

GENERAL SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we determine the general solutions of the functional equation $f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv) = g(x, y)h(u, v)$, where $f, g, h : \mathbb{K}^2 \rightarrow \mathbb{K}$ are unknown functions and $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$, where $\mathbb{K} = \mathbb{R}$ (resp. \mathbb{C}) the real (resp. complexe) field. We also treat the functional equation $f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) = g(x, y, z, t)h(u, v, w, s)$, where $f, g, h : \mathbb{K}^4 \rightarrow \mathbb{K}$ are unknown functions. We do not require any continuity condition on the unknown functions. This paper may be considered as a continuation to the paper [2].

RESUMEN. Se determinan las soluciones generales de la ecuación funcional $f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv) = g(x, y)h(u, v)$, donde $f, g, h : \mathbb{K}^2 \rightarrow \mathbb{K}$ se desconocen y $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$, donde $\mathbb{K} = \mathbb{R}$ (resp. \mathbb{C}) es el campo real (resp. complejo). También se trabaja con la ecuación $f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) = g(x, y, z, t)h(u, v, w, s)$, donde $f, g, h : \mathbb{K}^4 \rightarrow \mathbb{K}$ se desconocen. No se requiere ninguna condición de continuidad sobre las funciones desconocidas. Este artículo se puede considerar una continuación de [2].

KEY WORDS AND PHRASES. Functional equations, multiplicative functions, determinant of a matrix.

PALABRAS CLAVES. Ecuaciones funcionales, funciones multiplicativas, determinante de una matriz.

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1. INTRODUCTION

1.1. Position of the problems. Throughout this paper, \mathbb{K} will be the real or complex field and $\lambda, \mu \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$ will be two fixed scalars. The purpose of this paper is to find the general solutions of the functional equations

$$f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv) = f(x, y) f(u, v) \quad E(2; \lambda)$$

and the associated Pexider equation :

$$f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv) = g(x, y) h(u, v) \quad E_p(2; \lambda)$$

where $f, g, h : \mathbb{K}^2 \rightarrow \mathbb{K}$ are unknown functions. We also determine the general solutions of the functional equations:

$$\begin{aligned} f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) \\ = f(x, y, z, t) f(u, v, w, s) \end{aligned} \quad E(4; \lambda, \mu)$$

and the associated Pexider equation:

$$\begin{aligned} f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) \\ = g(x, y, z, t) h(u, v, w, s), \end{aligned} \quad E_p(4; \lambda, \mu)$$

where $f, g, h : \mathbb{K}^4 \rightarrow \mathbb{K}$ are unknown functions.

The methods used here to solve these equations are elementary, purely algebraic and do not require any regularity assumptions on the unknown functions.

1.2. Remarks and motivations. Suppose that $\lambda = n$ is an integer greater or equal to two, and set

$$f_n(x, y) := (x + (n - 1)y)(x - y)^{n-1}, \quad \forall x, y \in \mathbb{K}.$$

Then f_n is a particular solution to the equation $E(2; n)$. Indeed, for all $x, y \in \mathbb{K}$, consider the matrix $A_n(x, y) := [a_{ij}(x, y)]_{1 \leq i, j \leq n}$ defined by $a_{ii}(x, y) = x$ and $a_{ij}(x, y) = y$ for $i \neq j$. Then it is easy to see that

$$A_n(x, y)A_n(u, v) = A_n(xu + (n - 1)yv, xv + yu + (n - 2)yv). \quad (1)$$

Taking the determinant in both sides of (1), we see that $f_n(x, y) := \det(A_n(x, y))$ is a solution of the functional equation $E(2; n)$. However, a simple computation, will give

$$\det(A_n(x, y)) = (x + (n - 1)y)(x - y)^{n-1}.$$

When $n = 2$, the equation $E(2; 2)$ becomes

$$f(xu, yu, xv + yu) = f(x, y)f(u, v), \quad (2)$$

which was studied in the paper [2]. We recall that the paper [2] was mainly concerned with the investigation of real solutions to the equation (2) and its Pexiderised.

Our work may be viewed as a continuation to the paper [2]. It completes the paper [2] by investigating new functional equations still connected to the determinant of some matrices as shown above.

2. SOLUTIONS OF THE EQUATION $E(2; \lambda)$ AND $E_p(2; \lambda)$

A Mapping $M : \mathbb{K} \rightarrow \mathbb{K}$ is said to be *multiplicative* if $M(x \cdot y) = M(x)M(y)$ for all $x, y \in \mathbb{K}$.

Theorem 2.1. *The function $f : \mathbb{K}^2 \rightarrow \mathbb{K}$ satisfies the functional equation $E(2; \lambda)$ for all $x, y, u, v \in \mathbb{K}$ and $\lambda \in \mathbb{K}^*$ if and only if*

$$f(x, y) = M_1(x + (\lambda - 1)y)M_2(x - y), \quad \forall x, y \in \mathbb{K}, \quad (3)$$

where M_1 and M_2 are multiplicative functions defined on \mathbb{K} .

Proof. Let $f : \mathbb{K}^2 \rightarrow \mathbb{K}$ be a solution of the functional equation $E(2; \lambda)$. For all $x, y \in \mathbb{K}$, we set

$$F(x, y) := f\left(\frac{x + (\lambda - 1)y}{\lambda}, \frac{x - y}{\lambda}\right). \quad (4)$$

It is easy to see that for all $x, y \in \mathbb{K}$, we have

$$f(x, y) = F(x + (\lambda - 1)y, x - y). \quad (5)$$

Let $X, Y, U, V \in \mathbb{K}$. Then there exist $x, y, u, v \in \mathbb{K}$ uniquely determined such that

$$X = x + (\lambda - 1)y, \quad \text{and} \quad Y = x - y \quad (6)$$

and

$$U = u + (\lambda - 1)v, \quad \text{and} \quad V = u - v. \quad (7)$$

With these new variables, one has the following identities:

$$XU + (\lambda - 1)YV = \lambda[xu + (\lambda - 1)yv], \quad (8)$$

and

$$XU - YV = \lambda[xv + yu + (\lambda - 2)yv]. \quad (9)$$

Since f is a solution to $E(2; \lambda)$ then, from (8) and (9) above, we have:

$$\begin{aligned} F(XU, YV) &= f\left(\frac{XU + (\lambda - 1)YV}{\lambda}, \frac{XU - YV}{\lambda}\right) \\ &= f(xu + (\lambda - 1)yv, xv + yu + (\lambda - 2)yv) \\ &= f(x, y) f(u, v) \\ &= F(x + (\lambda - 1)y, x - y) F(u + (\lambda - 1)v, u - v) \\ &= F(X, Y) F(U, V), \end{aligned}$$

for all $X, Y, U, V \in \mathbb{K}$. We conclude that the mapping F satisfies

$$F(xu, yv) = F(x, y) F(u, v), \quad \forall x, y, u, v \in \mathbb{K}. \quad (10)$$

By setting $y = v = 1$ in (10), we see that the mapping $M_1 : x \mapsto F(x, 1)$ is multiplicative. By a similar way, the mapping $M_2 : y \mapsto F(1, y)$ is multiplicative. We deduce that for all $x, y \in \mathbb{K}$, we have

$$F(x, y) = F(x, 1)F(1, y). \quad (11)$$

By (5) and the equation (11), we get

$$f(x, y) = M_1(x + (\lambda - 1)y) M_2(x - y), \quad \forall x, y \in \mathbb{K}. \quad (12)$$

Conversely, if f is given as in (12), then it is easy to see that f satisfies equation $E(2; \lambda)$. Thus our theorem is completely proved. \square

In the next theorem we give the general solution of the Pexider equation associated to the functional equation $E(2; \lambda)$.

Theorem 2.2. *The functions $f, g, h : \mathbb{K}^2 \rightarrow \mathbb{K}$ satisfy the functional equation*

$$f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv) = g(x, y) h(u, v), \quad \forall x, y \in \mathbb{K},$$

if and only if

$$f = 0, g = 0, \text{ and } h \text{ is arbitrary} \quad (13)$$

or

$$f = 0, h = 0, \text{ and } g \text{ is arbitrary} \quad (14)$$

or

$$\begin{aligned} f(x, y) &= \alpha\beta M_1(x + (\lambda - 1)y) M_2(x - y), \\ g(x, y) &= \beta M_1(x + (\lambda - 1)y) M_2(x - y), \\ h(x, y) &= \alpha M_1(x + (\lambda - 1)y) M_2(x - y), \end{aligned} \quad (15)$$

where $M_1, M_2 : \mathbb{K} \rightarrow \mathbb{K}$ are multiplicative mappings and α, β are arbitrary nonzero constants.

Proof. Letting $u = 1$ and $v = 0$ in $E_p(2; \lambda)$, we have for some constant α

$$f(x, y) = \alpha g(x, y) \quad \forall x, y \in \mathbb{K}. \quad (16)$$

Similarly, letting $x = 1$ and $y = 0$ in $E_p(2; \lambda)$, we get for some constant β

$$f(u, v) = \beta h(u, v), \quad \forall u, v \in \mathbb{K}. \quad (17)$$

If either $\alpha = 0$ or $\beta = 0$, we get

$$f = 0, g = 0, \text{ and } h \text{ is arbitrary}$$

or

$$f = 0, h = 0, \text{ and } g \text{ is arbitrary}$$

So, let us suppose that $\alpha \neq 0$ and $\beta \neq 0$. Then it follows from (16), (17) and $E_p(2; \lambda)$ that

$$\frac{f(ux + (\lambda - 1)vy, xv + yu + (\lambda - 2)yv)}{\alpha\beta} = \frac{f(x, y)}{\alpha\beta} \frac{f(u, v)}{\alpha\beta},$$

for all $x, y, u, v \in \mathbb{K}$. From Theorem 2.1, we have

$$f(x, y) = \alpha\beta M_1(x + (\lambda - 1)y) M_2(x - y), \quad \forall x, y \in \mathbb{K}, \quad (18)$$

where $M_1, M_2 : \mathbb{K} \rightarrow \mathbb{K}$ are two multiplicative mappings. Using the relations (16) and (17), we get

$$g(x, y) = \beta M_1(x + (\lambda - 1)y) M_2(x - y),$$

and

$$h(x, y) = \alpha M_1(x + (\lambda - 1)y) M_2(x - y),$$

for all $x, y \in \mathbb{K}$. This completes the proof of the theorem. \square

3. SOLUTIONS OF THE EQUATIONS $E(4; \lambda, \mu)$ AND $E_p(4; \lambda, \mu)$

Theorem 3.1. *The function $f : \mathbb{K}^4 \rightarrow \mathbb{K}$ satisfies the functional equation $E(4; \lambda, \mu)$ if and only if*

$$f(x, y, z, t) = M_1(x + (\lambda - 1)y) M_2(x - y) M_3(z + (\mu - 1)t) M_4(z - t), \quad \forall x, y, z, t \in \mathbb{K}, \quad (19)$$

where $M_1, M_2, M_3, M_4 : \mathbb{K} \rightarrow \mathbb{K}$ are multiplicative mappings.

Proof. We let $z = w = 1$ and $t = s = 0$ in $E(4; \lambda, \mu)$. Then for all $x, y, u, v \in \mathbb{K}$, we get

$$f(xu + (\lambda - 1)yv, xv + yu + (\lambda - 2)yv, 1, 0) = f(x, y, 1, 0) f(u, v, 1, 0). \quad (20)$$

It follows from Theorem 2.1, that

$$f(x, y, 1, 0) = M_1(x + (\lambda - 1)y) M_2(x - y), \quad \forall x, y \in \mathbb{K}. \quad (21)$$

where $M_1, M_2 : \mathbb{K} \rightarrow \mathbb{K}$ are two multiplicative mappings.

Similarly, we set $u = x = 1$ and $y = v = 0$ in $E(4; \lambda, \mu)$, then for all $z, t, w, s \in \mathbb{K}$, we have

$$f(1, 0, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) = f(1, 0, z, t) f(1, 0, w, s). \quad (22)$$

Hence by Theorem 2.1, we obtain

$$f(1, 0, z, t) = M_3(z + (\mu - 1)t) M_4(z - t), \quad \forall z, t \in \mathbb{K}. \quad (23)$$

It is easy to see that for all $x, y, z, t \in \mathbb{K}$ we have

$$f(x, y, z, t) = f(x, y, 1, 0) f(1, 0, z, t), \quad \forall x, y, z, t \in \mathbb{K}. \quad (24)$$

Finally, using (21), (23) and (24), we have the desired solution and the proof of Theorem 2.2 is now complete. \square

Theorem 3.2. *The functions $f, g, h : \mathbb{K}^4 \rightarrow \mathbb{K}$ satisfy the functional equation*

$$\begin{aligned} f(xu + (\lambda - 1)yv, xv + yu + (\lambda - 2)yv, zw + (\mu - 1)ts, zs + tw + (\mu - 2)ts) \\ = g(x, y, z, w) h(u, v, w, s), \end{aligned}$$

if and only if

$$f = 0, g = 0, \text{ and } h \text{ is arbitrary} \quad (25)$$

or

$$f = 0, h = 0, \text{ and } g \text{ is arbitrary} \quad (26)$$

or

$$\begin{aligned} f(x, y, z, t) &= \alpha\beta M_1(x + (\lambda - 1)y)M_2(x - y)M_3(z + (\mu - 1)t)M_4(z - t), \\ g(x, y, z, t) &= \beta M_1(x + (\lambda - 1)y)M_2(x - y)M_3(z + (\mu - 1)t)M_4(z - t), \\ h(x, y, z, t) &= \alpha M_1(x + (\lambda - 1)y)M_2(x - y)M_3(z + (\mu - 1)t)M_4(z - t), \end{aligned} \quad (27)$$

where $M_1, M_2, M_3, M_4 : \mathbb{K} \rightarrow \mathbb{K}$ are multiplicative mappings and α, β are arbitrary nonzero constants.

The proof of this theorem is similar to the proof we made for Theorem 3.1. So we omit the details.

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