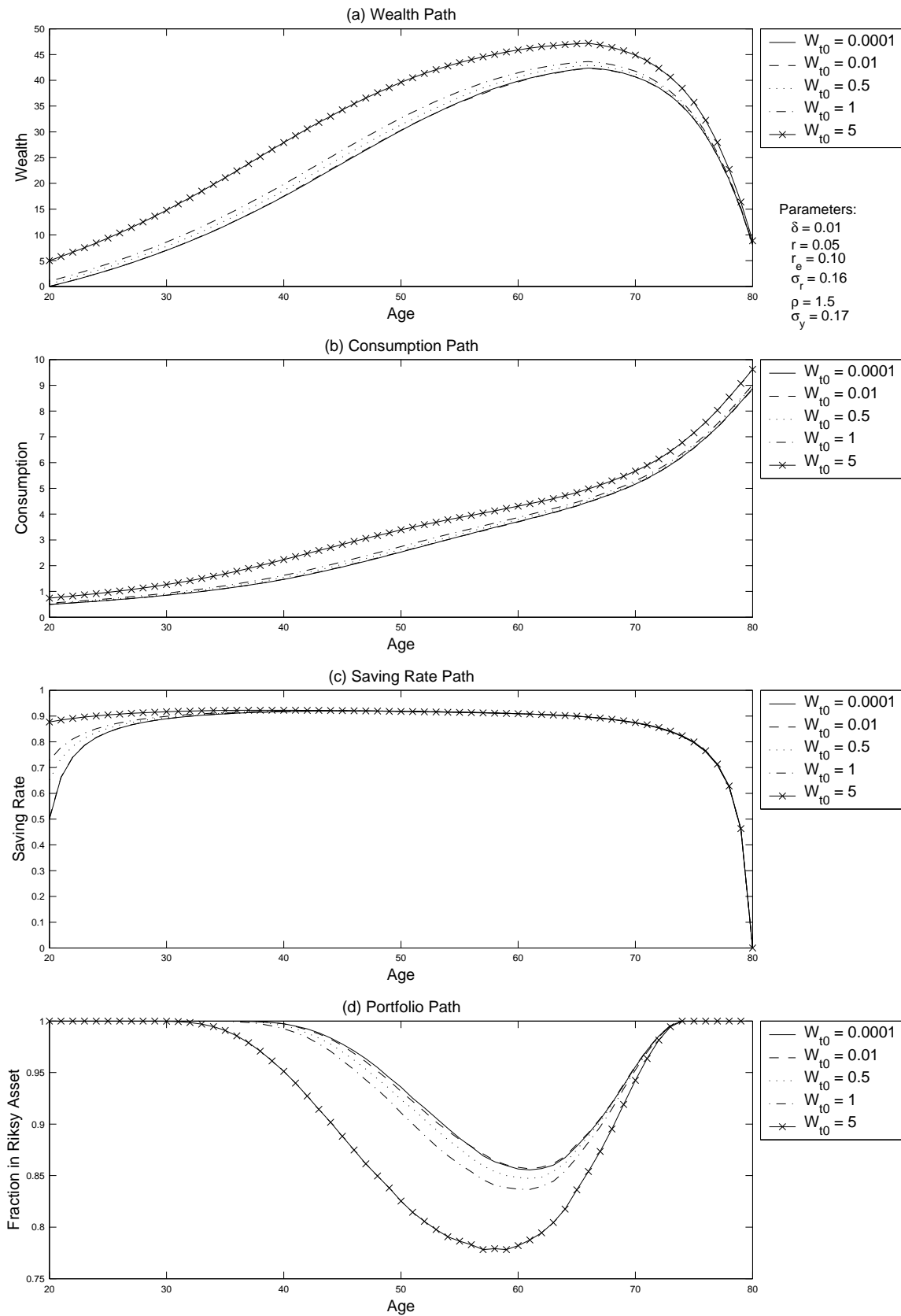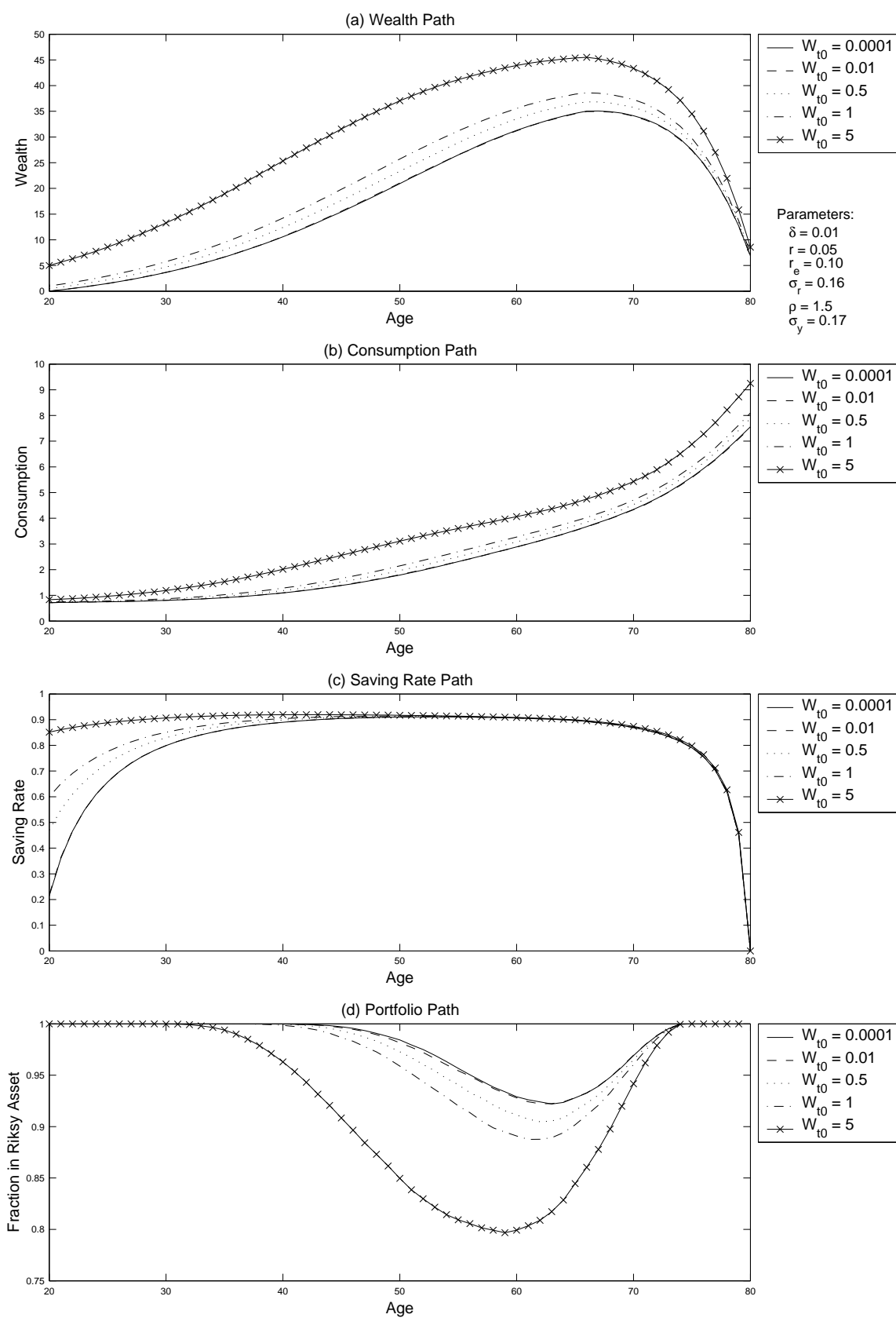
Figure 3: Contemporary Certainty, with Risky Asset

(a) Wealth Path

Parameters:
$\delta = 0.01$
$r = 0.05$
$r_e = 0.10$
$\sigma_r = 0.16$
$\rho = 1.5$
$\sigma_y = 0.17$

(b) Consumption Path

(c) Saving Rate Path

(d) Portfolio Path

Legend (all panels):
$W_{t0} = 0.0001$
$W_{t0} = 0.01$
$W_{t0} = 0.5$
$W_{t0} = 1$
$W_{t0} = 5$

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Figure 4: Contemporary Uncertainty, with Risky Asset

(a) Wealth Path

Legend:
- $W_{t0} = 0.0001$
- $W_{t0} = 0.01$
- $W_{t0} = 0.5$
- $W_{t0} = 1$
- $W_{t0} = 5$

Parameters:
$\delta = 0.01$
$r = 0.05$
$r_e = 0.10$
$\sigma_r = 0.16$
$\rho = 1.5$
$\sigma_y = 0.17$

(b) Consumption Path

Legend:
- $W_{t0} = 0.0001$
- $W_{t0} = 0.01$
- $W_{t0} = 0.5$
- $W_{t0} = 1$
- $W_{t0} = 5$

(c) Saving Rate Path

Legend:
- $W_{t0} = 0.0001$
- $W_{t0} = 0.01$
- $W_{t0} = 0.5$
- $W_{t0} = 1$
- $W_{t0} = 5$

(d) Portfolio Path

Legend:
- $W_{t0} = 0.0001$
- $W_{t0} = 0.01$
- $W_{t0} = 0.5$
- $W_{t0} = 1$
- $W_{t0} = 5$

low-initial-wealth. This is: $\frac{c_t(W_{20}=5)}{c_t(W_{20}=0)} - 1$ at $t = 20$ and then at $t = 80$. We observe that the benchmark model does not generate additional differences in consumption either with contemporary certainty or with contemporary uncertainty.[13]

When only patience is added to the model, contemporary uncertainty has a significant effect in generating differences in consumption, where the consumption of the wealthy increases to 44% more than the consumption of the poor, having started at 11% more only. When only the risky asset feature is introduced, the contemporary certainty case would allow the consumption gap to shrink, but the contemporary uncertainty case generates the contrary effect, increasing the differences from 33% at the beginning of the life-cycle to 59% at the end of it (having reached 70% at the age 52). When only labour income uncertainty is switched on in the model, the poor individuals have a strong reason for precautionary saving, accumulating very fast at the beginning of the life cycle to create a buffer stock. The result is that initial consumption differences are large, but the gaps shrink significantly for both contemporary certainty and uncertainty (from 38% to 12% under contemporary certainty, and from 44% to 11% under contemporary uncertainty). Finally, when only retirement is switched on in the model, no changes in consumption differences are observed under contemporary certainty, or under contemporary uncertainty.

Table 2: Consumption Differences under Different Saving Motives

Excess consumption of wealthier over the life-cycle ($\rho = 1.5$) $\left(\frac{c_{20}(W_{20}=5)}{c_{20}(W_{20}=0)} - 1 \quad to \quad \frac{c_{80}(W_{20}=5)}{c_{80}(W_{20}=0)} - 1 \right)$		
	Contemporary Certainty	Contemporary Uncertainy
Benchmark model	0.25 to 0.25	0.27 to 0.27
+ Patience	0.24 to 0.26	0.11 to 0.44
+ Risky asset	0.47 to 0.34	0.33 to 0.59[a]
+ Labour income uncertainty	0.38 to 0.12	0.44 to 0.11
+ Retirement	0.25 to 0.25	0.27 to 0.27

[a] Has a hump to 0.70 at age 52.

Figure 5 presents the paths for the differences in consumption for the models with and without risky asset and with and without contemporary uncertainty. All models include patience, labour income uncertainty and retirement. Panel (a) presents

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[13]Observe that in the simplest case, with no retirement period, lifetime wealth is $LW = W_{t=20} + \sum_{t=20}^{t=80} \frac{y_t}{(1+r)^{t-20}} \approx W_{t=20} + 19.98$ for r=5%. Then, the ratio of lifetime wealth is $\frac{LW(W_{t=20}=5)}{LW(W_{t=20}=0)} \approx 1.25$. We can see that this reflects directly on consumption ratios.

the results for risk aversion index equal to 1.5, and panel (b) for risk aversion index equal to 2.5. The consumption inequality index corresponds to the fraction of excess consumption of the high-initial-wealth over the low-initial-wealth over the life-cycle, i.e., $\frac{c_t(W_{20}=5)}{c_t(W_{20}=0)} - 1$ from $t = 20$ until $t = 80$.[14] The baseline case implies contemporary certainty and no availability of risky asset. We can observe that the consumption inequality index remains flat over the life-cycle on a value approximately equal to 0.25, which is the ratio of lifetime wealth minus one. Recall that differences in lifetime wealth arrive only because of differences in initial wealth (labour income profile is identical for all individuals). Any deviation from that ratio is a consequence of the consumption/saving decision process under uncertainty.

Focusing on panel (a), without risky assets there is no increase in consumption differences under contemporary certainty (baseline case), but a significant increase under contemporary uncertainty. When risky asset is included in the model, under contemporary certainty the difference in consumption starts higher, but shrinks significantly towards the end of the life-cycle, being much lower than it is without risky asset. However, when contemporary uncertainty is included over the risky asset model, there is much less differences at the beginning of the life-cycle, a big jump in the middle of the life-cycle, and a significant decrease towards the end of the life-cycle.

This set of results can be explained as follows. At the beginning of the life-cycle, the individuals behave mainly as buffer-stock agents (precautionary saving), and only after the age of 40 do they start saving for retirement;[15] patience is affecting the whole life-cycle similarly. Since poorer individuals are those who are more affected by labour income uncertainty, contemporary uncertainty plays a significant role in preventing them from saving more than they would do otherwise. Then, only after the initially-poor have accumulated enough wealth so that labour income uncertainty is of no relevance to them, they will finally take advantage of the risky asset and accumulate wealth at a faster pace to catch up a bit with the wealthier in terms of consumption.

Now turn to panel (b) of Figure 5. The main result, compared to panel (a), is that the considerable difference in consumption created initially by risky asset

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[14]This is the same index used in Table 2.

[15]This is well documented by Carroll (1997) and Gourinchas and Parker (2002).

has a tendency to remain, particularly under contemporary uncertainty. Higher risk aversion generates much lower saving rates at the beginning of the life cycle under contemporary uncertainty, generating lower paths of wealth accumulation for all groups. This implies that the poor will never manage to accumulate enough wealth to catch up with the saving rates of the wealthier. Thus, the differences in consumption will not diminish towards the end of the life cycle.

The results of Figure 5 are summarised in Table 3 jointly with the results of the model with risk aversion equal to 3.5. The results confirm the tendency observed above, although the effect of precautionary saving is boosted by the risk aversion, which is reflected in the case of contemporary uncertainty without risk asset, which starts at 32%, then has an inverted hump to 22%, to end the life-cycle with a difference of 42%. When the risky asset is included in the model, its effect on the saving rate oversets precautionary saving differences, resulting in higher differences in consumption that remain high towards the end of the life-cycle.

Table 3: Consumption Differences under Different Levels of Risk Aversion

Excess consumption of wealthier over the life-cycle $\left(\frac{c_{20}(W_{20}=5)}{c_{20}(W_{20}=0)} - 1 \quad to \quad \frac{c_{80}(W_{20}=5)}{c_{80}(W_{20}=0)} - 1 \right)$		
	Contemporary Certainty	Contemporary Uncertainy
$\rho = 1.5$		
Patience + Lab.Inc.Uncertainty + Retirement	0.25 to 0.25	0.13 to 0.44
Patience + Lab.Inc.Uncertainty + Retirement + Risky Asset	0.49 to 0.08	0.16 to 0.22[a]
$\rho = 2.5$		
Patience + Lab.Inc.Uncertainty + Retirement	0.25 to 0.26	0.23 to 0.46
Patience + Lab.Inc.Uncertainty + Retirement + Risky Asset	0.43 to 0.25[b]	0.30 to 0.61[c]
$\rho = 3.5$		
Patience + Lab.Inc.Uncertainty + Retirement	0.26 to 0.26	0.32 to 0.42[d]
Patience + Lab.Inc.Uncertainty + Retirement + Risky Asset	0.42 to 0.40[e]	0.40 to 0.87

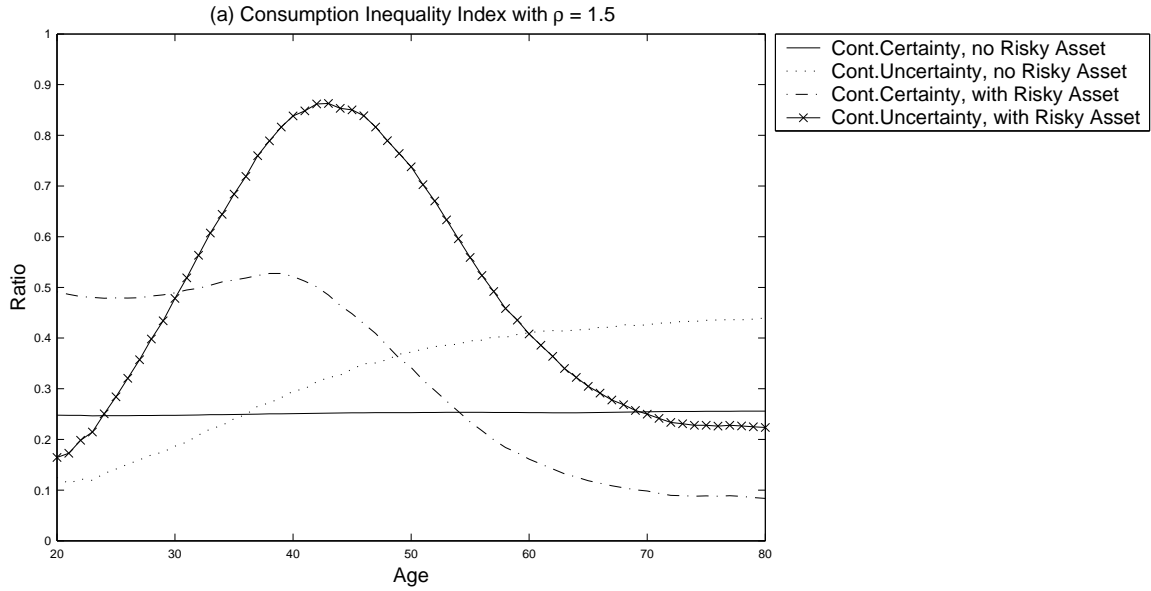[a] Has a hump to 0.86 at age 42.

[b] Has a hump to 0.48 at age 44.

[c] Has a hump to 0.80 at age 52.

[d] Has an inverted hump to 0.22 at age 24.

[e] Has a hump to 0.49 at age 50.

What is interesting to remark is that under contemporary certainty there is no increase in consumption differences over the life-cycle. In fact, the gap is reduced in models with risky asset. On the contrary, differences in consumption increase for all models when contemporary uncertainty is switched on. Moreover, that effect can be boosted when risky asset is introduced for high levels of risk aver-

Figure 5: Consumption Inequality

(a) Consumption Inequality Index with $\rho = 1.5$

Legend:
— Cont.Certainty, no Risky Asset
⋯ Cont.Uncertainty, no Risky Asset
−·− Cont.Certainty, with Risky Asset
−×− Cont.Uncertainty, with Risky Asset

(b) Consumption Inequality Index with $\rho = 2.5$

Legend:
— Cont.Certainty, no Risky Asset
⋯ Cont.Uncertainty, no Risky Asset
−·− Cont.Certainty, with Risky Asset
−×− Cont.Uncertainty, with Risky Asset

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sion. Then, under contemporary certainty, the risky asset plays an equaliser role, whereas under contemporary uncertainty it increases inequalities.

Recall that these results assume that there is equal access to risky assets, where low-wealth individuals invest most of their savings in early life (their consumption does not depend too much on accumulated wealth, so that their portfolios are less conservative). If this was not the case, meaning that the risky asset is only accessible for the wealthier, inequality can be boosted by the inability of the low-wealth individuals to generate wealth from investing in risky assets. This implies that augmenting the model to account for entry costs into financial markets may be particularly interesting (see section 5).

4   Borrowing Against Illiquid Savings

In section 3, our life-cycle model with contemporary uncertainty produced a high demand for borrowing. However, borrowing was not permitted. In this section, we augment the model so that borrowing becomes admissible. We allow the individual to borrow after contemporary uncertainty is resolved. The individual can borrow only against his savings, not against his future labour income as a flow of his human capital. The problem then has two state variables for the borrowing decision: available resources, and available borrowing. This complicates significantly the numerical solutions as we will see, so we limit our model with borrowing to the case without risky asset.

The possibility of borrowing makes saving no longer illiquid, it becomes *partially illiquid*. This is particularly important for our analysis of inequalities in consumption. The fact that borrowing becomes available may break the link through which contemporary uncertainty generates an increase in consumption differences over the life-cycle. Certainly, the cost of liquidation becomes critical. The model without borrowing can be considered as the extreme case of the model with borrowing where the cost of borrowing is infinity.

The main idea is to formulate a model which accounts for the fact that individuals hold savings in some sort of illiquid assets and have a (costly) means to

partially liquidate it. This can be interpreted as a case where individuals save in saving accounts or similar investments and at the same time borrow from their credit cards, at much higher interest rates.

The central element of the model is that we assume that there are different types of uncertainties that are resolved at different moments, and that saving and borrowing decisions are made under different information sets.

The individual's problem is to maximise his utility function at each point in time t. Additive preferences are assumed, so that the problem is:

$$\max_{\{s_t, b_t\}} \quad U(c_t, W_t) = \sum_{t=0}^{T} \beta^t E_t \left[u(c_t) \right] \tag{10}$$

subject to

$$c_t \quad = \quad W_t + \tilde{y}_t - \tilde{x}_t - s_t + b_t \tag{11}$$

$$W_{t+1} \quad = \quad s_t R - b_t R_b \tag{12}$$

$$b_t \quad \leq \quad \frac{s_t R}{R_b} \tag{13}$$

$$R_b \quad \geq \quad R \tag{14}$$

$$s_t \quad \leq \quad W_t + y_t \tag{15}$$

$$\min\{\tilde{y}_t\} \quad > \quad \max\{\tilde{x}_t\}, \tag{16}$$

where β is the discount factor, W_t is wealth in period t (assets), $\tilde{y}_t$ is labour income shock in period t, $\tilde{x}_t$ is negative shock in period t, s_t is saving in period t, b_t is borrowing in period t, R is the asset return rate assumed to be certain and constant over time, and R_b is the cost of borrowing. Observe that borrowing is restricted to the liquidation of savings, which would normally be costly $(R_b > R)$, meaning that it is not possible to borrow against future labour income. The borrowing constraint can be interpreted as a solvency constraint that prevents the households from going bankrupt in every state of the world. Moreover, the restrictions on savings and on shocks ensure that consumption will always be positive (even at wealth equal to zero).

Labour income is random for $t < t_r$, where t_r is retiring age, namely 65 years of age. When $t \geq t_r$, labour income becomes pension $y_t = \gamma \cdot y_p$, where γ is the fraction of permanent income y_p received. Notice that pension has no uncertain component.

The timing of the model is as follows:

1. At the beginning of period t, labour income uncertainty is resolved (the individual knows $\tilde{y}_t$). This defines information set I_{t_0}.

2. The saving decision s_t has to be made.

3. At the beginning of *interim* period (sub-period) t_1, negative shock uncertainty is resolved (individual knows $\tilde{x}_t$). This defines information set I_{t_1}.

4. The individual decides to borrow b_t.

5. At the end of sub-period t_1 consumption occurs.

Given the timing of the model, it implies that saving and borrowing decisions are made under different information sets, meaning that there are two maximisation problems. These maximisation problems are actually nested one inside the other. When choosing saving, the individual is assumed to know how much he would borrow in any possible contingency under information set I_{t_1}.

Then, the problem is to solve the following Bellman equation:

$$V(W_t) = \max_{\{s_t\}}\left\{ E_{t_0}\left[\max_{\{b_t\}}\left\{u(c_t) + \beta E_{t_1}[V(W_{t+1})]\right\}\right]\right\} \tag{17}$$

where V is the value function.

Solving first the inner maximisation problem, we define the first condition for the borrowing decision, taking saving as given. Define $V_1(W_t) = u(c_t) + \beta E_{t_1}[V(W_{t+1})]$ as the inner expression to be maximised. Then, the first order condition for borrowing is:

$$\frac{\partial V_1(W_t)}{\partial b_t} = u'(c_t) + \beta R_b \cdot E_{t_1}[V'(W_{t+1})] = 0. \tag{18}$$

By solving (18) we obtain $b_t^* = b_t^*(W_t + y_t - x_t - s_t, s_t)$, which is the optimum borrowing given the information set I_{t_1} (note that the tildes over y_t and x_t have been removed as they become known). Observe that the state variables for choosing b_t are $(W_t + y_t - x_t - s_t)$ and s_t. Recall that these state variables represent how much is available for consumption, $(W_t + y_t - x_t - s_t)$, and how much is available for borrowing, s_t. For simplicity, the state variables can be simplified to $(W_t + y_t - x_t)$ and s_t.

Having solved for the borrowing decision, the individual chooses his optimum level of saving. The first order condition for saving is:

$$\frac{\partial V(W_t)}{\partial s_t} = E_{t_0}[u'(c_t)]\left(-1 + \frac{\partial b_t^*}{\partial s_t}\right) \tag{19}$$

$$+ \beta \cdot E_{t_0}[V'(W_{t+1})]\left(R - R_b \frac{\partial b_t^*}{\partial s_t}\right) = 0.$$

However, using the law of iterated expectations, the first order condition (20) can be rewritten as:

$$E_{t_0}[u'(c_t)](-1) + \beta R \cdot E_{t_0}[V'(W_{t+1})] + E_{t_0}\left[\frac{\partial b_t^*}{\partial s_t}\left(\underbrace{u'(c_t) + \beta R_b \cdot E_{t_1}[V'(W_{t+1})]}_{\frac{\partial V_1(W_t)}{\partial b_t} = 0}\right)\right] = 0.$$

Then, the first order condition for saving reduces to

$$E_{t_0}[u'(c_t)](-1) + \beta R \cdot E_{t_0}[V'(W_{t+1})] = 0. \tag{20}$$

Another way of understanding this envelope condition for saving is taking the total derivative, so that

$$\frac{dV(W_t)}{ds_t} = \frac{\partial V(W_t)}{\partial V_1(W_t)} \cdot \frac{\partial V_1(W_t)}{\partial b_t^*} \cdot \frac{\partial b_t^*}{\partial s_t} + \frac{\partial V(W_t)}{\partial s_t}.$$

From the first order condition of borrowing (18) we know that $\frac{\partial V_1(W_t)}{\partial b_t^*} = 0$, then the envelope condition implies that

$$\frac{dV(W_t)}{ds_t} = \frac{\partial V(W_t)}{\partial s_t}. \tag{21}$$

On the other hand, the envelope condition for W_t is

$$\frac{\partial V(W_t)}{\partial W_t} = V'(W_t) = E_{t_0}[u'(c_t)] \tag{22}$$

then

$$V'(W_{t+1}) = E_{(t+1)_0}[u'(c_{t+1})]. \tag{23}$$

Using the envelope condition (23) in (18) and (20), and applying the law of iterated expectations, we obtain the corresponding Euler equations for borrowing and consumption:

$$u'(c_t) = \beta R_b \cdot E_{t_1}[u'(c_{t+1})] \tag{24}$$

$$E_{t_0}[u'(c_t)] = \beta R \cdot E_{t_0}[u'(c_{t+1})]. \tag{25}$$

4.1 Main Findings

We run the borrowing model with all saving motives switched on (patience, labour income uncertainty and retirement) for different values of risk aversion (1.5, 2.5, and 3.5) and different sizes for the borrowing cost wedge (2%, 5%, and 10%). The rest of the parameters are the same as in section 3, i.e., the discount rate 1%, the risk free return rate 5%, and the corresponding shocks and probabilities as in Table 1.

Figure 6 and Figure 7 present the results for borrowing rates equal to 7% and 15% repectively (2% and 10% borrowing cost wedge). We can observe that, when allowed to borrow against their illiquid savings, individuals use that possibility rather intensively, particularly those who are relatively more affected by contemporary uncertainty, namely the less wealthy. At the beginning of the life-cycle, individuals have accumulated less wealth, so they are more exposed to the negative shocks, then they borrow much more intensively than they do later in life when they have already accumulated enough wealth.

The borrowing rate is slightly decreasing over the life-cycle for high-initial-wealth individuals, starting at around 2% and declining until it reaches around 1% for a borrowing cost equal to 7%. Unlike them, low-initial-wealth individuals borrow between 4% and 7% early in life, and tend to catch up with the high-initial-wealth individuals around age 35 for risk aversion equal to 1.5, age 45 for risk aversion equal to 2.5, and never catch up completely when risk aversion is equal to 3.5 (we do not present the graphs for risk aversion 2.5 and 3.5 because of lack of space). This is because higher risk aversion generates significantly lower saving rates, so that individuals accumulate less wealth over the life cycle. Thus, the low-initial-wealth individuals will never be able to accumulate wealth enough to behave like the high-initial-wealth individuals.

When the borrowing cost is increased to 15%, things do not change very much. In fact, the only noticeable change is the fall in borrowing rate paths. For high-initial-wealth individuals, the borrowing rate slightly decreases over the life-cycle, starting at around 1% and declining until it is around 0.5%. However, low-initial-wealth individuals borrow between 3.5% and 6% early in life and tend to catch up with the high-initial-wealth individuals around age 35 for risk aversion equal

to 1.5 at levels just below 1%. This is accompanied by a very small decrease in saving rate, so that the overall effect is that accumulation paths are not altered.

The borrowing rates translate into actual borrowing on the basis of how much the individual has saved. Since actual savings are increasing over the life cycle, so are the actual borrowing paths. Low-initial-wealth individuals never borrow more than high-initial-wealth individuals in actual terms. Actual borrowing is smaller when the wedge for the borrowing cost is increased. For a risk aversion equal to 1.5 and a borrowing cost wedge equal to 2%, actual borrowing starts the life-cycle at 0.1/0.05 for high/low-initial-wealth individuals, peaks to 0.3/0.2 at age 55/60 (this is when wealth peaks as well for both types of individuals), and then decreases gradually towards the end of the life cycle. Since higher risk aversion imply lower levels of wealth, actual borrowing is also lower (0.1/0.05 early in life, and peaks to 0.21/0.13 at age 55/60).

Compared to the model without borrowing (Figure 2), we observe higher saving rates specially for low-initial-wealth groups early in life. The result is that we obtain higher wealth accumulation paths, which in the end generates less consumption inequality.

Figure 8 presents the results of differences in consumption compared to previous models. We can observe that, in terms of consumption inequality, there is not much difference between the models with high and low borrowing wedge. The most remarkable result is that, when borrowing is allowed, the differences in consumption paths remain flat as in the case of no contemporary uncertainty. This indicates that borrowing against savings, even at high costs, can completely offset the effect of contemporary uncertainty in terms of inequality of consumption paths.

Differences in consumption under borrowing are summarised for other models in Table 4. Compared to the case where there is no borrowing (the last column), we observe that all the cases under borrowing generate a decreasing gap in consumption paths between high-intial-wealth and low-initial-wealth individuals. This is what we expected, as borrowing allows less wealthy individuals (the more affected by contemporary uncertainty) to keep higher saving rates instead of avoiding them

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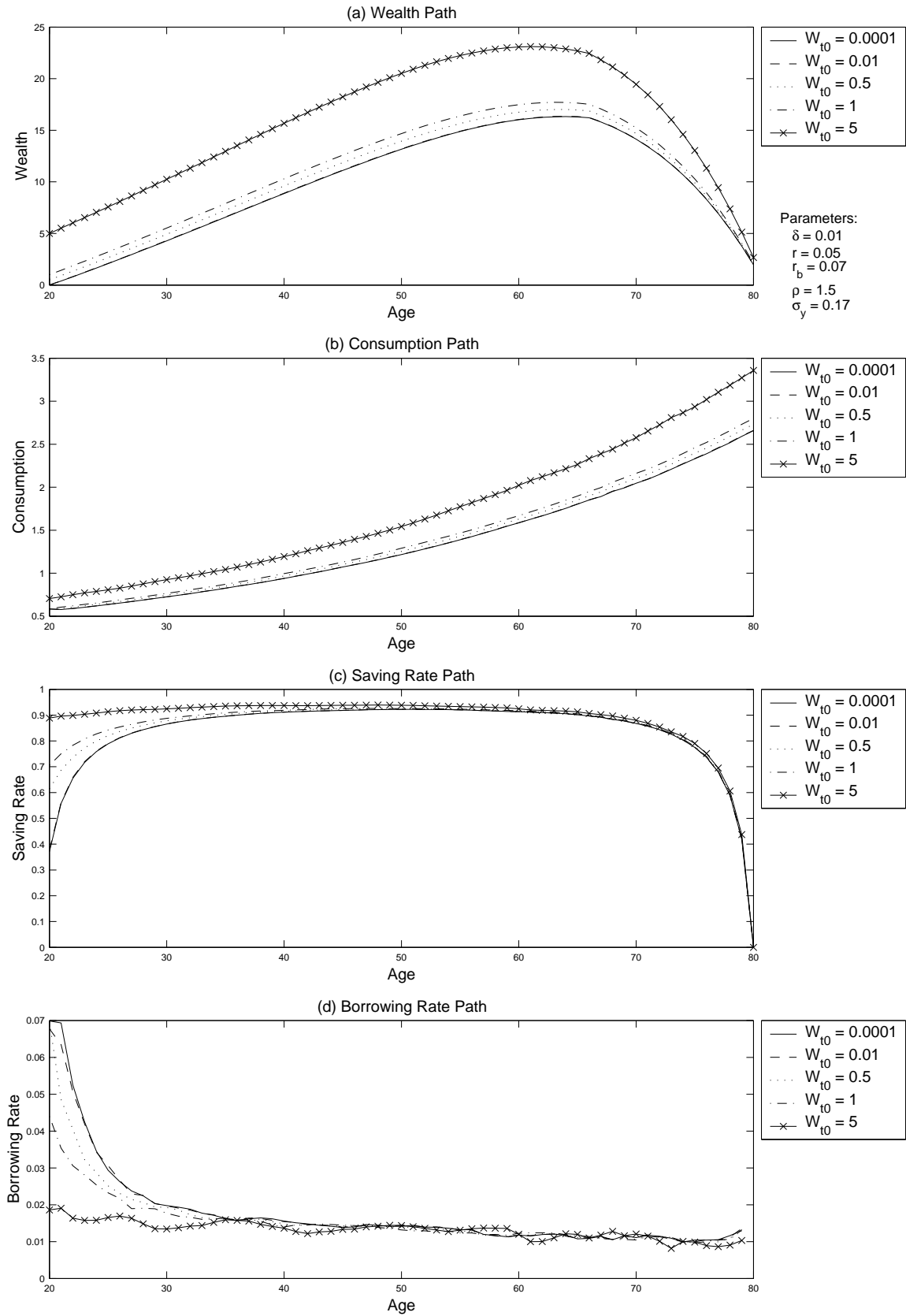Figure 6: Contemporary Uncertainty, no Risky Asset, with Borrowing ($r_b = 7\%$)

(a) Wealth Path

(b) Consumption Path

(c) Saving Rate Path

(d) Borrowing Rate Path

Parameters:
$\delta = 0.01$
$r = 0.05$
$r_b = 0.07$
$\rho = 1.5$
$\sigma_y = 0.17$

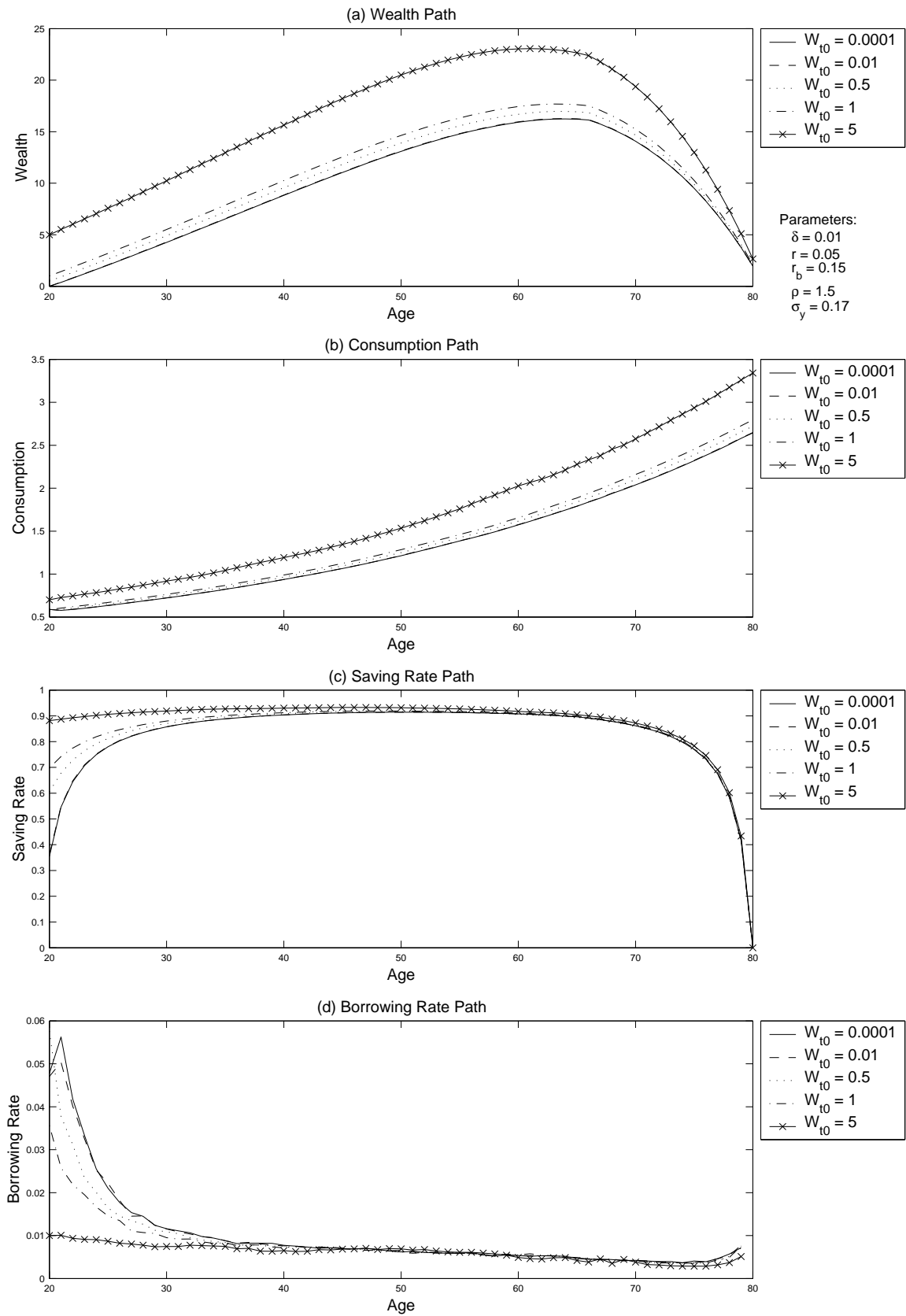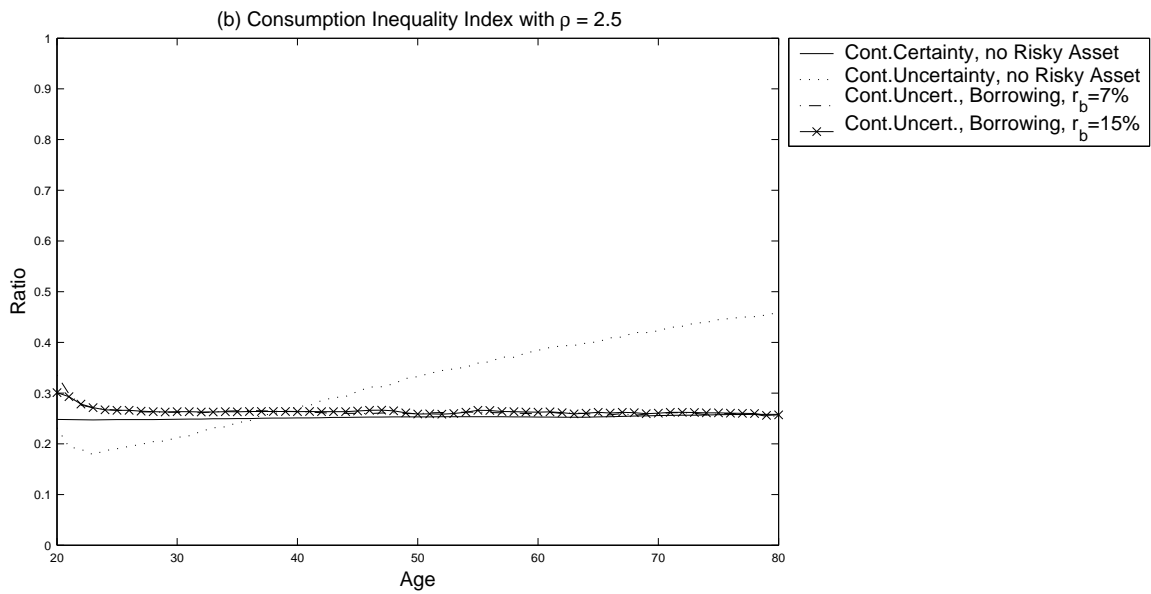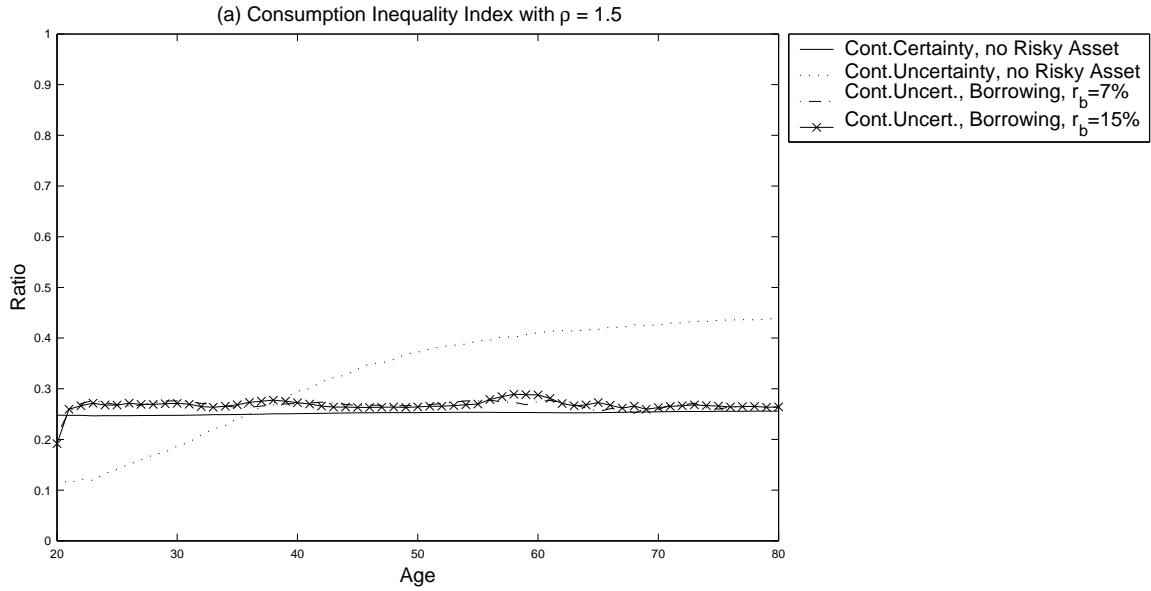Figure 7: Contemporary Uncertainty, no Risky Asset, with Borrowing ($r_b = 15\%$)

Figure 8: Consumption Inequality with Borrowing

because of the fear of not having enough consumption at present after the negative shock is resolved.

It is also interesting to notice how, as the borrowing cost increases, differences in consumption also increase (or at least do not diminish as much in cases with a higher risk aversion). Similarly to the case without borrowing, the higher the risk aversion the more relevant becomes the role of labour income uncertainty, generating higher saving rates for low-wealth individuals so that differences in consumption are initially higher and then decrease significantly.

The main conclusion is that borrowing, even at a high cost, allows to prevent the increase in the inequality effect of contemporary uncertainty. This result emphasises the role of financial markets in the sense that spreading the access to borrowing would be beneficial to decrease consumption differences over the life-cycle.

Table 4: Consumption Differences under Different Risk Aversion and Borrowing Wedge

	Excess consumption of wealthier over the life-cycle $\left(\frac{c_{20}(W_{20}=5)}{c_{20}(W_{20}=0)} - 1 \quad to \quad \frac{c_{80}(W_{20}=5)}{c_{80}(W_{20}=0)} - 1 \right)$			
Risk Aversion Coefficient	$r_b = 7\%$	$r_b = 10\%$	$r_b = 15\%$	No borrowing[a]
$\rho = 1.5$	0.21 to 0.26	0.20 to 0.26	0.19 to 0.26	0.13 to 0.44
$\rho = 2.5$	0.33 to 0.25	0.33 to 0.26	0.30 to 0.26	0.23 to 0.46
$\rho = 3.5$	0.45 to 0.24	0.43 to 0.24	0.40 to 0.24	0.32 to 0.42[b]

[a] "No borrowing" corresponds to the model in section 3 without risky asset.

[b] Has an inverted hump to 0.22 at age 24.

5 Extensions of the Model

5.1 Borrowing and Risky Asset

One obvious extension would be to include borrowing possibilities in the model with risky asset. As we mentioned above, this complicates significantly the numerical solution of the maximisation problem as it would imply three control variables and two state variables for the borrowing decision. It is certainly feasible, but it goes beyond the scope of the present study.

However, it is interesting to analyse ex ante what would be the interesting new features of such a model. We mentioned in section 3 that contemporary uncertainty does not affect the portfolio decision directly, it does so through lower saving rates. However, if risky asset was included in the model with borrowing, there would be an interaction between the portfolio choice and the borrowing decision. This is because borrowing creates a link between future consumption and present consumption. There would be a borrowing limit based on the portfolio choice to satisfy solvency in every state of the world. The less conservative the portfolio allocation the tighter the borrowing limit. Then, we would expect that such a model would deliver more conservative portfolios, *ceteris paribus*, for those more affected by the contemporary uncertainty, namely the less wealthy. The final outcome in terms of consumption differences would be that the less wealthy individuals would be less able to take advantage of the risky asset to accumulate wealth, widening the gap between high-initial-wealth and low-initial-wealth individuals over the life-cycle. We leave further analysis on this topic to further research.

5.2 Liquid and Illiquid Asset

Given the discussion about the underlying assumptions of the general model with contemporary uncertainty presented in section 3, it is interesting to discuss how a model would look like if that assumption was relaxed and we were to allow for two types of assets: one liquid and one illiquid.

Carmichael and Dissou (2000) have proposed a model of endogenous growth with a similar distinction of assets. In their model, there is a liquid asset that has no return (like "keeping the money under the mattress"), and another asset with a positive return (saving or "productive investment"). The liquid asset can be used to buffer shocks in consumption given by health expenses in the short run. The illiquid asset can only be liquidated in the long run.

Assume that the only way to transfer resources from one period to another is by buying liquid or illiquid assets. Calling l_t the liquid asset and s_t the illiquid asset, a similar model could be build in terms of the life-cycle framework with a timing like:

1. At the beginning of period t, labour income uncertainty is resolved (the individual knows $\tilde{y}_t$). This defines information set I_{t_0}.

2. The individual buys liquid and/or illiquid assets l_t and s_t, so that $l_t = W_t + y_t - s_t$. Portfolio choice ϕ_t for the illiquid asset has to be made.

3. At *interim* period (sub-period) t_1, expenses shock uncertainty is resolved (the individual knows $\tilde{x}_t$). This defines information set I_{t_1}.

4. At the end of sub-period t_1 the individual chooses how much to consume c_t out of his available resources $l_t - x_t$. The remaining liquid assets $(l_t - x_t - c_t)$ transfer into period $t + 1$ without any return.

The individual's problem would become to maximise his utility function:

$$\max_{\{s_t,\phi_t,c_t\}} U(c_t, W_t) = \sum_{s=0}^{T-t} \beta^s E_t \left[u(c_{t+s}) \right] \tag{26}$$

subject to

$$l_t = W_t + y_t - s_t$$
$$W_{t+1} = s_t \cdot [\phi_t \tilde{R}_{t+1} + (1 - \phi_t)R_f] + (l_t - x_t - c_t),$$
$$c_t \leq l_t + x_t$$

where x_t is unknown (random variable $\tilde{x}_t$) at the time s_t is chosen.

This model has three control variables to solve for: s_t, ϕ_t, and c_t. Control variable s_t requires one state variable $W_t + y_t$. Control variable ϕ_t requires state variable s_t. Finally, control variable c_t, requires to know state variable $l_t - x_t$. As can be noticed, the problem becomes more complex and falls outside the scope of the current research. However, it would be particularly interesting to include the liquid asset in the model as it adds significant realism. A model like this one could also include borrowing, so that the individual could borrow against his savings, being able to keep less liquid assets.

5.3 Exploring How to Augment the Portfolio Model to Include "Minimum Amounts" for Stock Investments

As we concluded in section 3, limitations of low-wealth individuals in accessing financial markets can boost already present differences in wealth accumulation

and consumption paths. Contemporary uncertainty diminishes saving rates, so that the possible amount to be invested in risky assets becomes smaller. Financial market imperfections that imply fixed entry costs (to be able to hold risky assets) become particularly relevant. As we mentioned above, entry costs to the financial markets have received significant attention in the literature. The literature is vast in cases where a lump sum entry cost is required to be able to buy risky assets. However, there is a particular form of entry cost that has not yet been explored: the existence of "minimums for stock investment."

Investing in stock markets usually requires a "minimum amount" of investment. This is without considering any possible fixed entry cost. Managing investment accounts is usually costly (fixed cost), so that the broker has the incentive to prefer large amounts of investments to small amounts. In incomplete markets, such as the financial markets of developing countries, it is common to find restrictions on minimum amounts of investment. This situation can be particularly harmful for low-wealth individuals who are willing to take advantage of the risk premium. This type of model might be interesting to analyse in future research. We draw a scheme of the set-up of a model of these characteristics.

The general model in section 3.1 (equations (1) to (3)) can be augmented to fit a financial imperfection like "minimum amounts" for stock investments. The idea is that there is a minimum amount of money that the individual can invest in stocks. If the individual does not want to invest that much money in stocks, then he can only access the risk free asset. The individual's problem becomes to maximise his utility function

$$\max_{\{s_t, \phi_t\}} \quad U(c_t, W_t) = \sum_{s=0}^{T-t} \beta^s E_t \left[u(c_{t+s}) \right] \tag{27}$$

subject to

$$c_t \quad = \quad W_t + \tilde{y}_t - \tilde{x}_t - s_t$$

$$W_{t+1} \quad = \quad s_t \cdot \left\{ \mathbf{1}(s_t \phi_t \geq m)[\phi_t \tilde{R}_{t+1} + (1 - \phi_t)R_f] + [1 - \mathbf{1}(s_t \phi_t \geq m)]R_f \right\},$$

where the function $\mathbf{1}(s_t \phi_t \geq m)$ equals 1 if $s_t \phi_t \geq m$ and 0 otherwise. m is the minimum amount of investment in stocks. Notice that the individual will always have to choose what proportion of his savings he would invest in stocks (ϕ_t) and then compare his desired investment amount to the minimum investment amount (m) to see whether he can enter the stock market or not.

A model like this presents a discontinuity in the derivative of the state variable W_{t+1}, then a numerical solution must be derivative free.[16] The basic constraints of the utility maximising problem can be incorporated by assigning very low utility values to those cases.[17]

Alan (2005) uses a different method to incorporate the discontinuity in the derivative of the state variable when augmenting a similar model with fixed entry costs. First, she solves the problem for investing in risky and riskless assets for each node. Second, she solves the model for the case where the individual can only invest in the risk free asset. Third, she compares the values of both optimizations and the rule that results in a higher value is picked.

A key question is whether the introduction of a "minimum amount" to invest could induce some individuals in the margin to have a riskier portfolio just in order be able to keep on investing in stocks. If that is the case, a strategy like the one used by Alan may not be useful. This is because those individuals will act neither as a person who invests in the stock market but is not in the margin nor as a person who does not invest at all in the stock market.

This situation is interesting because the individual's decision of portfolio choice is in fact affected by the financial market imperfection of a "minimum amount" for stock investment. This is different from the fixed-entry-cost models where the individual who decides which proportion to invest in stocks does not seem to be affected by the entry cost. The entry cost simply makes him decide whether or not to invest in stocks at all.

A behaviour in which the individual adopts a riskier portfolio simply in order to be able to stay in the stock market may be affected by other background risks (the tempering effect of background risk) in the same way as the normal risk taking decision in portfolio choice. Certainly, risk aversion level will also determine how big this effect could be.

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[16]The Nelder-Mead Simplex Algorithm is a derivative free grid-search method that is implemented in software such as Mathematica and Matlab.

[17]This is usually called the "penalty function method".

Contemporary uncertainty becomes more relevant for individuals who hold very low levels of wealth, when the individual is more affected in relative terms by uncertain contemporary shocks. Hence, this paper also contributes to answering the question about "Why do the poor save so little?", although it is not our main objective. Within the context of the life-cycle model, Hubbard, Skinner and Zeldes (1995), and Dynan, Skinner and Zeldes (2004) blame the asset-based means-tested social insurance programmes for the extremely low saving rate of the poor. This paper then adds yet another explanation, supporting the life-cycle model with the inclusion of contemporary uncertainty. In this context, social programmes that guarantee a minimum consumption for those below a certain level of wealth could actually make individuals in those who are programmes save more and catch up which richer groups in their wealth and consumption paths.[18]

6   Conclusions

We conclude that adding contemporary uncertainty to a life-cycle model with a standard parametrisation generates increasing gaps in expected consumption paths between high-initial-wealth and low-initial-wealth individuals over the life-cycle. Moreover, this increasing gap is boosted when risky assets (with a corresponding risk premium) are introduced, especially under high risk aversion.

The effect of risky assets is the opposite under contemporary certainty and contemporary uncertainty. Under contemporary certainty, the presence of a risky asset allows the generation of converging consumption paths over the life-cycle. On the contrary, under contemporary uncertainty the presence of a risky asset increases the gap between consumption paths. Since individuals use risk premium to increase their wealth levels, different access to risky asset critically determines differences in consumption. If the risky asset is only accessible to the wealthier, inequality can be boosted by the inability of the low-wealth individuals to generate wealth by investing in risky assets. This remarks the relevance of accounting for entry costs in future research.

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[18]Financial exclusion has also accounted in press articles like Chapman (2004).

The use of borrowing by individuals who at the same time hold assets (savings, illiquid in our case), stresses the relevance of this type of model for understanding what has been called the "borrowing puzzle", where individuals hold assets and at the same time borrow from their credit cards or other source of expensive credit.

Borrowing, even at a high cost, allows the prevention of the increase in inequality effect of contemporary uncertainty. This result emphasises the role of financial markets in the sense that spreading the access to borrowing would be beneficial in decreasing consumption differences over the life-cycle.

Our findings may help us to understand why the people who hold low levels of wealth (namely the poor) save so little. This supports social programmes that guarantee a minimum consumption for those below a certain level of wealth, since this would allow them to catch up with wealthier individuals. This argument is the contrary of what has been used in the literature, which blames those programmes for making saving non-attractive.

There are a number of possible extensions to the life-cycle model with contemporary uncertainty. In terms of making the model more realistic, liquid assets could be included jointly with borrowing and portfolio choice. Portfolio choice could also be made more interesting by adding entry barriers such as "minimum amounts" to investing in risky assets. On the other hand, attempts to match the stylised facts of a real economy could enhance the value of the research.

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Appendix A Algorithm to Solve the Model of Portfolio Choice with Contemporary Uncertainty

A.1 Algorithm

We solve the model backwards for T periods, accounting for a retirement period where income becomes certain, being a fraction of the last permanent income. The information set in t_0 defines the state variable $(W_t + y_t)$ which determines the control variables s_t and ϕ_t.

We solve the model in two steps. First we solve the portfolio decision (equation (6)) as a function of absolute saving, obtaining a function $\phi_t(s_t)$ (we actually obtain only values of function $\phi_t(s_t)$ for different nodes in a grid). Then, using function $\phi_t(s_t)$, by means of interpolation, we solve the saving problem (equation (5)).

Solving (5) also requires interpolating the function $V'(W_{t+1})$. We do this by using linear interpolation, using the solution from the next period (recall that we solve backwards), i.e., we interpolate $V'(W_{t+1})$ at points $(W_{t+1} + y_{t+1})|_t$ by using the solution at $t+1$ for nodes in $(W_{t+1} + y_{t+1})|_{t+1}$. Then solving for the saving rate implies that, for each value of s_t we try as a solution, we need to interpolate both $\phi_t(s_t)$ and $V'(W_{t+1})$.(*Piecewise Cubic Hermite Interpolating Polynomial* was also attempted, giving no significant improvement over linear interpolation and implying substantially more computational cost).[19]

Expectations are discretised by using Gauss-Hermite quadrature (numerical integration). This method is a standard in the literature[20], generating a set of finite shocks and corresponding probability weights.

The algorithm is as follows:

1. Define shocks and weights for labour income, expenses shock, and return rate of risky asset.

2. Define a grid of nodes for $(W_t + y_t)$.

3. Define a grid of nodes for s_t.

4. Solve the first order condition for portfolio allocation (6) using `MATLAB` nonlinear equation solver `Fzero`[21] for values in grid s_t. We use a logarithmic transformation for portfolio choice in order to give the non-linear equation

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[19] Judd (1998) suggests to use Schumaker Quadratic Shape-Preserving Splines. However, this scheme has similar advantages and the same drawbacks as Piecewise Cubic Hermite Interpolating Polynomial.

[20] See Haliassos and Michaelides (2002) and Judd (1998, pp.265) for further discussion of numerical methods which are similar to the present one and for different quadrature methods.

[21] `Fzero` uses secant and inverse quadratic interpolation methods, and bisection as a last resource. Observe that bisection, although very slow, guarantees to find a solution provided it exists in a given interval.

solver the freedom to choose values from the real line. (We use four-digits precision). Since the second derivative of the argument to be maximised in the Bellman equation (4) is negative, we deal with corner solutions by assigning a solution $\phi_t = 1$ when (6) is always strictly positive in the domain $\{0, 1\}$, and by assigning a solution $\phi_t = 1$ when (6) is always strictly negative in the domain $\{0, 1\}$. In solving (6) we interpolate $V'(W_{t+1})$ from the results in the next period (the last period has the terminal condition that $V(W_T) = 0$.

5. Save the results of ϕ_t alongside the corresponding nodes of s_t.

6. Solve the first order condition for the saving decision (5) using `MATLAB` non-linear equation solver `Fzero`. We use a logarithmic transformation and solve for the saving rate defined as $sr_t = \frac{s_t}{W_t + y_t}$. Evaluating the first order condition requires us to interpolate values of portfolio allocation for each value of the saving rate proposed as a solution for every node in grid $(W_t + y_t)$.

7. Evaluate the value function at the solutions and store the results altogether with marginal utilities to continue recursion.

A.2 Simulations

Once the model is solved, simulations are run to show the results. First, income, expenses and return rate of risky asset shocks are generated. In order to ensure that the conditions under which the model has been solved hold, the random shocks are matched to discretised values of the shocks used in the solution of the model. This could be interpreted as if the shocks were an independent identically distributed Markov process. The matching is done by generating a uniformly distributed random variable between zero and one and fitting it between the weights of the nodes of the quadrature scheme.

More precisely, the generic random variable (the actual shock) is distributed lognormal, $\tilde{x}_t \sim LN(\mu_x, \sigma_x^2)$, hence the quadrature scheme for numerical integration of the expectations gives a set of pairs $\{\tilde{x}^{\{i\}}, w_x^{\{i\}}\}$ for $i = 1, \dots, n_x$, where $w_x^{\{i\}}$ is the corresponding weight for node $\tilde{x}^{\{i\}}$, $\sum_{i=1}^{n_x} w_x^{\{i\}} = 1$, and n_x is the number of nodes used in the quadrature scheme.

For the choice of the actual shock, a uniform random variable $\tilde{x}_u \sim U(0, 1)$ is created. Then, the actual shock is:

$$\tilde{x}_t = \sum_{k=1}^{n_x} \left[\mathbf{1} \left(\sum_{i=1}^{k-1} w_x^{\{i\}} < \tilde{x}_u \right) \times \mathbf{1} \left(\tilde{x}_u \le \sum_{i=1}^{k} w_x^{\{i\}} \right) \times \tilde{x}^{\{k\}} \right], \qquad (28)$$

where function $\mathbf{1}(\cdot)$ equals one if the expression within brackets is true and zero otherwise.

Next, the saving rate is obtained by interpolating for the corresponding level of the state variable $W_t + y_t$. Then the actual saving is computed and used to interpolate

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the corresponding portfolio choice ϕ_t. Then, consumption and wealth levels are computed. Each simulation is then stored, and average values are computed for plotting.

Appendix B   A Life-Cycle Model with Contemporary Uncertainty and Borrowing

B.1   Solving the Model

We solve the model backwards for T periods, accounting for a retirement period where income becomes certain, being a fraction of the last permanent income. Solving the model requires two maximisation procedures, where each of them have different state variables. The information set in t_0 defines the state variable $(W_t + y_t)$, whereas the information set t_1 defines the state variables $(W_t + y_t - x_t - s_t)$ and s_t. For these reasons we need a different state variable grid for each information set.

However, to make a borrowing decision, the individual needs to know not only what is available for consumption, $(W_t + y_t - x_t - s_t)$, but also how much he can borrow, $\frac{s_t R}{R_b}$. This defines a maximisation problem with two state variables, which for convenience we will define as $(W_t + y_t - x_t)$ and s_t.[22]

The most difficult part of the problem is to obtain function b_t^*. When saving is equal to zero it is not possible to borrow at all, hence the borrowing rate is set equal to zero. Then, the borrowing rate will be equal to zero when saving is small enough so that the remaining available resources are sufficient to cope with the negative shock. Finally, the borrowing rate is positive when saving is high enough to make borrowing valuable. This results in a borrowing rate function as a function of the saving rate that is initially flat at zero, and then becomes upward sloping, increasing at a decreasing rate (see Figure 9 for a function without uncertainty as an example).

Interpolation when the borrowing rate function is strictly positive is rather simple as the function is rather smooth. However, near the kink where the borrowing rate changes from zero to positive, interpolation becomes tricky. For this purpose, it is crucial to determine as accurately as possible which one is the kink point. If the kink point is not identified with precision, we could end up with an interpolating function such as the one in Figure 10. It is possible to observe that, with a coarse grid, when the interpolation function has a positive slope, the function is initially convex and then concave, which can generate serious interpolation inconveniences.

In order to identify the kink point, we define the saving rate that corresponds

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[22]The convenience in defining the state variable as $(W_t + y_t - x_t)$ instead of $(W_t + y_t - x_t - s_t)$ will become more evident when explaining the interpolation procedure.

Figure 9: Borrowing Function without Uncertainty
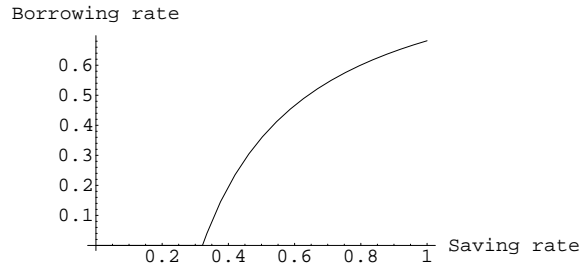
to that point as $\hat{s}_t^{(i)}$, where i indicates a particular level of $W_t + y_t - x_t$ (i.e., a particular node in that grid). Hence, we use a rather fine grid for saving and borrowing (600 nodes for saving and 400 for borrowing). This is very costly computationally, but it has the advantage of giving a very fine mesh for the two-dimension interpolation, a critical part of solution of the model.[23] The result of solving the first order condition for borrowing (18) is a function in two variables $b_t^*(W_t + y_t - x_t, s_t)$. This function will then be used to solve for the saving rate.

Figure 10: Borrowing Function with a Coarse Grid
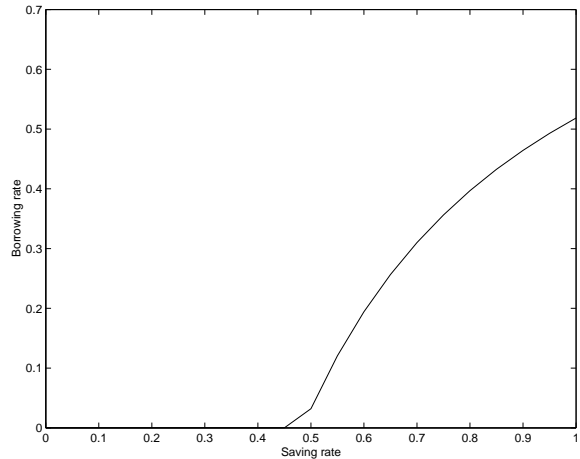
Solving for (18) also requires interpolating the function $V'(W_{t+1})$. We do this by using spline interpolation using the solution from the next period (recall that we solve backwards), i.e., we interpolate $V'(W_{t+1})$ at points $(W_{t+1} + y_{t+1})|_t$ by using the solution at $t + 1$ for nodes in $(W_{t+1} + y_{t+1})|_{t+1}$.

Solving for the saving rate implies that, for each value of s_t we try as a solution, we need to interpolate both $b_t^*(W_t + y_t - x_t, s_t)$ and $V'(W_{t+1})$. Interpolating $b_t^*(W_t + y_t - x_t, s_t)$ requires a two-dimension interpolation, which is computationally

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[23]In a 1.8GHz computer with 512Mb RAM it takes around 3 hours to solve. This compares to the 30 minutes required to solve the model without borrowing.

more costly than a one-dimension interpolation, and can be particularly inaccurate near a kink point such as $\hat{s}_t$. Spline, Piecewise Cubic Hermite Interpolation Polynomial, and other polynomial based schemes proved to be quite inaccurate given the nature of the interpolation function. The main problem is that borrowing cannot be larger than saving, actually $b_t \leq \frac{s_t R}{R_b}$. Then, for different values of state variable s_t there are different maximums of borrowing. To overcome this problem, we chose linear two-dimension interpolation, having to interpolate with only 3 points (instead of 4 as normally), near the maximum possible borrowing.

Expectations are discretised by using Gauss-Hermite quadrature (numerical integration). This method is a standard in the literature[24], generating a set of finite shocks and corresponding probability weights.

B.2 Algorithm for Solving the Model in Backwards Recursion

1. Define shocks and weights for labour income and negative shock. `nsy` and `nsx` are the corresponding number of shocks.

2. Define a grid of length `n` for $(W_t + y_t)$ called `Wyn`.

3. Define a grid for $(W_t + y_t - x_t)$, called `Wyxn`, as `Wyn-x(j)`, so that `Wyxn` is (n x nsx). Define a fine grid `Sn` of 600 nodes.

4. Define a function to evaluate the Value Function at a certain level of s_t and b_t called `VF`.

5. Evaluate `VF(b=0)` and `VF(b=e)`, where `e` is a very small number. Build `VF(b=e)-VF(b=0)` to find the corresponding $\hat{s}_t$=`shat` in grid `Srn` where `VF(b=e)-VF(b=0)` becomes > 0.

6. Solve the first order condition for borrowing (18) using `MATLAB` non-linear equation solver `Fzero`[25] for values in grid `Sn` larger than those in `shat`. (We use four-digits precision). Since (18) has a negative slope, we deal with corner solutions by assigning a solution `brt`=1 when (18) is always strictly positive in the domain $\{0,1\}$, and by assigning a solution `brt`=0 when (18) is always strictly negative in the domain $\{0,1\}$. In order to avoid numerical problems, we assign a small value, i.e, 1×10^{-5} to consumption values below that level (including negative values). This will imply that the solver will discard such a solution because the marginal utility would be extremely large. $V'(W_{t+1})$ is approximated by the marginal utility of consumption in the next period.

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[24] See Haliassos and Michaelides (2002) and Judd (1998, pp.265) for further discussion of numerical methods that are similar to the present one, and for different quadrature methods.

[25] `Fzero` uses secant and inverse quadratic interpolation methods, and bisection as a last resource. Observe that bisection, although very slow, guarantees to find a solution provided it exists in a given interval.

7. Solve the first order condition for saving (20) using `MATLAB` non-linear equation solver `Fzero`. We use a logarithmic transformation and solve for the saving rate defined as $sr_t = \frac{s_t}{W_t+y_t}$ to allow the maximisation routine to choose values from the real line. Evaluating the first order condition requires us to interpolate values for borrowing for each value of the saving rate proposed as a solution for every node in grid `Wyxn`. We do this by two-dimension linear interpolation as explained above.

8. Evaluate the value function at the solutions and store the results altogether with marginal utilities to continue the recursion.

B.3 Simulations

The simulations scheme is similar to the one used in section A.2. The only difference is that we need the two state variables $\texttt{Wyt} = W_t + y_t - x_t$ and saving (`st`) to perform two-dimension linear interpolation to obtain the corresponding borrowing (`bt`). Then, consumption and wealth levels are computed. Each simulation is then stored and the average values are computed for plotting.