Semiosis as a
Multimodal Process

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RESUMEN

Las aproximaciones semióticas clásicas resultan ser muy estrechas para investigar los fenómenos didácticos del salón de clase de matemáticas. Además de los recursos semióticos estándares utilizados por los alumnos y los maestros (como los símbolos escritos y el lenguaje hablado), otros recursos semióticos importantes son los gestos, las miradas, los dibujos y los modos extra-lingüísticos de expresíón. Sin embargo, estos últimos caben difícilmente en las definiciones clásicas de los sistemas semióticos. Para superar esta dificultad, en este artículo adopto una perspectiva vygotskiana y presento una noción extendida de sistema semiótico, el haz semiótico*,* que se revela particularmente útil para incluir todas los recursos semióticos que encontramos en los procesos de aprendizaje de las matemáticas. En este artículo subrayo algunos puntos críticos en la descripción usual de los sistemas semióticos; discuto acerca del paradigma multimodal y encarnado que ha venido emergiendo en los últimos años en investigaciones realizadas en psicolinguística y neurociencia y analizo los gestos desde un punto de vista semiótico. Luego, introduzco la noción de paquete semiótico y lo ejemplifico a través de un estudio de casos.

- **PALABRAS CLAVES:** Recursos semióticos, encarnamiento, multimodalidad, gestos, inscripciones.

ABSTRACT

Classical semiotic approaches are too narrow to investigate the didactical phenomena in the mathematics classroom. In addition to the standard semiotic resources used by students and teachers (e.g. written symbols and speech), other important semiotic ressources include also gestures, glances, drawings and extra-linguistic modes of expressions. However, these semiotic ressources fit with difficulties within the constraints of the classical definitions of semiotic systems. To overcome such difficulties I adopt a vygotskian approach and present an enlarged notion of semiotic system, the *semiotic bundle*, which reveals particularly useful for framing all the semiotic resources we find in the learning processes in mathematics. The paper stresses some critical points in the usual description of the semiotic systems; it discusses the multimodal and embodied paradigm, which is emerging in these last years from researches in psycholinguistics and neuroscience and analyses gestures from a semiotic point of view. Then it introduces the notion of semiotic bundle and exemplifies it through a case study.

KEY WORDS: Semiotic resources, embodiment, multimodality, gestures, inscriptions.

RESUMO

As aproximações clássicas semióticas resultam ser muito limitadas para investigar os fenômenos didáticos de sala de aula de matemática. Além aos recursos padrão dos semióticos usados pelos estudantes e pelos professores (como os símbolos escritos e a língua falada), outros recursos importantes dos semióticos são os gestos, os olhares, os desenhos e as maneiras extra-lingüísticas da expressão. Não obstante, estes últimos não se adaptam bem nas definições clássicas dos sistemas dos semióticos. A fim superar esta dificuldade, neste artigo eu adoto um perspectiva vygotskiana e apresento uma noção estendida do sistema do semiótico ao pacote semiótico que é particularmente útil incluir todos os recursos dos semióticos que nós encontramos nos processos da aprendizagem da matemática. Neste artigo eu enfatizo alguns pontos críticos na descrição usual dos sistemas semióticos. Discuto sobre o paradigma multimodal e personificado que tem emergido nos últimos anos das investigações feitas na psicolinguística e na neurociência e analiso os gestos sob um ponto da vista do semiótico. Logo, eu introduzo a noção do pacote do semiótico e a exemplifico com um estudo dos casos.

PALAVRAS CHAVE: Recursos semióticos, significação, multimodalidade, gestos, inscrições.

RÉSUMÉ

Les approches sémiotiques classiques sont trop étroites pour étudier les phénomènes didactique de la salle de classe de mathématiques. En plus des ressources sémiotiques traditionnelles (comme les symboles écrits et la langue) utilisées par les élèves et les enseignants, d'autres ressources sémiotiques importantes comprennent les gestes, les regards, les dessins et les modes extra-langagiers d'expression. Ces dernières rentrent difficilement dans les définitions classiques des systèmes sémiotiques. Afin de surmonter cette difficulté, dans cet article j'adopte une perspective vygotskienne et je présente une notion élargie de système sémiotique, le faisceau sémiotique, qui s'avère particulièrement utile afin d'inclure toutes les ressources sémiotiques que nous rencontrons dans les processus d'apprentissage des mathématiques. Dans cet article je souligne quelques points critiques concernant la description usuelle des systèmes sémiotiques; j'offre une discussion du paradigme multimodal et incarné lequel a émergé ces dernières années dans le cadre des recherches menées en psycholinguistique et neuroscience. Suite à cela j'analyse les gestes d'un point de vue sémiotique. Après j'introduis la notion de paquet sémiotique et l'exemplifie à travers une étude de cas.

MOTS CLÉS: Ressources sémiotiques, incarnement, multimodalité, gestes, inscriptions.

Introduction.

Semiotics is a powerful tool for interpreting didactical phenomena. As Paul Ernest points out,

> *"Beyond the traditional psychological concentration on mental structures and functions 'inside' an individual it considers the personal appropriation of signs by persons within their social contexts of learning and signing. Beyond behavioural performance this viewpoint also concerns patterns of sign use and production, including individual creativity in sign use, and the underlying social rules, meanings and contexts of sign use as internalized and deployed by individuals. Thus a semiotic approach draws together the individual and social dimensions of mathematical activity as well as the private and public dimensions. These dichotomous pairs of ideas are understood as mutually dependent and constitutive aspects of the teaching and learning of mathematics, rather than as standing in relations of mutual exclusion and opposition."* (Ernest, 2006, p.68)

However, the classical semiotic approach places strong limitations upon the structure of the semiotic systems it considers. They generally result in being too narrow for interpreting the complexity of didactical phenomena in the classroom. As we shall discuss below, this happens for two reasons:

(i) As observed by L. Radford (2002), there are a variety of semiotic resources used by students and teachers, like gestures, glances, drawings and extra-linguistic modes of expression, which do not satisfy the requirements of the classical definitions for semiotic systems as discussed in literature (e.g. see Duval, 2001).

(ii) The way in which such different registers are activated is multimodal. It is necessary to carefully study the relationships within and between registers, which are active at the same moment and their dynamics developing in time. This study can only partially be done using the classic tools of semiotic analysis.

To overcome these two difficulties, I adopt a Vygotskian approach for analyzing semiotic resources and present an enlarged notion of semiotic system, which I have called *semiotic bundle*. It encompasses all the classical semiotic registers as particular cases. Hence, it does not contradict the semiotic analysis developed using such tools but allows us to get new results and to frame the old ones within a unitary picture.

This paper is divided into three main chapters. *Chapter 1* summarizes some salient aspects of (classical) Semiotics: it shows its importance for describing learning processes in mathematics (§ 1.1), points out two opposite tendencies in the story of Semiotics, which reveal the inadequacy of the classical approach when it is used in the classroom (§1.2), and discusses the semiotic role of artefacts, integrating different perspectives from Vygotsky to Rabardel (§1.3).

Chapter 2 develops the new concept of semiotic bundle (§2.1), discusses the multimodal and embodied paradigm, which has emerged in recent years from research in psycholinguistics and neuroscience

(§2.2), and analyses gestures from a semiotic point of view (§2.3).

Chapter 3 introduces a case study, which concretely illustrates the use of semiotic bundles in interpreting the didactical phenomena.

A *Conclusion*, with some comments and open problems, ends the paper.

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1. The semiotic systems: a critical approach

1.1 Semiotics and mathematics

Charles S. Peirce points out a peculiar feature of mathematics which distinguishes it from other scientific disciplines:

> *"It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram, the relations of whose parts shall present a complete analogy with those of the parts of the object of*

reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra". (Hartshorne & Weiss, 1933, 3.363; quoted in Dörfler, n.d.).

In fact, mathematical activities can develop only through a plurality of palpable registers that refer to its ideal objects:

> *"…the **oral** register, the **trace** register (which includes all graphic stuff and writing products), the **gesture** register, and lastly the register of what we can call the **generic materiality**, for lack of a better word, namely the register where those ostensive objects that do not belong to any of the registers above reside"* [2].

(Bosch & Chevallard, 1999, p. 96, emphasis in the original)

These observations are the root of all semiotic approaches to mathematical thinking, some of which I shall briefly review below.

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[2] "...[le] registre de *l'oralité*, registre de *la trace* (qui inclut graphismes et écritures), registre de *la gestualité*, enfin registre de ce que nous nommerons, faute de mieux, *la matérialité quelconque*, où prendront place ces objets ostensifs qui ne relèvent d'aucun des registres précédemment énumérés. "

Peirce's observations point out different aspects of the semiotic approach:

(i) the introduction of signs, namely perceivable (spatio-temporal) entities, like *"icons or diagrams, the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning"*;

(ii) the manipulation of signs, namely *"experimenting upon this image in the imagination"* and/or *"manipulating it"* concretely and *"observing the effects of such manipulation"*;

(iii) the emergence of rules and of strategies of manipulation: *"in such activities we are guided by previous discoveries, which are embodied in the signs themselves"*, e.g. in the general formulae of algebra, and *"that become patterns to imitate in our procedure"*. Typical examples are the signs of Algebra and of Calculus, Cartesian graphs, arrow diagrams in Graph Theory or Category Theory, but also 2D figures or 3D models in Geometry. Generally speaking, such signs are *"kind[s] of inscriptions of some permanence in any kind of medium (paper, sand, screen, etc)"* (Dörfler, n.d.) that allow/support what has been sometimes called (e.g. Dörfler, *ibid.*) *diagrammatic reasoning*. The paper of Dörfler provides some examples, concerning Arithmetic, Algebra, Calculus and Geometry. Other examples, albeit with different terminology, are in Duval (2002, 2006).

However, as the quotation from Peirce shows, the semiotic activities are not necessarily limited to the treatment of inscriptions since they also deal with images that are acted upon in imagination (whatever it may mean): *"A sign is in a conjoint relation to the thing denoted and to the mind. If this relation is not of a degenerate species, the sign is related to its object only in consequence of a mental association, and depends upon a habit."* (Hartshorne & Weiss, 1933, 3.360).

I shall discuss this point below after having considered the more standard approaches to semiotic systems, which study inscriptions (signs in a more or less wide sense) and operations upon them. E.g., according to Ernest (2006, pp. 69-70), a *semiotic system* consists of three *components*:

1. A set of signs, the tokens of which might possibly be uttered, spoken, written, drawn or encoded electronically.

2. A set of rules of sign production and transformation, including the potential capacity for creativity in producing both atomic (single) and molecular (compound) signs.

3. A set of relationships between the signs and their meanings embodied in an underlying meaning structure.

An essential feature of a semiotic system has been pointed out by Duval (2002), who introduced the concept of *semiotic representations.* The signs, relationships and rules of production and transformation are semiotic representations insofar as they bear an intentional character (this is also evident in the quotation of Peirce). This intentional character is not intrinsic to the sign, but concerns people who are producing or using it. For example, a footprint in the sand generally is not a semiotic representation in this sense: a person who is walking on the beach has no interest in producing or not producing it; however, the footprint that Robinson Crusoe saw one day was the sign of an unsuspected inhabitant of the deserted island, hence he gave it a semiotic function and for him the footprint became a semiotic representation.

Other important aspects of semiotic systems are their *semiotic functions,* which can be distinguished as *transformational* or *symbolic* (see: Duval, 2002 and 2006; Arzarello et al., 1994).

The *transformational* function consists in the possibility of transforming signs within a fixed system or from one system to another, according to precise rules (algorithms). For example, one can transform the sign $x(x+1)$ into $(x^2 + x)$ within the algebraic system (register) or into the graph of a parabola from the Algebraic to the Cartesian system. Duval (2002, 2006) calls *treatment* the first type of transformation and the second one *conversion*. According to Duval (2002), conversions are crucial in mathematical activities:

> "The characteristic feature of mathematical activity is the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another."

The *symbolic* function refers to the possibility of interpreting a sign within a register, possibly in different ways, but without any material treatment or conversion on it. E.g. if one asks if the number $n(n+1)$ is odd or even one must interpret n and $(n+1)$ with respect to their oddity and see that one of the two is always even. This is achieved without any transformation on the written signs, but rather by interpreting differently the signs n, $(n+1)$ and their mutual relationships: the first time as odd-even numbers and then as even-odd numbers. The symbolic function of signs has been described by different authors using different words and from different perspectives: C.S. Peirce, C.K.Ogden & I. A. Richards (semiotics); G. Frege (logic); L. Vygotsky (psychology) and others: see Steinbring (2005, chapter 1) for an interesting summary focusing on the problem from the point of view of mathematics education. The symbolic function possibly corresponds to the activity of *"experimenting upon an image in the imagination"*, mentioned by Peirce. All of the aforementioned authors point out the triadic nature of this function, namely that it consists in a complex (semiotic) relationship among three different components (the so called semiotic triangle), e.g. using Frege´s terminology, among the Sense (Sinn), the Sign (Zeichen) and the Meaning (Bedeutung). Peirce spoke of *"a triple relation between the sign, its object and the mind"*; Frege (1969) was more cautious and avoided putting forward in his analysis what he called the *third world*, namely the psychological side.

Semiotic systems provide an environment for facing mathematics not only in its structure as a scientific discipline but also from the point of view of its learning, since they allow us to seek the cognitive functioning underlying the diversity of mathematical processes. In fact, approaching mathematical activities and products as semiotic systems also allows us to consider the cognitive and social issues which concern didactical phenomena, as illustrated by the quotation of Ernest in the Introduction.

Transformational and symbolic functions of signs are the core of mathematics and they are very often intertwined. I shall sketch here a couple of examples. An interesting historical example, where both transformational and symbolic functions of semiotic registers are present is the method of completing the square in solving second order equations. This can be done within the algebraic as well as the

geometric register. Another important example of the creative power of the symbolic function is given by the novelty of the Lebesgue integral (of a real function f in an interval $[a,b]$) with respect to the Riemann one. In the latter, one collects data forming the approximating integral sums subdividing the interval $[a,b]$ in intervals Δ_i, each of length δ_i less than some δ: the basic signs are the products $l_i\delta_i$, where l_i is some value of the function f in Δ_i (or its sup or inf in it) and the final sum $\sum l_i\delta_i$ is made considering the values i corresponding to all the intervals Δ_i of the subdivision. In the former, the subdivision is made considering, for each value l of f, the set Δ_l of x's such that $f(x) = l$: the basic signs are the products $l\left|\Delta_l\right|$, and the final sum $\sum l\left|\Delta_l\right|$, is made considering all the values l that the function assumes while x varies in $[a,b]$.

1.2 Two opposite tendencies

Within the main components of a semiotic system (signs and operations on them), there is a tension between two opposite modalities, which is particularly evident when a semiotic lens is used to analyse didactical processes and not only mathematical products. This tension is in fact a by-product of the two contrasting features of mathematics pointed out by Peirce, that is, its apodictic and observational aspects.

The first one consists in the strong tendency to formalize in mathematics:

"The more important for the mathematical practice is the availability of a calculus which operates on diagrams (function terms) and permits to evaluate derivatives, anti-derivatives and integrals according to established diagrammatic operation rules. … Here

again we find the striving for manipulable diagrams which can be taken to accurately reflect the related non-diagrammatic structures and processes." (Dörfler, n.d.)

Different crucial examples of this tendency are: the algebraic language, which (Harper, 1987) introduced suitable formalism to treat classes of arithmetic problems (equations included); Cartesian geometry, which allowed for the translation of the geometric figural register into the algebraic one; and arrow-diagrams in Category Theory. All such new inscriptional entries also allowed for new forms of reasoning and solving problems and hence had a strong epistemological and cognitive impact. A culminating case in this tendency toward formalization consists in the idea of formal system, elaborated by Hilbert (see Detlefsen, 1986).

The construction of a (formal) axiomatization in the sense of Hilbert's formalist program can be considered another method of translating into diagrams. Let us take, for instance, an axiom system for the structure of real numbers: it consists of formulas in a precise formal language together with the rules inference, e.g. first order predicate logic. These can be viewed as diagrams in the sense intended by Peirce. Proofs and theorems are then obtained by manipulating such diagrams and observing the outcomes of the manipulations (the logical deductions). One could therefore interpret (formal) axiomatization as a kind of diagrammatization (see Dörfler, n.d.).

Moreover, if one looks carefully at some logical ideas in Mathematical Logic developed at the turn of the twentieth century, the tendency toward formalism shows a further mathematical aspect of semiotic conversions, namely the idea of the interpretation of one theory into

another. As an example, I call to mind the second part of the book *Foundation of Geometry* (Hilbert, 1962), where Hilbert typically interprets geometrical objects and statements into real numbers or into some subfield of reals to build models where some specific axiom of geometry does not hold. The concept of interpretation is the logical and mathematical counterpart of the idea of conversion from one register to another. Its roots are in the conversion/interpretation of one model into another one: typically, the interpretation of a model for hyperbolic geometry within the Euclidean model, namely the Klein disk and the Poincaré disk or half-plain. The rationale behind such logical approaches is that the relationships among objects represented in different ways within different registers can be shown better in one register than in another, exactly because of the specificity of the register, possibly because of the symbolic function it promotes. For instance, we can note the validity or less of an axiom of geometry in the usual Euclidean model (first register) or in a model built using only a subfield of real numbers (second register). A very recent area of research that has developed in line with this approach is the project of *Reverse Mathematics* (Sympson, 1999), where typically an important theorem T is proved carefully within a formal system **S** using some logical hypothesis H. For example, the Heine-Borel theorem in Analysis using as logical hypothesis a (weak) form of König lemma. Reverse Mathematics then tries to answer to the following 'reverse' question: does it exist within **S** a proof of H using T as hypothesis? Namely, one tries to prove the equivalence between T and H within a suitable system **S**, namely the equivalence between sentences whose meaning is within two different registers (e.g. the analysis and the logical one).

The concept of interpretation has carefully refined the transformational and symbolic functions of mathematical signs during the years, from the pioneering semantic interpretations of geometrical models to the elaborate formal theories studied in Reverse Mathematics.

On the one hand, this approach has enlarged the horizon of semiotic systems from within mathematics (*inner enlargement*): think of the different models of reasoning induced by the Calculus inscriptions with respect to those pertaining to the algebraic ones, or to those induced by the «reasoning by arrows» in Category Theory. But on the other hand, it has also narrowed the horizon within which mathematical semiotic activities are considered, limiting them to their strictly formal aspects.

Unfortunately, this is not enough when cognitive processes must be considered, e.g. in the teaching-learning of mathematics. In such a context, it is the same notion of signs and of operations upon them that needs to be considered with a greater flexibility and within a wider perspective. In the classroom, one observes phenomena which can be considered as signs that enter the semiotic activities of students[3] but which are not signs as defined above and are not processed through specific algorithms. For example, observing students who solve problems working in group, their gestures, gazes and their body language in general are also revealed as crucial semiotic

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[3] *Semiotic Activity* is classically defined as any "communicative activity utilizing signs. This involves both sign 'reception' and comprehension via listening and reading, and sign production via speaking and writing or sketching." The main purpose of the paper is to widen this definition.

resources. Namely, non-written signs and non-algorithmic procedures must also be taken into consideration within a semiotic approach. Roughly speaking, it is the same notion of sign and of operations upon them that needs to be broadened. In fact, over the years, many scholars have tried to widen the classical formal horizon of semiotic systems, also taking into consideration less formal or non formal components.

While formalism represents the first tendency of the aforementioned tension in Semiotics, these broadening instances from outside mathematics constitute the other tendency (*outer enlargement*). This tendency can already be found in the complex evolution of the sign definition in Peirce and is also contained in some pioneering observations by Vygotsky concerning the relationships between gestures and written signs, such as the following:

> "The gesture is the initial visual sign that contains the child's future writing as an acorn contains a future oak. Gestures, it has been correctly said, are writing in air, and written signs frequently are simply gestures that have been fixed." (Vygotsky, 1978, p. 107; see also: Vygotsky, L. S. 1997, p. 133.).

This was also anticipated by Ludwig Wittgenstein, who changed his mind about the centrality of propositions in discourse and the role of gestures, passing from the *Tractatus* to the *Philosophische Untersuchungen,* as the following well known episode illustrates:

> "Wittgenstein was insisting that a proposition and that which it describes must have the same 'logical form', the same 'logical

> multiplicity', Sraffa made a gesture, familiar to Neapolitans as meaning something like disgust or contempt, of brushing the underneath of his chin with an outward sweep of the finger-tips of one hand. And he asked: 'What is the logical form of that?' Sraffa's example produced in Wittgenstein the feeling that there was an absurdity in the insistence that a proposition and what it describes must have the same 'form'. This broke the hold on him of the conception that a proposition must literally be a 'picture' of the reality it describes." (Malcom & Wright, 2001, p. 59)

But it is specifically in some recent research in the field of Mathematical Education that semiotic systems are being studied explicitly within a wider (outer) approach (e.g. see: Duval, 2002, 2006; Bosch & Chevallard, 1999; Steinbring, 2005, 2006; Radford, 2003a; Arzarello & Edwards, 2005). Such research deepens the original approaches by people like Peirce, Frege, Saussurre, Vygotsky and others.

I will sketch some examples: the semiotic means of objectification, the notion of semiotic systems (both due to Luis Radford), the concept of Representational Infrastructure (due to J. Kaput and to R. Noss) and the so-called extra-linguistic modes of expressions (elaborated by psycholinguists). Radford introduces the notion of *semiotic means of objectification* in Radford (2003a). With this seminal paper, Radford makes explicit the necessity of entertaining a wider notion of semiotic system. He underlines that:

> "Within this perspective and from a psychological viewpoint, the objectification of mathematical objects appears linked to the

individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity. To arrive at it, usually the individuals have recourse to a broad set of means. They may manipulate objects (such as plastic blocks or chronometers), make drawings, employ gestures, write marks, use linguistic classificatory categories, or make use of analogies, metaphors, metonymies, and so on. In other words, to arrive at the goal the individuals rely on the use and the linking together of several tools, signs, and linguistic devices through which they organize their actions across space and time."

Hence he defines this enlarged system as semiotic means of objectification, that is:

"These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities."

The semiotic means of objectification constitute many different types of signs (e.g. gestures, inscriptions, words and so on). They produce what Radford calls *contextual generalization*, namely a generalization which still refers heavily to the subject's actions in time and space and in a precise context, even if he/she is using signs that have a generalizing meaning. In contextual generalization, signs have a two-fold semiotic nature: they are going to become symbols but are still indexes. We use these terms in the sense of Peirce (see: Hartshorne, C. & Weiss, 1933): an index gives an indication or a hint on the object, like an image of the Golden Gate makes you think of the town of San Francisco *("it signifies its object solely*

by virtue of being really connected with it", Hartshorne & Weiss, 1933, 3.361). A symbol is a sign that contains a rule in an abstract way (e.g. an algebraic formula).

The semiotic means of objectification also embody important cultural features. In this sense, Radford speaks of *semiotic systems of cultural meanings* (Radford, this volume; previously called *Cultural Semiotic Systems*, Radford, 2003a), that is, those systems which make available varied sources for meaning-making through specific social signifying practices; such practices are not to be considered strictly within the school environment but within the larger environment of society as a whole, embedded in the stream of its history. Furthermore, cultural semiotic systems are an example of outer enlargement of the notion of semiotic system.

A similar example of enlargement of the notion of semiotic system is the concept of *representational infrastructure*, introduced by J. Kaput et al. (2002), which exploits some cultural and social features of signs. Discussing the appearance of new computational forms and literacies that are pervading the social and economic lives of individuals and nations alike, they write:

"…The real changes are not technical, they are cultural. Understanding them… is a question of the social relations among people, not among things. The notational systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization's infrastructure. As with

infrastructure in general, it functions best when it is taken for granted, invisible, when it simply 'works'." (Kaput et al., 2002, p. 51).

An example both of cultural semiotic system and of representational infrastructure, discussed in Radford (2003a) and in Kaput et al. (2002), consists in the developing of algebraic symbolism, which *"in more than one millennium gradually freed itself from written natural language and developed within a representational infrastructure"*.

As a last example of a broader notion of semiotic system, I refer to the distinction made by psycho-linguists between linguistic and extra-linguistic modes of expression. They describe the former as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs (Bara & Tirassa, 1999):

> *"Linguistic communication is the communicative use of a symbol system. Language is compositional, that is, it is made up of constituents rather than parts... Extra-linguistic communication is the communicative use of an open set of symbols. That is, it is not compositional: it is made up of parts, not of constituents. This makes for crucial differences from language..."*

1.3 The semiotic mediation of artefacts

In keeping with this perspective, artefacts as representational infrastructures also enter into semiotic systems. Realizing the semiotic similarity between signs and artefacts constitutes a crucial step in the story of outer semiotic enlargements. This similarity has two aspects. One is ergonomic and is properly focused if one considers the dialectic between artefact and instrument developed by Verillon & Rabardel (1995) who introduced the notion of *instrumental genesis*. The other is psychological and has been pointed out by Vygotsky, who described the dialectic relationships between signs and instruments by what he called *process of internalization*. I shall describe both in some detail since they allow us to understand more deeply the relevance of the outer enlargements sketched above and are at the basis of my definition of *semiotic bundle*, which I shall introduce below.

Let me start with the ergonomic theory of Verillon and Rabardel[4]: an artefact has its schemes of use (for example, the rules according to which one must manage a compass or a software) and as such it becomes an instrument in the hands of the people who are using it. This idea develops in a fresh way the notion of transformation on a semiotic system. In the ergonomic approach, the technical devices are considered with two interpretations. On the one side, an object has been constructed according to a specific knowledge that assures the accomplishment of specific goals; on the other side, a user interacts with this object, using it (possibly in different ways). The object in itself is called an artefact, that is, a particular object with its features realized for specific goals and it becomes an instrument, that is, an artefact with the various modalities of use, as elaborated by the individual who is using it. The instrument is conceived as the artefact together with the actions made by the subject, organized in collections of operations, classes of invariants and

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4   This part of the paper is taken from Arzarello & Robutti (2004).

utilizations schemes. The artefact, together with the actions, constitutes a particular instrument: thus, the same subject can use the same artefact as different instruments.

The pair instrument-artefact can be seen as a semiotic system in the wider sense of the term. The instrument is produced from an artefact introducing its rules of use and, as such, it is a semiotic representation with rules of use that bear an intentional character: it is similar to a semiotic representation. As semiotic representations, instruments can play a fundamental role in the objectification and in the production of knowledge. For example: the compass is an artefact which can be used by a student to trace a circle as the locus of points in a plane at the same distance from a fixed point. A cardboard disk can be used for the same purpose as the compass, but the concept of circle induced by this use may be different.

The transformation of the artefact into an instrument is made through suitable treatment rules, e. g. for the compass, the action of pointing it at a point and tracing a curve with a fixed ray; for the cardboard disk, the action of carefully drawing a line along its border. In a similar way, students learn to manage algebraic symbols: the signs of Algebra or of Analysis, e.g., $a^2 \text{-} b^2$ or $\mathrm{D}x^2$, are transformed according to suitable treatment rules, e.g. those producing $(a+b)(a\text{-}b)$ or $2x$. Just like an artefact becomes an instrument when endowed with its using rule, the signs of Algebra or of Analysis become symbols, namely signs with a rule (recall the Peirce notion quoted above), because of their treatment rules (see also the discussion about techniques and technologies in Chevallard, 1999).

In both cases, we get semiotic systems with their own rules of treatment. As the coordinated treatment schemes are elaborated by the subject with her/his actions on/with the artefacts/signs, the relationship between the artefact/signs and the subject can evolve. In the case of concrete artifacts, it causes the so-called process of *instrumental genesis*, revealed by the schemes of use (the set of organized actions to perform a task) activated by the subject. In the example above, the knowledge relative to the circle is developed through the schemes of use of the compass or of the cardboard. In the case of algebraic signs, the analogous of the instrumental genesis produced by syntactic manipulations may produce different types of knowledge relative to the numerical structures (see the notion of theory as emerging from the techniques and the technologies, discussed in Chevallard, 1999). Hence, the ergonomic analysis points to an important functional analogy between artefacts and signs [5].

Within a different perspective, Vygotsky had also pointed out a similar analogy between tools [6], which can support human labour, and signs, which can uphold the psychological activities of subjects:

> *"...the invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose and so on) is analogous to the invention of tools in one psychological*

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[5] A similar analogy is achieved within a different framework by Chevallard (1999).

[6] In the Cambridge Dictionary, a tool is defined as *"something that helps you to do a particular activity"*, an instrument is *"a tool that is used for doing something"*, while an artefact is an *"object"*. Following this definition, I consider the instrument as a specific tool.

respect. The signs act as instrument of psychological activity in a manner analogous to the role of a tool in labour." (Vygotsky, 1978, p. 52)

As I anticipated above, this common approach to signs and tools is based on the notion of *semiotic mediation* [7], which is at the core of the Vygotskian frame: for a survey see Bartolini & Mariotti (to appear) a paper from which I take some of the following comments.

Vygotsky pointed out both a functional analogy and a psychological difference between signs and instruments. The analogy is illustrated by the following quotation, which stresses their semiotic functions:

   "...the basic analogy between sign and tools rests on the mediating function that characterizes each of them" (*ibid.,* p. 54).

The difference between signs and tools is so described:

   "the tool's function is to serve as the conductor of human influence on the object of activity; it is externally *oriented...The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself: the sign is* internally *oriented."* (*ibid*., p. 55)

This distinction is central in the Vygotskyan approach, which points out the transformation from externally oriented tools to internally oriented tools (often called *psychological tools*) through the process of *internalization*. According to

Vygotsky, in the process of internalization, interpersonal processes are transformed into intrapersonal ones. The process of internalization (through which the 'plane of consciousness' is formed, see Wertsch & Addison Stone, 1985, p.162) occurs through semiotic processes, in particular by the use of semiotic systems, especially of language, in social interaction:

   "...the Vygotskian formulation involves two unique premises...First, for Vygotsky, internalisation is primarily concerned with social *processes. Second, Vygotsky's account is based largely on the analysis of the semiotic mechanisms, especially language, that mediate social and individual functioning....Vygotsky's account of semiotic mechanisms provides the bridge that connects the external with the internal and the social with the individual…Vygotsky's semiotic mechanisms served to bind his ideas concerning genetic analysis and the social origins of behaviour into an integrated approach…it is by mastering semiotic mediated processes and categories in social interaction that human consciousness is formed in the individual"* (Wertsch & Addison Stone, 1985, pp.163-166)

As Bartolini Bussi & Mariotti (Bartolini & Mariotti, to appear) point out, Vygotsky stresses the role and the dynamics of semiotic mediation: first, externally oriented, a sign or a tool is used in action to accomplish a specific task; then, the actions with the sign or the tool (semiotic activity, possibly under the guidance of an expert), generate new signs (words included), which foster the internalization process and produce a new

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[7] It is described in Vygotsky (1978, especially p. 40 and ff).

psychological tool, internally oriented, completely transformed but still maintaining some aspects of its origin.

Vygotsky describes such dynamics without any reference to mathematics; hence, his observations are general; many recent studies have adapted his framework to fit the specificity of mathematics (e.g. see Radford, 2003a; Bartolini & Mariotti, to appear).

2. A new theoretical frame: the semiotic bundle

2.1 Definition and examples

My framework is also specific for mathematics; it allows for better combining the two issues described above, the one from semiotics, in the spirit of the quoted Ernest definition of semiotic systems, and the other from psychology, according to the Vygotskian approach. Both pictures are essential for analyzing the learning processes in mathematics; they are here integrated within a wider model.

On the one hand, it is necessary to broaden the notion of semiotic system in order to encompass all the variety of phenomena of semiotic mediation in the classroom, as already suggested by Radford, who introduced a new notion of semiotic system:

The idea of semiotic system that I am conveying includes classical system of representations – e.g. natural language, algebraic formulas, two or three-dimensional systems of representation, in other terms, what Duval (2001) calls discursive and non-discursive registers – but also includes more general systems, such as gestures (which have an intuitive meaning

and to a certain extent a fuzzy syntax) and artifacts, like calculators and rulers, which are not signs but have a functional meaning. (Radford, 2002, p. 21, footnote 7).

On the other hand, the psychological processes of internalization, so important in describing the semiotic mediation of signs and tools, must fill a natural place within the new model.

A major step towards the common frame consists in reconsidering the notion of semiotic system along the lines suggested by Radford. Once we have a more suitable notion of semiotic system, we shall come back to the Vygotskian approach and show that this fresh notion encompasses it properly, allowing for a deeper understanding of its dynamics.

This fresh frame takes into account the enormous enlargement of the semiotic systems horizon, both from the inner and from the outer side that has been described above. Once the semiotic systems have been widened to contain gestures, instruments, institutional and personal practices and, in general, extra-linguistic means of expression, the same idea of operation within or between different registers changes its meaning. It is no longer a treatment or conversion (using the terminology of Duval) within or between semiotic representations according to algorithmic rules (e.g. the conversion from the geometric to the Cartesian register). On the contrary, the operations (within or between) must be widened to also encompass phenomena that may not be strictly algorithmic: for example, practices with instruments, gestures and so on.

At this point of the discussion, the above definition by Ernest can be widened to encompass all the examples we have

given. We thus arrive at the notion that I have called *semiotic bundle* (or bundle of semiotic sets). To define it, I need first the notion of *semiotic set*, which is a widening of the notion of semiotic system.

A semiotic set is:

a) A set of signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact.

b) A set of modes for producing signs and possibly transforming them; such modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject.

c) A set of relationships among these signs and their meanings embodied in an underlying meaning structure.

The three components above (signs, modes of production/transformation and relationships) may constitute a variety of systems, which span from the compositional systems, usually studied in traditional semiotics (e.g. formal languages) to the open sets of signs (e.g. sketches, drawings, gestures). The former are made of elementary constituents and their rules of production involve both atomic (single) and molecular (compound) signs. The latter have holistic features, cannot be split into atomic components, and the modes of production and transformation are often idiosyncratic to the subject who produces them (even if they embody deeply shared cultural aspects, according to the notion of *semiotic systems of cultural meanings* elaborated by Radford, quoted above). The word set must be interpreted in a very wide sense, e.g. as a variable collection.

A semiotic bundle is:

(i) A collection of semiotic sets.
(ii) A set of relationships between the sets of the bundle.

Some of the relationships may have conversion modes between them.

A semiotic bundle is a *dynamic structure* which can change in time because of the semiotic activities of the subject: for example, the collection of semiotic sets that constitute it may change; as well, the relationships between its components may vary in time; sometimes the conversion rules have a genetic nature, namely, one semiotic set is generated by another one, enlarging the bundle itself (we speak of *genetic conversions*).

Semiotic bundles are semiotic representations, provided one considers the intentionality as a relative feature (see the above comment on the sand footprint).

An example of semiotic bundle is represented by the unity speech-gesture. It has been a recent discovery that gestures are so closely linked with speech that "*we should regard the gesture and the spoken utterance as different sides of a single underlying mental process*" (McNeill, 1992, p.1), namely "*gesture and language are one system*" (*ibid.*, p.2). In our terminology, gesture and language are a semiotic bundle, made of two deeply intertwined semiotic sets (only one, speech, is also a semiotic system). Research on gestures has uncovered some important relationships between the two (e.g. match and mismatch, see Goldin-Meadow, 2003). A semiotic bundle must not be considered as a juxtaposition of semiotic sets; on the contrary, it is a unitary system and it is only for the sake of analysis that we distinguish its components as semiotic sets. It must be observed that if one limits oneself

to examining only the semiotic systems and their bundles, many interesting aspects of human discourse are lost: only by considering bundles of semiotic sets can new phenomena be discovered.

This wider approach is particularly fruitful when the processes and activities of people learning mathematics are scrutinized. In the research carried out by the Turin team[8] we investigate semiotic bundles made of several semiotic sets: e.g. gesture, speech and written inscriptions (e.g. mathematical symbols, drawings). The results consist in describing some of the relationships and conversion rules within such a complex bundle.

Semiotic bundles allow us to frame the Vygotskian notion of semiotic mediation sketched above in a more comfortable setting. The dynamics in the process of internalization, according to Vygotsky, is based on semiotic activities with tools and signs, externally oriented, which produce new psychological tools, internally oriented, completely transformed but still maintaining some aspects of their origin. According to Vygotsky, a major component in this internalization process is language, which allows for the transformations. Moreover, such transformations 'curtail' the linguistic register of speech into a new register: Vygotsky calls it *inner speech* and it has a completely different structure. This has been analyzed by Vygotsky in the last (7th) chapter of *Thought and Language*

(Vygotsky, 1992), whose title is *Thought and Word*. Vygotsky distinguishes two types of properties that allow us to distinguish the inner from the outer language: he calls them *structural* and *semantic properties*.

The structural properties of the inner language are its *syntactic reduction* and its *phasic reduction*: the former consists in the fact that inner language reduces to pure juxtaposition of predicates minimizing its syntactic articulation; the latter consists in minimizing its phonetic aspects[9], namely curtailing the same words.

According to Vygotsky's frame, the semantic properties of the inner language are based on the distinction made by the French psychologist Frederic Pauhlan between the *sense* and the *meaning* of a word and by *"the preponderance of the sense [smysl] of a word over its meaning [znachenie]"* (Vygotsky, 1978, p. 244):

> *"the sense is...the sum of all the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole, which has several zones of unequal stability. Meaning is only one of the zones of sense, the most stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense. "* (*ibid.,* p. 244-245).

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8   This is being done by our colleagues Luciana Bazzini and Ornella Robutti, by some doctoral and post-doc students, like Francesca Ferrara and Cristina Sabena, and by many teachers (from the elementary to the higher school level) that participate actively to our research, like Riccardo Barbero, Emilia Bulgarelli, Cristiano Dané, Silvia Ghirardi, Marina Gilardi, Patrizia Laiolo, Donatella Merlo, Domingo Paola, Ketty Savioli, Bruna Villa and others.

9   To make an analogy with the outer language, Vygotsky recalls an example, taken from Le Maitre (1905), p. 41: a child thought to the French sentence "Les montagnes de la Suisse sont belles" as "L m d l S s b" considering only the initial letters of of the sentence. Curtailing is a typical feature of inner language.

In inner language, the sense is always overwhelming the meaning. This prevailing aspect of the sense has two structural effects on inner language: the *agglutination* and the *influence*. The former consists in gluing different meanings (concepts) into one expression[10]; the latter happens when the different senses 'flow' together[11] into one unity.

To explain the properties of inner speech, Vygotsky uses analogies that refer to the outer speech and these give only some idea of what he means: in fact, he uses a semiotic system (written or spoken language) to describe something which is not a semiotic system. The grounding metaphors through which Vygotsky describes inner speech show its similarity to semiotic sets: properties like *agglutination* and *influence* make inner speech akin to some semiotic sets, like drawings, gestures and so on. Also, the syntactic phenomena of syntactic and phasic reduction mean that the so-called linear and compositional properties of semiotic systems are violated. Vygotsky's description through the lens of semiotic systems makes this aspect only partially evident.

The notion of semiotic bundle properly frames the most important point in Vygotsky's analysis, namely, the semiotic transformations that support the transformation from outer to inner speech (internalization). The core of Vygotsky's analysis, namely, the internalization process, consists exactly in pointing out a genetic conversion within a semiotic bundle: it generates a fresh semiotic component, the inner speech, from another existing one, the outer speech. The description is given using the structure of the former, which is clearly a semiotic system, to build grounding metaphors in order to give an idea of the latter, which is possibly a semiotic set. The whole process can be described as the enlarging of a bundle through a genetic conversion process.

The main point of this paper consists in using the notion of semiotic bundle to frame the mathematical activities that take place in the classroom. I will argue that learning processes happen in a multimodal way, namely in a dynamically developing bundle, which enlarges through genetic conversions and where more semiotic sets are active at the same moment. The enlargement consists both in the growing of (the number of) active semiotic sets within the bundle and in the increase of the number of relationships (and transformations) between the different semiotic sets.

Their mutual relationships will be analyzed through two types of lenses, which I have called synchronic and diachronic since they analyze the relationship among processes that happen simultaneously or successively in time. The two approaches, which will be discussed below, allow us to frame many results in a unitary way: some are already known but some are new. In particular, I shall investigate the role of gestures in the mathematical discourses of students[12]. I will argue that they acquire a specificity in the

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[10] Vygotsky makes the analogy with the outer language alluding to so-called agglutinating languages which put together many different words to constitute a unique word.

[11] To give an idea of influence, Vygotsky makes reference to *The Dead Souls* by N.V. Gogol whose title, by the end of the book, should mean to us *"not so much the defunct serfs as all the characters in the story who are alive physically but dead spiritually"* (*ibid.*, p. 247)

[12] Another research project that our group is pursuing concerns the role of teachers' gestures with respect to the learning processes of students: how they are shared by students and how they influence their conceptualization processes.

construction of meaning in mathematical activities because of the rich interplay among three different types of semiotic sets: speech, gestures and written representations (from sketches and diagrams to mathematical symbols). They constitute a semiotic bundle, which dynamically evolves in time.

To properly describe this interplay and the complex dynamics among the different semiotic sets involved in the bundle, I need some results from psychologists, who study gesture. In the next two sections (2.2 and 2.3) I will sketch out some of these.

2.2 Semiotic bundles and multimodality

In mathematics, semiotic representations are deeply intertwined with mental ones (see the discussion in Duval, 2006, pp. 106-107). On the one side, there is a genetic relationship between them: *«the mental representations which are useful or pertinent in mathematics are always interiorized semiotic representations»* (Duval, 2002, p.14). See also the discussion on the internalisation processes in Vygotsky.

On the other side, very recent discoveries in Neuropsychology underline the *embodied* and *multimodal* aspects of cognition. A major result of neuroscience is that *"conceptual knowledge is embodied, that is, it is mapped within the sensory-motor system"* (Gallese & Lakoff, 2005, p.456). *"The sensory-motor system not only provides structure to conceptual content, but also characterizes the semantic content of concepts in terms of the way in which we function with our bodies in the world"* (*ibid.*). The sensory-motor system of the brain is multimodal rather than modular; this means that

"an action like grasping…(1) is neurally enacted using neural substrates used for both action and perception, and (2) that the

modalities of action and perception are integrated at the level of the sensory-motor system itself and not via higher association areas." (ibid., p. 459).

"Accordingly, language is inherently multimodal in this sense, that is, it uses many modalities linked together—sight, hearing, touch, motor actions, and so on. Language exploits the pre-existing multimodal character of the sensory-motor system." (ibid., p. 456).

The paradigm of multimodality implies that *"the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context."* (Nemirovsky, 2003; p. 108).

Semiotic bundles are the real core of this picture: they fit completely with the embodied and the multimodal approach. At least one consequence of this approach is that the usual *transformations* and *conversions* (in the sense of Duval) from one register to the other must be considered as the basic producers of mathematical knowledge. Furthermore, its essence consists in the *multimodal interactions* among the different registers within a unique integrate system composed of different modalities: gestures, oral and written language, symbols, and so on (Arzarello & Edwards, 2005; Robutti, 2005). Also, the symbolic function of signs is absorbed within such a picture.

Once the multimodal nature of processes is on the table, manipulations of external signs and of mental images show a common psychological basis: transformational and symbolic functions are revealed as processes that have a deep common nature.

I will argue that if we mobilize a rich semiotic bundle with a variety of semiotic sets (and not only semiotic systems) with their complex mutual relationships (of transformation, conversion, symbolic functions as multimodal interactions among them) students are helped to construct integrated models for the mathematical knowledge they are supposed to learn and understand. In fact, mathematical activity is featured by the richness of the semiotic bundle that it activates. However, things may not be so in the school, where two negative phenomena can push the process in the opposite direction. I call them the Piaget and the Wittgenstein effect, respectively:

a) (*Piaget effect*). Piaget made the *search for isomorphisms* one of the key principles for analyzing knowledge development in children. This emphasis risks underestimating the relevance of the different registers of representation:

> « *Dismissing the importance of the plurality of registers of representation comes down to acting as if all representations of the same mathematical object had the same content or as if the content of one could be seen from another as if by transparency!"* (Duval, 2002, p.14).

b) (*Wittgenstein effect*). Recall the story about Sraffa and Wittgenstein. The author of *Tractatus* in the first phase of his research revealed a sort of *blindness to semiotic sets* (in that case, the gesture register). This is also the case for many mathematicians and teachers: they are possibly interested in semiotic systems as formal systems, while the wider semiotic sets are conceived as something that is not relevant for mathematical activities, especially at the secondary school level.

A consequence of these effects in the classroom is that only some semiotic systems are considered, while semiotic bundles (generally not even restricting oneself to the bundles of semiotic systems) are not taken into account. And even when different semiotic systems are considered, they are always conceived as signifiers of the same object. On the contrary, the representations within a semiotic bundle have their own specificity in promoting an integrated mental model according to the multimodal paradigm, as we shall show in the next chapter.

2.3 Gestures within semiotic bundles

Among the components of semiotic bundles, the semiotic set of gestures has an important role, especially when its relationship with speech and written signs are considered within a multimodal picture. Psychologists have mainly studied gestures in day to day conversation: I shall go over some of their findings in the remaining part of this chapter and I will describe the relationship of gestures (and speech) to written signs in Chapter 3. To do this, I will elaborate upon some of the papers in Arzarello & Edwards (2005), especially the Introduction, and I will also quote some results of Bucciarelli (in print).

Two main points from psychology are important to discuss the way gestures enter into the multimodal semiotic analysis within which we frame the understanding of mathematical concepts in students.

The first point concerns the so-called *Information Packaging Hypothesis*. It expands the idea that "*gestures, together with language, help constitute thought*" (McNeill, 1992, p. 245). According to McNeill (p. 594-5), gesture plays a role in cognition—not just in communication—since it is involved in the *conceptual planning* of the messages and plays a role

in speech production because it plays a role in the process of conceptualization. Gesture *"helps speakers organize rich spatio-motoric information into packages suitable for speaking [...] by providing an alternative informational organization that is not readily accessible to analytic thinking, the default way of organizing information in speaking"* (Kita, 2000).

Spatio-motoric thinking (constitutive of what Kita calls representational gestures) provides an alternative informational organization that is not readily accessible to analytic thinking (constitutive of speaking organization). Analytic thinking is normally employed when people have to organize information for speech production, since speech is linear and segmented (composed of smaller units); namely, it is a semiotic system. On the other hand, spatio-motoric thinking is instantaneous, global and synthetic, not analyzable into smaller meaningful units, namely, it is a semiotic set. This kind of thinking and the gestures that arise from it are normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery. Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> *"According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed*

representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought." (Goldin-Meadow, 2003, p. 178).

A second point, claimed by Bucciarelli (in press), concerns the relationships between Mental Models (see Johnson Laird, 1983, 2001) and gestures. Many studies in psychology claim that the learning of declarative knowledge involves the construction of mental models. Bucciarelli argues that gestures accompanying discourse can favour the construction of such models (and therefore of learning). In Cutica & Bucciarelli (2003) it is shown that when gestures accompany discourse the listener retains more information with respect to a situation in which no gestures are performed: *"The experimental evidence is in favour of the fact that gesture do not provide redundancy, rather they provide information not conveyed by words"* (Bucciarelli, in press).

Hence, gestures lead *"to the construction of rich models of a discourse, where all the information is posited in relation with the others"* (*ibid.*).

In short, the main contribution of psychology to the theory of semiotic bundles consists in this: the multimodal approach can favour the understanding of concepts because it can support the activation of different ways of coding and manipulating the information (e.g. not only in an analytic fashion) within the semiotic bundle. This can foster the construction of

a plurality of mental models, whose integration can produce deep learning.

Of course these observations are general and concern general features of learning. In the next chapter, I shall discuss how this general frame can be adapted to the learning of mathematics.

This attention to semiotic bundles underlines the fact that mathematics is inseparable from symbolic tools but also that it is *"impossible to put cognition apart from social, cultural, and historical factors"* (Sfard & McClain, 2002, p. 156), so that cognition becomes a *"culturally shaped phenomenon"* (*ibid*.). In fact, the embodied approach to mathematical knowing, the multivariate registers according to which it is built up and the intertwining of symbolic tools and cognition within a cultural perspective are the basis of a unitary frame for analyzing gestures, signs and artefacts. The existing research on these specific components finds a natural integration in such a frame (Arzarello & Edwards, 2005).

In the next chapter, I will focus the attention on the ways in which semiotic bundles are involved in the processes of building mathematical knowledge in the classroom.

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3. Semiotic bundles in mathematics learning.

3.1 Synchronic and diachronic analysis

In this chapter, I will illustrate how the notion of semiotic bundle can suitably frame the mathematising activities of young students who interact with each other while solving a mathematical problem. What we will see is a consequence of these social interactions, which can happen and develop because of the didactical situations to which the students are exposed. As I shall sketch below, they are accustomed to developing mathematics discussions during their mathematics hours. The richness of the semiotic bundle that they use depends heavily on such a methodology; in a more traditional classroom setting, such richness may not exist and this may be the cause of many difficulties in mathematical learning: see the comments in Duval (2002, 2006), already quoted, about this point.

The example under consideration concerns elementary school and has been chosen for two reasons: (1) it is emblematic of many phenomena that we have also found at different ages; (2) the simplicity of the mathematical content makes it accessible for everyone.

In the example, I shall show that students in a situation of social interaction use a variety of semiotic sets within a growing semiotic bundle and I shall describe the main mutual relationships among them. To do that, I will use two types of analysis, each focusing on a major aspect of such relationships. The first one is *synchronic analysis*, which studies the relationships among different semiotic sets activated simultaneously by the subject. The second is *diachronic analysis*, which studies the relationships among semiotic sets activated by the subject in successive moments. This idea has been introduced by the authors in Arzarello & Edwards (2005) under the names of *parallel* and *serial analysis*. I prefer the terminology "à la Saussurre" ([13]) because it underlines the

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[13] Saussure distinguishes between *synchronic* (static) linguistics and *diachronic* (evolutionary) linguistics. Synchronic linguistics is the study of language at a particular point in time. Diachronic linguistics is the study of the history or evolution of language.

time component that is present in the analysis. However, our time grain is at a different scale, that is, while Saussurre considers long periods of time concerning the historical evolution of at most two semiotic systems (spoken and written language), I consider the interactions among many different semiotic sets over very short periods of time.

Synchronic analysis, even if under a different name, is present in the study of gestures: e.g. the distinction made by Goldin-Meadow between matching and mismatching considers gesture and speech produced at the same moment and conveying equal or different information. Another example of synchronic analysis can be made in mathematics when considering the production of drawings (or formulas) and of speech by students who are solving a problem (see e.g. Arzarello, 2005; but the literature is full of examples). A further example is the semiotic node, discussed by Radford et al. (2003b).

Also, diachronic analysis is not completely new in the literature on signs: e.g. see the notion of mathematical objectification in Radford, or that of conversion in Duval, both discussed above. The power of diachronic analysis changes significantly when one considers the semiotic bundles. In fact, the relationship between sets and systems of signs cannot be fully analyzed in terms of translation or of conversion because of the more general nature of the semiotic sets with respect to the semiotic systems. The modes of conversion between a semiotic set and a semiotic system make evident a genetic aspect of such processes, since a genuine transformation (conversion) is a priori impossible. In fact, a transformation presupposes an action between two already existing systems like in the translation from one language to another.

In our case, on the contrary, there is a genesis of signs from a set or a system to a system or a set. The fresh signs with the new set (system) are often built preserving some features of the previous signs (e.g. like the icon of a house preserves some of the features of a house according to certain cultural stereotypes). The preservation generally concerns some of the extralinguistic (e.g. iconic) features of the previous signs, which are generating new signs within the fresh semiotic set (or system); possibly, the genesis continues with successive conversions from the new sets (systems) into already codified systems. Hence, the process of conversion described by Duval concerns mainly the last part of the phenomenon, which involves the transformation between already existing systems. Our analysis shows that such process starts before and has a genetic aspect, which is at the root of the genesis of mathematical ideas.

The main point is that only considering semiotic sets allows us to grasp such a phenomenon, possibly through a diachronic analysis. In fact, nothing appears if one considers only semiotic systems or considers synchronic events.

One could think that such a genesis is far from the sophisticated elaborations of more advanced mathematics. But things are not so; I have examples of this genesis concerning the learning of Calculus (see: Arzarello & Robutti, to appear).

The two analyses, synchronic and diachronic, allows us to focus on the roles that the different types of semiotic sets involved (gestures, speech, different inscriptions, from drawings to arithmetic signs) play in the conceptualization processes of pupils. The general frame is that of multimodality, sketched above.

3.2 The example [14]

The activity involves pupils attending the last year of primary school (5th grade, 11 y.o.); the teacher gives them a mathematical story that contains a problem to solve, taken from the legend of Penelope's cloth in Homer's Odyssey. The original text was modified to get a problem-solving situation that necessitated that the students face some conceptual nodes of mathematics learning (decimal numbers; space-time variables). The text of the story, transformed, is the following:

> *… On the island of Ithaca, Penelope had been waiting twenty years for the return of her husband Ulysses from the war. However, on Ithaca a lot of men wanted to take the place of Ulysses and marry Penelope. One day the goddess Athena told Penelope that Ulysses was returning and his ship would take 50 days to arrive in Ithaca. Penelope immediately summoned the suitors and told them: "I have decided: I will choose my bridegroom among you and the wedding will be celebrated when I have finished weaving a new piece of cloth for the nuptial bed. I will begin today and I promise to weave every two days; when I have finished, the cloth will be my dowry." The suitors accepted. The cloth had to be 15 spans in length. Penelope immediately began to work, but one day she would weave a span of cloth, while the following day, in secret, she would undo half a span… Will Penelope choose another husband? Why?*

When the Penelope's story was submitted to the students (Dec. 2004- Feb. 2005) they were attending the last year of primary school (5th grade). Later, in April-May 2005, in the same school six more teachers submitted the story to their classrooms, as part of an ongoing research project for the Comenius Project DIAL-Connect (Barbero et al., in press). Students were familiar with problem solving activities, as well as with interactions in group. They worked in groups in accordance with the didactical contract that foresaw such a kind of learning. The methodology of the mathematical discussion was aimed at favouring the social interaction and the construction of shared knowledge. As part of the didactical contract, each group was also asked to write a description of the process followed to reach the problem solution, including doubts, discoveries, heuristics, etc. The students' work and discussions were videotaped and their written notes were collected. The activity consisted of different steps that we can summarize as follows. First, the teacher reads the story and checks the students' understanding of the text; the story is then delivered to the groups. Different materials are at the students' disposal, among which paper, pens, colours, cloth, scissors, glue. In a second phase, the groups produce a written solution. The teacher invites the groups to compare the solutions in a collective discussion; she analyses strategies, difficulties, misconceptions, thinking patterns and knowledge content to be strengthened. Then, a poster with the different groups' solutions is produced. In the final phase, the students are required to produce a number table and a graph representing the story; they work individually using Excel to construct the table and the graph of the problem solution. Again, they discuss about different solutions and share conclusions.

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[14] This part of the paper is partially taken from Arzarello et al. (2006), with the permission of the other authors.

The part of the activity analyzed below is a small piece of the initial phase (30'); it refers to a single group composed of five children: D, E, M, O, S, all of them medium achievers except M, who is weak in mathematical reasoning.

3.3 Analysis: a story of signs under the lenses of diachronic and synchronic analysis.

The main difficulty of the Penelope problem is that it requires two registers to be understood and solved: one for recording the time, and one for recording the successive steps of the cloth length. These registers must be linked in some way, through some relationship (mathematicians would speak of a function linking the variables time and cloth length). At the beginning, these variables are not so clear for the students. So, they use different semiotic sets to disentangle the issue: gestures, speech, written signs. They act with and upon them; they interact with each other; they repeatedly use the text of the story to check their conjectures; they use some arithmetic patterns.

We see an increasing integration of these components within a semiotic bundle: in the end, they can grasp the situation and objectify a piece of knowledge as a result of a complex semiotic and multimodal process. We shall sketch some of the main episodes and will comment a few key points in the final conclusion (numbers in brackets indicate time).

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Episode 1. The basic gestures (synchronic analysis).

After reading the text, the children start rephrasing, discussing and interpreting it. To give sense to the story, they focus on the action of weaving and unraveling a span of cloth which is represented by different gestures: a hand sweeping across the desk (Fig. 1), the thumb and the index extended (Fig. 2), two hands displaced parallel on the desk (Figs. 3 and 4). Some gestures introduced by one student are easily repeated by the others and become a reference for the whole group.

This is the case of the two parallel hands shown in Figs. 3 and 4. Attention is focused on the action, and the gestures occur matching either the verbal clauses or the "span", as we can see from the following excerpt:

(6'58'') **S:** She makes a half (*hand gesture in Fig. 2*), then she takes some away (*she turns her hand*), then she makes… (*again, her hand is in the position of Fig. 2*) […]

| *Figure 1* | *Figure 2* | *Figure 3* | *Figure 4* |

E: "It is as if you had to make a piece like this, it is as if you had to make a piece of cloth like this, she makes it (*gesture in Fig. 3*). Then you take away a piece like this (*gesture in Fig. 5*), then you make again a piece like this (*gesture in Fig. 3*) and you take away a piece like this (*gesture in Fig. 5*)"

O: "No, look… because… she made a span (*Fig. 4*) and then, the day after, she undid a half (*O carries her left hand to the right*), and a half was left… right? … then the day after…"

D: (*D stops O*) "A half was always left"

Figure 5

The dynamic features of gestures that come along with speech condense the two essential elements of the problem: time passing and Penelope's work with the cloth. Their existence as two entities is not at all explicit at this moment, but, through gesturing, children make the problem more tangible. The function of gestures is not only to enter into the problem, but also to create situations of discourse whose content is accessible to everyone in the group. The rephrasing of similar words and gestures by the students (see the dispositions of the hands in Fig. 4) starts a dynamics for sharing various semiotic sets, with which the group starts to solve the problem. At the moment, the semiotic bundle is made up of their gestures, gazes and speech .

Episode 2. A new semiotic set: from gestures to written signs (diachronic analysis).

After having established a common understanding of what happens in Penelope's story, the children look for a way to compute the days. S draws a (iconic) representation of the work Penelope does in a few days, actually using her hand to measure a span on paper. The previous gesture performed by different students (Figs. 3-5) now becomes a written sign (Fig. 6). As had happened before with words and gestures, the drawing is also imitated and re-echoed by the others (Fig. 7): even these signs, generated by the previous gestures, contribute to the growth of the semiotic bundle. The use of drawings makes palpable to the students the need of representing the story using two registers. See the two types of signs in Figs. 7-8: the vertical parallel strokes (indicating spans of cloth) and the bow sign below them (indicating time).

Episode 3. The mutimodality of semiotic sets I: towards a local rule (diachronic + synchronic analysis).

In the following excerpts, the children further integrate what they have produced up to now (speech, gestures and written representations) and also use some arithmetic; their aim is to grasp the rule in the story of the cloth and to reason about it. They can now use the written signs as *"gestures that have been fixed"* (Vygotsky, 1978; p. 107) and represent the story in a condensed way (see Fig. 8); moreover, they check their conjectures reading again the text of the problem:

(10'30'') **S:** From here to here it is two spans (*she traces a line, mid of Fig. 8*). If I take half, this part disappears (*she traces the horizontal traits in Fig. 8*) and a span is left; therefore in two days she makes a span

O: No, in four days, in four, because…

S: In four days she makes two spans, because (*she traces the curve under the traits in Fig. 8*)…plus this

O: In four days she makes one, because (*she reads the text*), one day she wove a span and the day after she undid a half…

As one can see in Fig. 7, S tries to represent on paper Penelope's work of weaving and also of unraveling, which causes troubles, because of the necessity of marking time and length in different ways. These two aspects naturally co-existed in gestures of Figg. 1-3. O finds the correct solution (4 days for a span),

but the group does not easily accept it and O gets confused. The drawing introduced by S (Fig. 8) represents the cloth, but with holes; due to the inherent rigidity of the drawing, students easily see the span, but not half a span. A lively discussion on the number of days needed to have a span begins. Numbers and words are added to the drawings (Figs. 9-10) and fingers are used to compute (Fig. 11). New semiotic resources enter the scene within different semiotic sets which are integrating each other more and more, not by juxtaposition or translation but by integration of their elements: they all continue to be active within the semiotic bundle, even later, as we shall see below.

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Episode 4. The multimodality of semiotic sets II: towards a global rule (diachronic analysis).

Once the local question of "how many days for a span" is solved, the next step is to

Figure 6

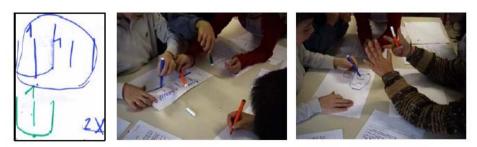*Figure 7*

Figure 8

Figure 9

Figure 10

Figure 11

solve the problem globally. To do that, the rule of "4 days for a span" becomes the basis (Fig.12) of an iterative process:

(13'30") **O, E:**… it takes four days to make a whole span (*E traces a circle with the pen all around*: Fig. 12)

D: and another four to make a span (*D shows his fingers*) and it adds to 8 (*D counts with fingers*)

S: so, we have to count by four and arrive at 50 days (*forward strategy*: Fig. 13) [...] (14'25") **O**: no, wait, for 15 spans, no, 4 times 15

S: no, take 15, and always minus 4, minus 4, minus 4 (or: 4 times 5), minus 2, no, minus 1 [*backward strategy*: Fig. 14]

Two solving strategies are emerging here: a *forward strategy* (counting 4 times 15 to see how many days are needed to weave the cloth) and a *backward strategy* (counting "4 days less" 15 times to see if the 50 days are enough to weave the cloth). The two strategies are not so clear to the children and conflict with each other.

In order to choose one of them, the children use actual pieces of paper, count groups of four days according to the forward strategy and so they acquire direct control over the computation. Only afterwards do they compute using a table and find that 60 days are needed for 15 spans of cloth. In this way, they can finally answer the question of the problem and write the final report: Penelope will not choose another bridegroom.

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Conclusions

The story of signs described in the example illustrates the nature of semiotic bundles. The first signs (gestures, gazes and speech) constitute a first basic semiotic bundle, through which the children start their semiotic activities. Through them, the bundle is enriched with new semiotic sets (drawings and numbers) and with a variety of fresh relationships among them. The enlargement occurs through genetic conversions, namely through a genetic process, where the previous semiotic sets (with their mutual relationships) generate new semiotic components and change because of this genesis, becoming enriched with fresh mutual relationships. By so doing, not only do the students produce new semiotic sets, but the sense—in the Vygotskian meaning of the word—of the older ones is transformed, still maintaining some aspects of their origin. All these processes develop within a gradually growing and multimodal cognitive environment that we have analyzed through the lens of the semiotic bundle.

The story of the bundle starts with the gesture of the two hands displaced parallel on the desk (episode 1). This gesture later generates a written iconic representation

| *Figure 12* | *Figure 13* | *Figure 14* | *Figure 15* |

(episode 2), successively enriched by numerical instances (episode 3) and by arithmetic rules (episode 4), expressed through speech and (new and old) gestures. Gesture, speech, written signs and arithmetic representations grow together in an integrated way supporting the semiotic activities within the semiotic bundle which enlarges more and more. Students develop their semiotic activities and share them: it is exactly through such activities that they can grasp the problem, explore it and elaborate solutions.

All the components are active in a multimodal way up to the end. This is even evident when the students discuss how to write the solution in the final report (Fig. 15: 27' 32"). Gestures and speech intervene first as cognitive means for understanding the story of the cloth; later as means of control for checking the conjectures on the rule. Information is condensed in gestures, entailing a global understanding of the story. The two variables (time and cloth development), first condensed in gesture (an agglutination example in the sense of Vygotsky), generate two different signs in the fresh semiotic set (drawings) that they themselves have generated within the semiotic bundle: it is exactly this disentanglement that allows children to grasp the story separating its structural elements. On its own, speech objectifies the structure of the story, first condensing the local rule in a sentence (episode 3), then exploiting the general rule as an iterative process (episode 4).

The semiotic objectification in this story happens because of the semiotic activities within the semiotic bundle. It is evident that it constitutes an integrated semiotic unity; the activity within it does not consist of a sequence of transcriptions from one register

to another, as posited in other studies (e.g. Duval, 1993). On the contrary, it develops in a growing, holistic and multimodal way, which, in the end, produces the objectification of knowledge.

The lenses of semiotic bundles allow us to frame the semiotic phenomena that occur in the classroom within a unitary perspective. Moreover, a semiotic bundle also incorporates dynamic features, which can make sense of the complex genetic relationships among its components, e.g. the genetic conversions and the Vygotskian internalization processes.

This study leaves many problems open: I list only some of those I am interested in studying in the near future:

1. Elsewhere (Arzarello, in press), I introduced the notion of *Space of Action, Production and Communication* (APC-space) as an environment in which cognitive processes develop through social interaction; its components are: culture, sensory-motor experiences, embodied templates, languages, signs, representations, etc. These elements, merged together, shape a multimodal system through which didactical phenomena are described. An interesting problem consists in studying the relationships between the semiotic bundles and the APC-space.

2. The time variable is important in the description of semiotic bundles, e.g. it is relevant to the diachronic and synchronic analysis. What are the connections between this frame and the didactic phenomena linked to students 'inner times[15]', like those described in Guala & Boero (1999)? There, the authors

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[15]   I thank Paolo Boero for suggesting this problem to me.

list different types of inner times in students' problem solving activities (the 'time of past experience', the 'contemporaneity time', the 'exploration time', the 'synchronous connection time'), which make sense of their mental dynamics. Of course, such activities can be analyzed with semiotic lenses. How do the different inner times enter into a semiotic bundle? Which kinds of conversions or treatments can they generate from one semiotic set to another or within the same semiotic set?.

3. In the processes of students who build new knowledge, there are two dual directions in the genetic conversions within the semiotic bundle: from semiotic sets to semiotic systems (e.g. from gestures to drawings and symbols) or the opposite. The episodes in Penelope's story are an example of the first type, while Vygotsky describes the second type in the internalization processes (a similar example is described in Arzarello & Robutti, to appear). It would be interesting to clarify the nature of this duality of processes.

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